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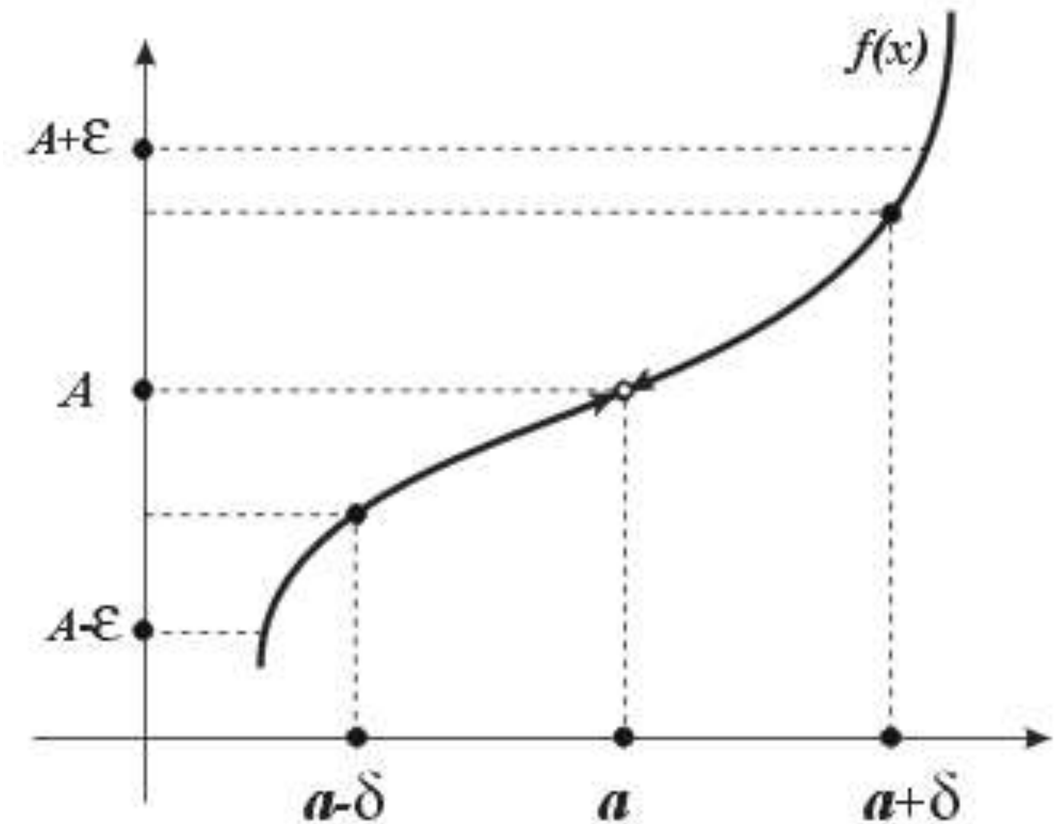
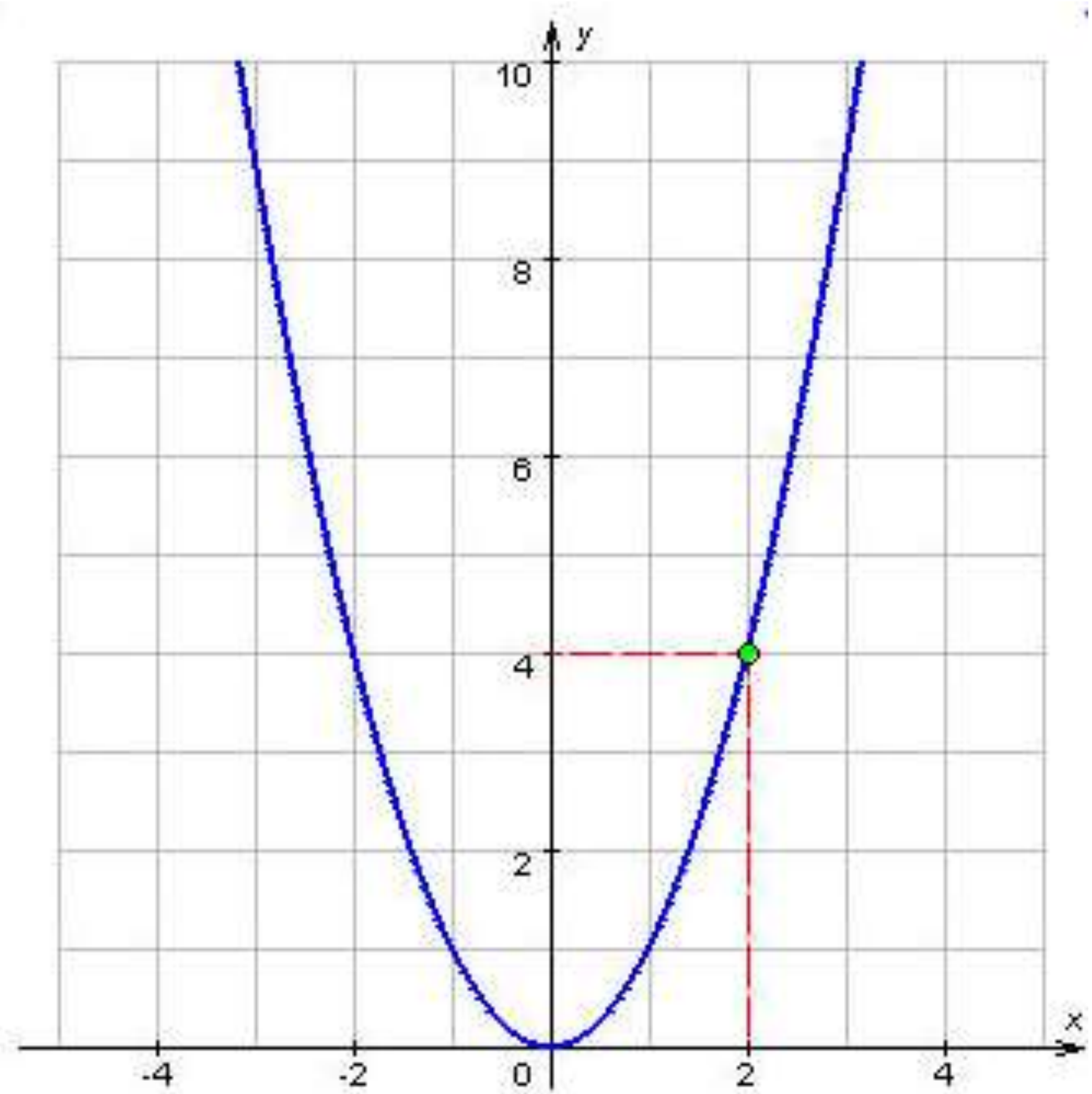
Theme:
Limit of a function Continuity of a function. Differential calculus of functions of one variable. Analysis of economic indicator interrelationships

