GUIDELINES

GAME THEORY

Example 1. Determine the lower and the upper prices of the game and existence of a saddle point for given payoff matrix:

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \Pi = & \begin{pmatrix} 4 & 5 & 3 \\ 6 & 7 & 4 \\ 5 & 2 & 3 \end{pmatrix} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \end{array}$$

Solution. According to the problem statement we have:
the lower price of the game is $\alpha = \max\limits_i(\alpha_i) = \max\limits_i \min\limits_j c_{ij} = \max\limits_i(3,4,2) = 4$.

the upper price of the game is $\beta = \min\limits_j(\beta_j) = \min\limits_j \max\limits_i c_{ij} = \min\limits_j(6,7,4) = 4$.

Therefore, $\alpha = \beta = 4$, then ***the game has the saddle point***, the game price $\nu = \alpha = \beta = 4$. The optimal solution is given by using the pure strategies A_2 and B_3, i.e. A_2 is the most profitable strategy of player A; B_3 is the most profitable strategy of player B.

Probabilities of strategies for player A are denoted as x_1, x_2, x_3 (A_1, A_2, A_3) and for player B are denoted as y_1, y_2, y_3 (B_1, B_2, B_3). In this case we have: $X_{opt} = (0,1,0)$ and $Y_{opt} = (0,0,1)$, i.e. we take only strategies A_2 and B_3.

Answer: $X_{opt} = (0,1,0)$, $Y_{opt} = (0,0,1)$, the game price $\nu = 4$ (units of money)., we take strategy A_2 with the probability 1; strategy B_3 with the probability 1.

For a payoff matrix with size $m \times n$ ($m \neq 2, n \neq 2$), we decrease its size with the help of exclusion of unprofitable strategies.

Example 2. Simplify this matrix and exclude unprofitable strategies:

$$\begin{pmatrix} 7 & 6 & 5 & 4 & 2 \\ 5 & 4 & 3 & 2 & 3 \\ 5 & 6 & 6 & 3 & 5 \\ 2 & 3 & 3 & 4 & 4 \end{pmatrix}_{4\times 5},$$

According to the problem statement we have:

$$\alpha = \max(2,2,3,2) = 3, \ \beta = \min(7,6,6,4,5) = 4, \quad \alpha \neq \beta.$$

***We obtain that this game has no a saddle point*.**

All elements A_2 are less than A_3, i.e. A_2 is more unprofitable for the first player, and A_2 can be excluded. All elements A_4 are less than A_3, then A_4 can be excluded.

For the second player: we compare B_1 and B_4, and exclude B_1; we compare B_2 and B_4, and exclude B_2; we compare B_3 and B_4, and exclude B_3. In the result of transformations we obtain the new matrix:

$$\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}_{2\times 2}.$$

GRAPHICAL METHOD

If a game has size $2\times n$, $m\times 2$ or 2×2 we can use the graphical method.

Let's consider a general scheme of graphical solving the game in mixed strategies:

1. To plot a system of strategies considering the behavior of player A or player B, which has 2 strategies, because a system of coordinates has 2 axes.
2. To mark the values of y_1, …, y_n and x_1, …, x_m from 0 to 1, because a probability changes from 0 to 1.
3. To plot the straight lines which correspond to another player.
4. To find two strategies for corresponding player which are intersected at the point of maximin (player A) or minimax (player B).
5. To calculate the probabilities of the optimal strategies and the game price.

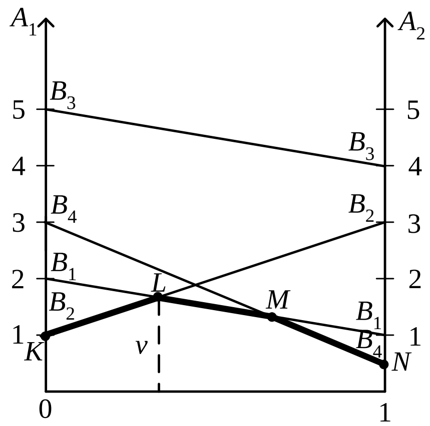
Using an optimal strategy allows obtaining the payoff (or the loss) which equals the game price: $\alpha \leq v \leq \beta$.