Part 1. Limit of a function. Continuity of a function.

Cauchy’s definition. The number $A \in R$ is called **the *limit of the function*** $y = f(x)$ at $x \to a$ (***x*** approaches ***a***) and designated as $\lim\limits_{x \to a} f(x) = A$ if for any ε there exists δ such that any $0 < |x - a| < \delta$ will satisfy the inequality $|f(x) - A| < \varepsilon$.





Example.

Find

$$\lim_{x \to 2} x^2$$

Definition. The number is called ***the right (left) limit of the function*** $y = f(x)$ at the point $x = a$ if for any converging to sequence $x_1, x_2, \ldots, x_n, \ldots$ of elements of which are larger (smaller) than a, the corresponding sequence $f(x_1), f(x_2), \ldots, f(x_n), \ldots$ of values of the function converges to ***A****.*

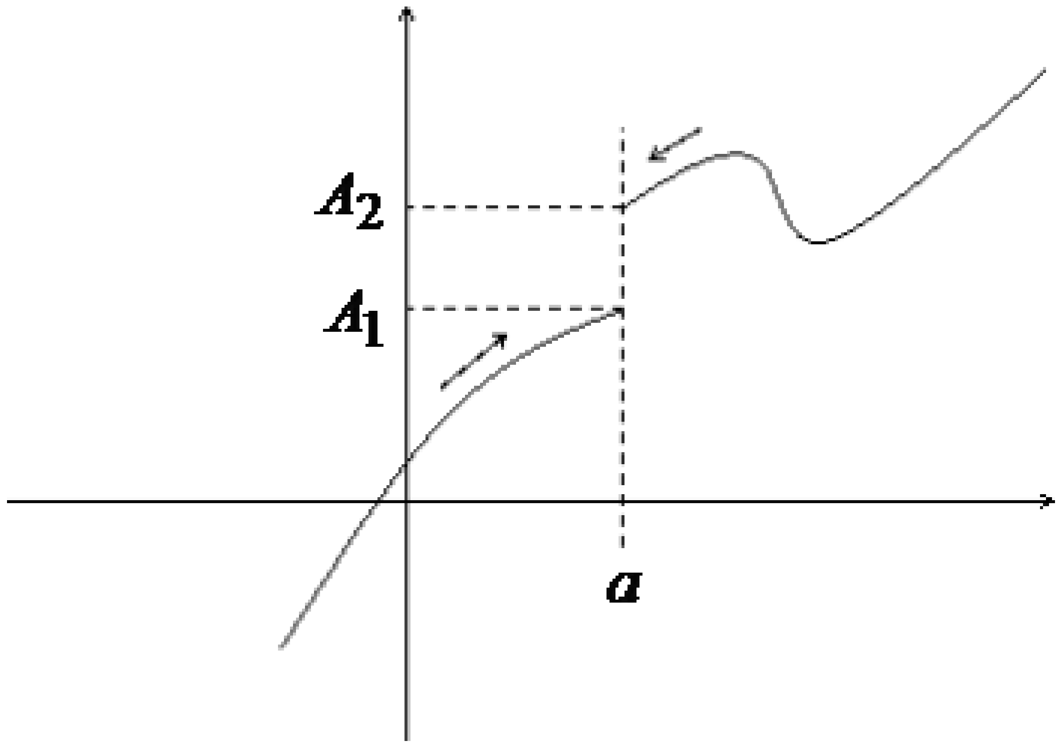
For the left limit of the function the following notation is used:

$$\lim_{x \to a-0} f(x) = A_1$$

Accordingly, the right limit of the function is denoted as follows:

$$\lim_{x \to a+0} f(x) = A_2$$

Definition*.* The right (left) limit of the function is called *one-sided limit*.



$$\lim_{x\to a-0} f(x) = A_1 \qquad\qquad \lim_{x\to a+0} f(x) = A_2$$

Continuity of a function

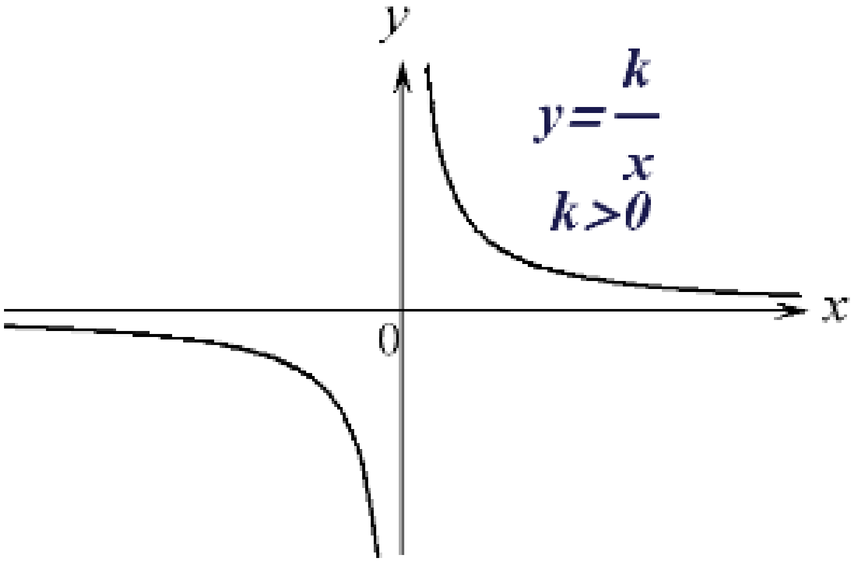
A function $y = f(x)$ is called at a point x_0 if