Example 3. Solve the matrix game and define the game price, if the payoff matrix is given by:

$$\Pi = \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{pmatrix} 2 & 1 & 5 & 3 \\ 1 & 3 & 4 & 0{,}5 \end{pmatrix} \end{array} \begin{array}{c} \\ A_1 \\ A_2 \end{array}$$

Solution. Since player A has 2 strategies, then we can use the graphical method for this problem with the given payoff matrix.

Let's plot a system of strategies and consider the behavior of player A. Let's plot strategies of player B and *choose the principle of maximin, i.e. the lower polyline and the upper point on this polyline (the point with maximal ordinate)*. We denote this *lower polyline as KLMN* and the upper point will be point L **(figure 1).**



Let's consider this point L, which is intersection of the strategies B_1 and B_2. Then we have the new matrix:

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \Pi = & \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \end{array} \begin{array}{c} A_1 \\ A_2 \end{array}.$$

or

	y_1	y_2
x_1	2	1
x_2	1	3

Let's consider strategies of player A. Let's denote x_1 as the probability that player A uses strategy A_1; x_2 as the probability that player A uses strategy A_2.

For these probabilities and the game price we have the system:

$$2 \cdot x_1 + 1 \cdot x_2 = v \tag{1}$$

(it means: if player B uses strategy B_1, then player A can choose strategy A_1 with coefficient 2 or strategy A_2 with coefficient 1)

$$1 \cdot x_1 + 3 \cdot x_2 = v \tag{2}$$

(it means: if player B uses strategy B_2, then player A can choose strategy A_1 with coefficient 1 or strategy A_2 with coefficient 3)

$$x_1 + x_2 = 1 \tag{3}$$

(the sum of probabilities of the complete group of events must be equal to 1).

We obtain: $X_{opt}=(x_1,x_2)=\left(\frac{2}{3},\frac{1}{3}\right)$ and $\nu=2\cdot x_1+1\cdot x_2=2\cdot\frac{2}{3}+1\cdot\frac{1}{3}=\frac{5}{3}$ (units of money).

Let's consider strategies of player B. Let y_1 be the probability that player B uses strategy B_1; y_2 be the probability that player B uses strategy B_2.

For these probabilities and the game price we have the system:

$$2\cdot y_1+1\cdot y_2=\nu \qquad (4)$$

(it means: if player A uses strategy A_1, then player B can choose strategy B_1 with coefficient 2 or strategy B_2 with coefficient 1)

$$1\cdot y_1+3\cdot y_2=\nu \qquad (5)$$

(it means: if player A uses strategy A_2, then player B can choose strategy B_1 with coefficient 1 or strategy B_2 with coefficient 3)

$$y_1+y_2=1 \qquad (6)$$

(the sum of probabilities of the complete group of events must be equal to 1).

Let's obtain: $Y_{opt}=(y_1,y_2,y_3,y_4)=\left(\frac{2}{3},\frac{1}{3},0,0\right)$. In this solution y_3 and y_4 are equal to 0, because strategies B_3 and B_4 are excluded from consideration by the graphical method.

Let's calculate the game price ν from (4):

$\nu=2\cdot y_1+1\cdot y_2=2\cdot\frac{2}{3}+1\cdot\frac{1}{3}=\frac{5}{3}$ (units of money).

Example 4. Solve the matrix game and define the game price, if the payoff matrix is given by:

$$\Pi=\begin{array}{cc}
\begin{array}{cc} B_1 & B_2 \end{array} & \\
\begin{pmatrix} 2 & 5 \\ 7 & 1 \\ 3 & 7 \\ 4 & 6 \end{pmatrix} & \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}
\end{array}.$$

Solution. Since player B has 2 strategies, then we can use the graphical method for this problem with the given payoff matrix. Let's plot a system of strategies for player B and consider the behavior of player A.