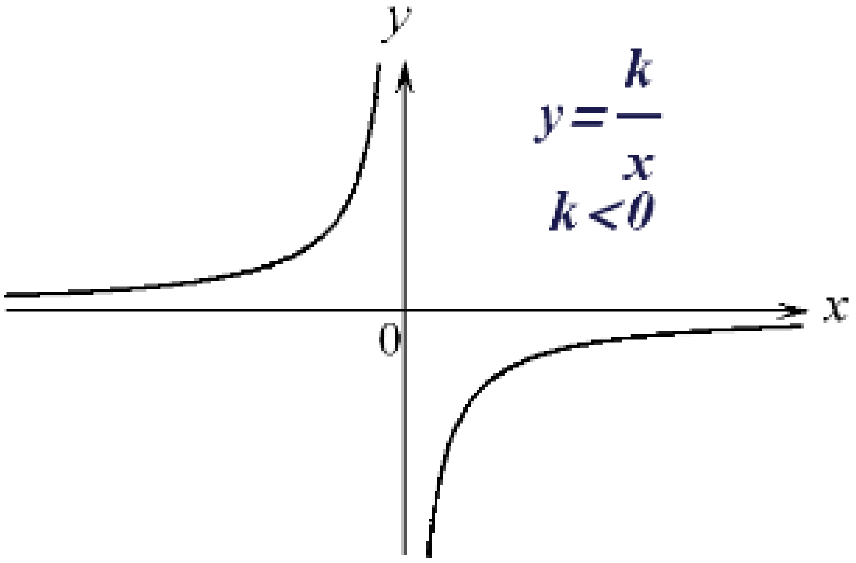
$$\lim_{x \to x_0} f(x) = f(x_0)$$

Continuity of a function

$$\lim_{x\to x_0-0} f(x) = \lim_{x\to x_0} f(x) = \lim_{x\to x_0+0} f(x)$$

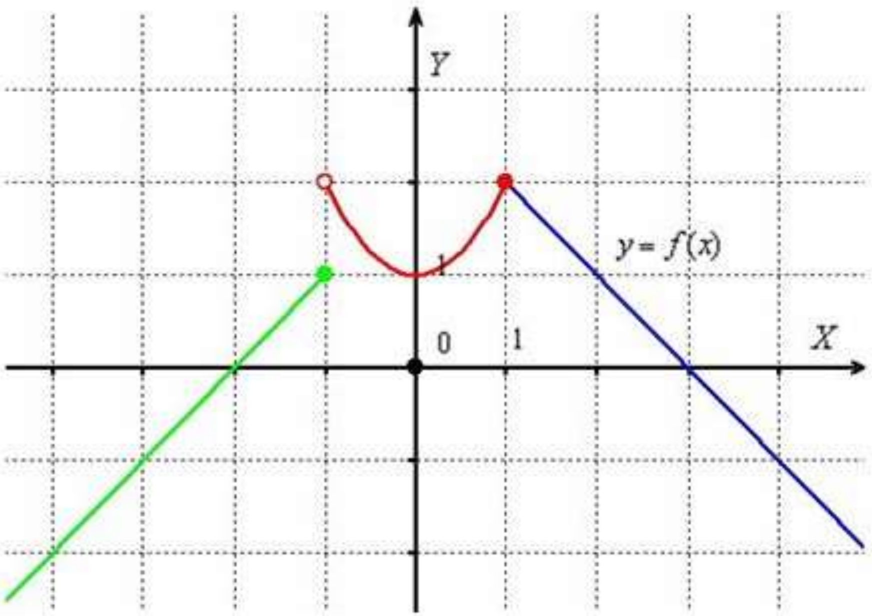
$$f(x_0-0) = f(x_0) = f(x_0+0)$$


y
y=k/x
k>0
x
0


y
y=k/x
k<0
x
0

Example

$$f(x)=\begin{cases} x+2, & x\le -1 \\ x^2+1, & -1<x\le 1 \\ -x+3, & x>1 \end{cases}$$



Basic "rules-theorems" of calculation of limits. Some properties of limits

If we know that the functions $f(x)$ and $g(x)$ have limits and these limits are finite, then:

1. If a function is constant, i.e. $f(x) = C = const$

   then $\lim\limits_{x \to a} f(x) = C$

2. A limit of the sum is equal to the sum of limits, i.e.

$$\lim_{x \to a} \left(f(x) + g(x) \right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

Basic "rules-theorems" of calculation of limits. Some properties of limits

If we know that the functions $f(x)$ and $g(x)$ have limits and these limits are finite, then:

3. A limit of the product is equal to the product of limits, i.e.

$$\lim_{x\to a}\left(f(x)\cdot g(x)\right)=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$

4. A limit of the quotient is equal to the quotient of limits, i.e.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$, if $\lim\limits_{x\to a} g(x)\neq 0$

Some properties of limits:

1) $\lim_{x\to\infty} C\cdot x = \infty$

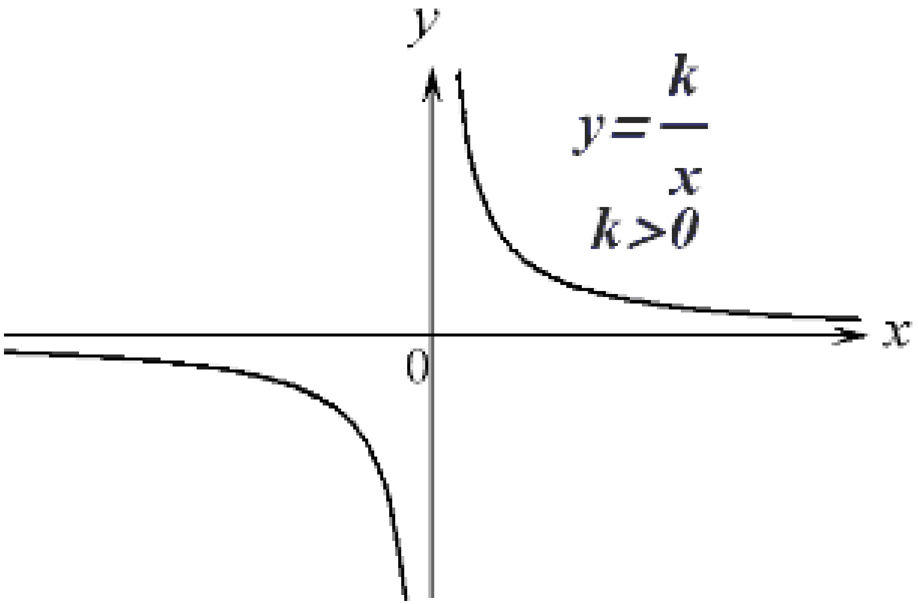
2) $\lim_{x\to\infty} \frac{C}{x} = 0$

3) $\lim_{x\to\infty} \frac{x}{C} = \infty$

4) $\lim_{x\to 0} \frac{C}{x} = \infty$

5) $\lim_{x\to 0} C\cdot x = 0$

6) $\lim_{x\to 0} \frac{x}{C} = 0$



The 1-st and the 2-nd remarkable limits and their consequences

The 1-st remarkable limit:

$$\lim_{x \to 0} \frac{\sin x}{x} = \left\| \frac{0}{0} \right\| = 1$$

The 2-nd remarkable limit:

$$\lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^{x} = \left\| 1^{\infty} \right\| = e$$

$$e \approx 2{,}718$$

$$\lim_{x \to 0} \left(1 + x \right)^{\frac{1}{x}} = \left\| 1^{\infty} \right\| = e$$

Techniques of calculation of limits

At calculating limits of a function we use the rule of limiting transition under the sign of a continuous function. This rule is formulated as follows:

$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right)$$

Example

Find the limit:

Example 1:

$$\lim_{x \to 1}\left(x^2 - 3x + 5\right) = 1 - 3 + 5 = 3.$$

Example

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Example 1:

$$\lim_{x\to 1}\left(x^2-3x+5\right)=1-3+5=3.$$

USE OCTAVE

octave:1> syms x

octave:2> f=x^2-3*x+5

f = (sym)

```
   2
  x  - 3·x + 5
```

octave:3> limit(f,1)

ans = (sym) 3

Example

Find the limit:

$$\lim_{x\to 2}\frac{2x^3+1}{x^2-3}$$

Solution. Let's substitute the limiting value x=2 and get:

$$\lim_{x\to 2}\frac{2x^3+1}{x^2-3}=\frac{2\cdot 2^3+1}{2^2-3}=17.$$

All elementary functions are *continuous* in their domains of definition. While calculating limits, first of all, we have to replace an argument of a function by its limiting value and find out whether there is indetermination.

Indeterminate forms expressions are the following:

$$\left\|\frac{0}{0}\right\|, \left\|\frac{\infty}{\infty}\right\|, \left\|\infty \cdot 0\right\|, \left\|\infty - \infty\right\|, \left\|\infty^{0}\right\|, \left\|0^{0}\right\|, \left\|1^{\infty}\right\|$$

If after the substitution of a limiting value of an argument we obtain an *indeterminate form,* then we should carry out some identical transformations that will *eliminate the indetermination* and then the sought limit is calculated.

Let us sequentially consider the standard cases of evaluation of the indefinite expressions.

Calculation of a limit of a rational function

$$\lim_{x \to a} \frac{P(x)}{Q(x)}$$

where

$$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

$$Q(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_{m-1} x + b_m$$

are polynomials of the orders *n* and *m*

$$n, m \in N$$

1) $$\lim_{x \to a} \frac{P(x)}{Q(x)} = \left\| \frac{0}{0} \right\|$$

To eliminate the indetermination let us carry out identical transformations, i.e. to pick out in the numerator and the denominator the factor approaching 0, namely $(x - a)$

In this case we should remember the following:

1) The consequence of the ***Bezout's theorem***:

if a is the root of the polynomial $P_n(x)$, i.e. $P_n(a)=0$

then $P_n(x)$ is divided by $(x-a)$ the binomial without a remainder:

$$P_n(x)=(x-a)\cdot R_{n-1}(x)$$

In this case we should remember the following:

1) The consequence of the *Bezout's theorem:* if a is the root of the polynomial $P_n(x)$, i.e. $P_n(a)=0$, then $P_n(x)$ is divided by the binomial $(x-a)$ without a remainder:

$$P_n(x)=(x-a)\cdot R_{n-1}(x)$$

2) The square trinomial $P_2(x)=ax^2+bx+c$, where $D\geq 0$ ($D=b^2-4ac$ is the discriminant of $ax^2+bx+c=0$), can be presented as a product of linear factors:

$$ax^2+bx+c=a(x-x_1)(x-x_2)$$

where x_1 and x_2 are roots of the square trinomial.

3) ***Vieta*** theorem

$$x_1+x_2=-\frac{b}{a}$$

$$x_1\cdot x_2=\frac{c}{a}$$

Example 9: Calculate the limit:

$$\lim_{x\to 2}\frac{x^2-4}{x^2-x-2}=\left[\frac{0}{0}\right]=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x+1)}=\lim_{x\to 2}\frac{x+2}{x+1}=\frac{4}{3}$$

Example. Let's calculate

$$A = \lim_{x \to 2} \frac{x^2 - 6x + 8}{x^2 - 8x + 12} =$$

Example. Let's calculate

$$A=\lim_{x\to 2}\frac{x^2-6x+8}{x^2-8x+12}=\left\|\frac{2^2-6\cdot 2+8}{2^2-8\cdot 2+12}=\frac{0}{0}\right\|$$

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$$A=\lim_{x\to 2}\frac{x^2-6x+8}{x^2-8x+12}=\left\|\frac{2^2-6\cdot 2+8}{2^2-8\cdot 2+12}=\frac{0}{0}\right\|$$

Solution. In this case we have to eliminate the indetermination, i.e. to pick out in the numerator and the denominator the factor tending to zero. It is obvious that is the root of the polynomials in the numerator and the denominator.

Let's find roots of the square trinomial $ax^2+bx+c=0$ and present it as

$$ax^2+bx+c=a(x-x_1)(x-x_2)$$

where

$$x_1=\frac{-b+\sqrt{D}}{2a}\qquad x_2=\frac{-b-\sqrt{D}}{2a}$$

$$D=b^2-4ac$$

Let’s find roots of the polynomials in the numerator and the denominator:

$$x^2-6x+8=0$$

$$D=b^2-4ac=(-6)^2-4\cdot1\cdot8=36-32=4$$

$$x_1=\frac{-b+\sqrt{D}}{2a}=\frac{-(-6)+\sqrt{4}}{2\cdot1}=\frac{6+2}{2}=\frac{8}{2}=4$$

$$x_2=\frac{-b-\sqrt{D}}{2a}=\frac{-(-6)-\sqrt{4}}{2\cdot1}=\frac{6-2}{2}=\frac{4}{2}=2$$

$$a(x-x_1)(x-x_2)=(x-4)(x-2)$$

$$x^2 - 8x + 12 = 0$$

$$D = b^2 - 4ac = (-8)^2 - 4 \cdot 1 \cdot 12 = 64 - 48 = 16$$

$$x_1 = \frac{-b + \sqrt{D}}{2a} = \frac{-(-8) + \sqrt{16}}{2 \cdot 1} = \frac{8 + 4}{2} = \frac{12}{2} = 6$$