Let's plot strategies of player A and *choose the principle of minimax, i.e. the upper polyline and the lower point on this polyline (the point with mimimal ordinate)*. We denote this upper *polyline as KLMN* and the lower point will be point *L* **(figure 2).** Let's consider this point *L*, which is intersection of the strategies A_2 and A_4. Then we have the new matrix:

$$\begin{array}{c} \\ \Pi = \end{array} \begin{array}{cc} B_1 & B_2 \\ \multicolumn{2}{c}{\begin{pmatrix} 7 & 1 \\ 4 & 6 \end{pmatrix}} \end{array} \begin{array}{c} \\ A_2 \\ A_4 \end{array}.$$

or

	y_1	y_2
x_2	7	1
x_4	4	6

Let's consider strategies of player A. Let's denote x_2 as the probability that player A uses strategy A_2; x_4 as the probability that player A uses strategy A_4.

For these probabilities and the game price we have the system:

$$7 \cdot x_2 + 4 \cdot x_4 = v \quad (1)$$

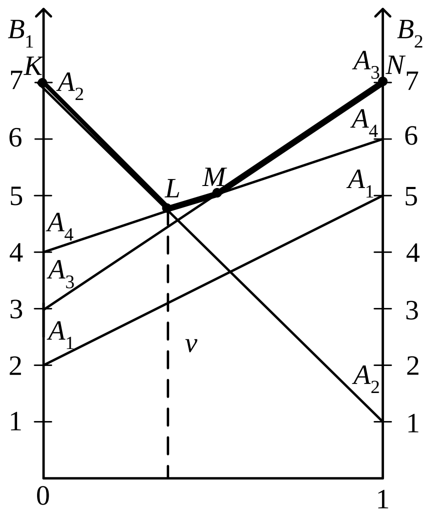
(if player B uses strategy B_1, then player A can choose strategy A_2 with 7 or strategy A_4 with 4)

$$1 \cdot x_2 + 6 \cdot x_4 = v \quad (2)$$

(if player B uses strategy B_2, then player A can choose strategy A_2 with 1 or strategy A_4 with 6)

$$x_2 + x_4 = 1 \quad (3)$$

(the sum of probabilities of the complete group of events must be equal to 1).



Let's obtain: $X_{opt} = (x_1, x_2, x_3, x_4) = \left(0, \frac{1}{4}, 0, \frac{3}{4}\right)$. In this solution x_1 and x_3 are equal to 0, because strategies A_1 and A_3 are excluded from consideration by the graphical method. Let's calculate the game price ν from (2): $\nu = 1 \cdot x_2 + 6 \cdot x_4 = 1 \cdot \frac{1}{4} + 6 \cdot \frac{3}{4} = \frac{19}{4} = 4{,}75$ (units of money).

Let's consider strategies of player B. Let y_1 be the probability that player B uses strategy B_1; y_2 be the probability that player B uses strategy B_2.

For these probabilities and the game price we have the system:

$$7 \cdot y_1 + 1 \cdot y_2 = \nu \quad (4)$$

(<u>it means:</u> if player A uses strategy A_2, then player B can choose strategy B_1 with coefficient 7 or strategy B_2 with coefficient 1)

$$4 \cdot y_1 + 6 \cdot y_2 = \nu \quad (5)$$

(<u>it means:</u> if player A uses strategy A_4, then player B can choose strategy B_1 with coefficient 4 or strategy B_2 with coefficient 6)

$$y_1 + y_2 = 1 \quad (6)$$

(the sum of probabilities of the complete group of events must be equal to 1).

Let's obtain: $Y_{opt} = (y_1, y_2) = \left(\frac{5}{8}, \frac{3}{8}\right)$. Let's calculate the game price ν from (4): $\nu = 7 \cdot y_1 + 1 \cdot y_2 = 7 \cdot \frac{5}{8} + 1 \cdot \frac{3}{8} = \frac{38}{8} = \frac{19}{4}$ (units of money).