$$x_2 = \frac{-b - \sqrt{D}}{2a} = \frac{-(-8) - \sqrt{16}}{2 \cdot 1} = \frac{8 - 4}{2} = \frac{4}{2} = 2$$

$$a(x - x_1)(x - x_2) = (x - 6)(x - 2)$$

Let’s substitute into the sought limit:

$$A=\lim_{x\to 2}\frac{x^2-6x+8}{x^2-8x+12}=\left\|\frac{2^2-6\cdot 2+8}{2^2-8\cdot 2+12}=\frac{0}{0}\right\|=$$

$$=\lim_{x\to 2}\frac{(x-2)(x-4)}{(x-2)(x-6)}=\lim_{x\to 2}\frac{x-4}{x-6}=\frac{2-4}{2-6}=\frac{-2}{-4}=\frac{1}{2}$$

2) $$\lim_{x\to\infty}\frac{P(x)}{Q(x)}=\left\|\frac{\infty}{\infty}\right\|$$

$$P(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$$

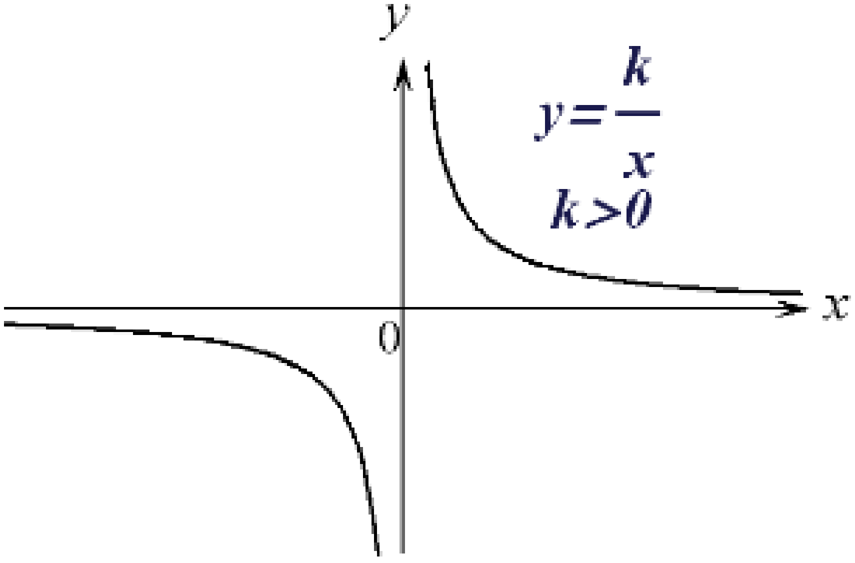
$$Q(x)=b_0x^m+b_1x^{m-1}+\ldots+b_{m-1}x+b_m$$

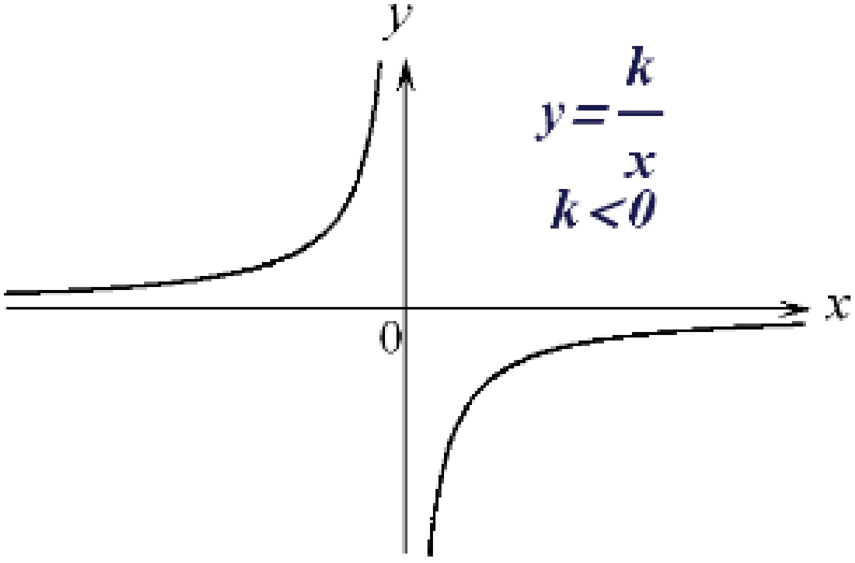
We find x to the greatest power in the numerator and the denominator and divide each summand of the fraction. After this we substitute $x=\infty$ into the limit and use the following limit

$$\lim_{x\to\infty}\frac{C}{x^\alpha}=0$$

Let’s remember

$$\lim_{x \to \infty} \frac{C}{x^{\alpha}} = 0$$





Example 12.1: Calculate the limit:

$$\lim_{x\to\infty}\frac{(x+10)^2}{8x^2+10}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{x^2+20x+100}{8x^2+10}=\lim_{x\to\infty}\frac{1+\frac{20}{x}+\frac{100}{x^2}}{8+\frac{10}{x^2}}=\frac{1}{8}$$

Example 13:

$$\lim_{x\to\infty}\frac{x^2+2x-3}{x^3+7x+1}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{2}{x^2}-\frac{3}{x^3}}{1+\frac{7}{x^2}+\frac{1}{x^3}}=\frac{0}{1}=0$$

Example 14.1: Calculate the limit:

$$\lim_{x\to\infty}\frac{2x^2-3x-5}{x+1}=\frac{\infty}{\infty}=$$

$$=\lim_{x\to\infty}\frac{\dfrac{2x^2-3x-5}{x^2}}{\dfrac{x+1}{x^2}}=\lim_{x\to\infty}\frac{2-\dfrac{3}{x}^{\to 0}-\dfrac{5}{x^2}^{\to 0}}{\dfrac{1}{x}^{\to 0}+\dfrac{1}{x^2}^{\to 0}}=\frac{2}{0}=\infty$$

Example 2. Let's calculate $\lim\limits_{x\to\infty}\dfrac{3x^3+4x^2+7}{2x^3-3x^2+11}$

Solution. Let's replace x by $x=\infty$.

$$\lim_{x\to\infty}\frac{3x^3+4x^2+7}{2x^3-3x^2+11}=\left\|\frac{\infty}{\infty}\right\|=$$

Example 2. Let's calculate $\lim\limits_{x\to\infty}\frac{3x^3+4x^2+7}{2x^3-3x^2+11}$

Solution. Let's replace x by $x=\infty$.

$$\lim_{x\to\infty}\frac{3x^3+4x^2+7}{2x^3-3x^2+11}=\left\|\frac{\infty}{\infty}\right\|=$$

Let's eliminate the indetermination $\left\|\frac{\infty}{\infty}\right\|$. For this we divide the numerator and the denominator by the factor *x* to the greatest power, i.e. x^3

$$=\lim_{x\to\infty}\frac{\frac{3x^3}{x^3}+\frac{4x^2}{x^3}+\frac{7}{x^3}}{\frac{2x^3}{x^3}-\frac{3x^2}{x^3}+\frac{11}{x^3}}=$$

Example 2. Let's calculate $\lim\limits_{x\to\infty} \dfrac{3x^3+4x^2+7}{2x^3-3x^2+11}$

Solution. Let's replace x by $x=\infty$.

$$\lim_{x\to\infty} \frac{3x^3+4x^2+7}{2x^3-3x^2+11} = \left\| \frac{\infty}{\infty} \right\| =$$

Let's eliminate the indetermination $\left\| \frac{\infty}{\infty} \right\|$. For this we divide the numerator and the denominator by the factor x to the greatest power, i.e. x^3

$$= \lim_{x\to\infty} \frac{\dfrac{3x^3}{x^3}+\dfrac{4x^2}{x^3}+\dfrac{7}{x^3}}{\dfrac{2x^3}{x^3}-\dfrac{3x^2}{x^3}+\dfrac{11}{x^3}} = \lim_{x\to\infty} \frac{3+\dfrac{4}{x}+\dfrac{7}{x^3}}{2-\dfrac{3}{x}+\dfrac{11}{x^3}} =$$

Let's use the following limit $\lim_{x \to \infty} \frac{C}{x^{\alpha}} = 0$

We have

$$= \lim_{x \to \infty} \frac{3 + \frac{4}{x} + \frac{7}{x^3}}{2 - \frac{3}{x} + \frac{11}{x^3}} = \frac{3+0+0}{2-0+0} = \frac{3}{2}$$

Thus,

$$\lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)}=\begin{cases}0, & if \quad n<m\\ \dfrac{a_0}{b_0}, & if \quad n=m\\ \infty, & if \quad n>m\end{cases}$$

$$P(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$$

$$Q(x)=b_0x^m+b_1x^{m-1}+\ldots+b_{m-1}x+b_m$$

Calculation of limits of functions with irrational expressions

While calculating the limits of functions which have an irrational expression, which vanishes as $x \to a$ we should pick out the factor $x - a \to 0$. We can do it by getting rid of irrationality in the numerator and the denominator by multiplying the given fraction by the correspondent conjugate factor. At that the following formula is often used:

$$a^2 - b^2 = (a - b)(a + b)$$

$$a^3 \pm b^3 = (a \pm b)\left(a^2 \mp ab + b^2\right)$$

TASK

Example 17.1:

$$\lim_{x\to 6}\frac{\sqrt{x-2}-2}{x-6}=\left[\frac{0}{0}\right]=\lim_{x\to 6}\frac{(\sqrt{x-2}-2)(\sqrt{x-2}+2)}{(x-6)(\sqrt{x-2}+2)}=$$

$$=\lim_{x\to 6}\frac{x-2-4}{(x-6)(\sqrt{x-2}+2)}=\lim_{x\to 6}\frac{1}{\sqrt{x-2}+2}=\frac{1}{4}$$

TASK

Example 17.2:

$$\lim_{x\to 2}\frac{\sqrt{3x-2}-2}{\sqrt{2x+5}-3}=\left(\frac{0}{0}\right)=\lim_{x\to 2}\frac{\left(\sqrt{3x-2}-2\right)\left(\sqrt{3x-2}+2\right)}{\left(\sqrt{2x+5}-3\right)\left(\sqrt{3x-2}+2\right)}=$$

$$=\lim_{x\to 2}\frac{3x-2-4}{\left(\sqrt{2x+5}-3\right)\left(\sqrt{3x-2}+2\right)}=\lim_{x\to 2}\frac{3(x-2)\left(\sqrt{2x+5}+3\right)}{\left(\sqrt{2x+5}-3\right)\left(\sqrt{2x+5}+3\right)\left(\sqrt{3x-2}+2\right)}=$$

$$=\lim_{x\to 2}\frac{3(x-2)\left(\sqrt{2x+5}+3\right)}{(2x+5-9)\left(\sqrt{3x-2}+2\right)}=\lim_{x\to 2}\frac{3(x-2)\left(\sqrt{2x+5}+3\right)}{2(x-2)\left(\sqrt{3x-2}+2\right)}=$$

$$=\frac{3}{2}\lim_{x\to 2}\frac{\sqrt{2x+5}+3}{\sqrt{3x-2}+2}=\frac{3}{2}\cdot\frac{3+3}{2+2}=\frac{18}{8}=\frac{9}{4}$$

TASK

Let's calculate the following limit:

$$\lim_{x \to 8} \frac{\sqrt{9+2x}-5}{x-8}$$

Example 3. Let's calculate $$\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{x^2-6x-16}$$

Solution. We have

Example 3. Let's calculate $\lim\limits_{x\to 8} \dfrac{\sqrt{9+2x}-5}{x^2-6x-16}$

Solution. We have

$$\lim_{x\to 8} \frac{\sqrt{9+2x}-5}{x^2-6x-16} = \left\|\frac{0}{0}\right\|$$

Example 3. Let’s calculate $\lim\limits_{x\to 8}\dfrac{\sqrt{9+2x}-5}{x^2-6x-16}$

Solution. We have

$$\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{x^2-6x-16}=\left\|\frac{0}{0}\right\|$$

As $x\to 8$ then $x-8\to 0$. Let us pick out the factor $(x-8)$ in the numerator and the denominator.

The numerator has the irrational function $\sqrt{9+2x}-5=a-b$

To get rid of irrationality in the numerator and the denominator we multiply the given fraction by the correspondent conjugate factor

$$a+b=\sqrt{9+2x}+5$$

We obtain the product $(a-b)(a+b)$

$$(a-b)(a+b)=a^2-b^2$$

Thus in the numerator we get the following expression:

$$\left(\sqrt{9+2x}+5\right)\left(\sqrt{9+2x}-5\right)$$

Thus calculation of the given limit is as follows:

$$\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{x^2-6x-16}=\left\|\frac{0}{0}\right\|=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}-5\right)\left(\sqrt{9+2x}+5\right)}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

Thus calculation of the given limit is as follows:

$$\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{x^2-6x-16}=\left\|\frac{0}{0}\right\|=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}-5\right)\left(\sqrt{9+2x}+5\right)}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$a^2-b^2=(a-b)(a+b)$$

Thus calculation of the given limit is as follows:

$$\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{x^2-6x-16}=\left\|\frac{0}{0}\right\|=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}-5\right)\left(\sqrt{9+2x}+5\right)}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}\right)^2-5^2}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

Thus calculation of the given limit is as follows:

$$\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{x^2-6x-16}=\left\|\frac{0}{0}\right\|=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}-5\right)\left(\sqrt{9+2x}+5\right)}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}\right)^2-5^2}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=\lim_{x\to 8}\frac{9+2x-25}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

Thus calculation of the given limit is as follows:

$$\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{x^2-6x-16}=\left\|\frac{0}{0}\right\|=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}-5\right)\left(\sqrt{9+2x}+5\right)}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}\right)^2-5^2}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=\lim_{x\to 8}\frac{9+2x-25}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{2x-16}{(x-8)(x+2)\left(\sqrt{9+2x}+5\right)}=$$

Thus calculation of the given limit is as follows:

$$\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{x^2-6x-16}=\left\|\frac{0}{0}\right\|=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}-5\right)\left(\sqrt{9+2x}+5\right)}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}\right)^2-5^2}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=\lim_{x\to 8}\frac{9+2x-25}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{2x-16}{\left(x-8\right)(x+2)\left(\sqrt{9+2x}+5\right)}=\lim_{x\to 8}\frac{2(x-8)}{\left(x-8\right)(x+2)\left(\sqrt{9+2x}+5\right)}=$$

Thus calculation of the given limit is as follows:

$$\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{x^2-6x-16}=\left\|\frac{0}{0}\right\|=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}-5\right)\left(\sqrt{9+2x}+5\right)}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}\right)^2-5^2}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=\lim_{x\to 8}\frac{9+2x-25}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{2x-16}{\left(x-8\right)(x+2)\left(\sqrt{9+2x}+5\right)}=\lim_{x\to 8}\frac{2(x-8)}{\left(x-8\right)(x+2)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{2}{(x+2)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}\right)^2-5^2}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=\lim_{x\to 8}\frac{9+2x-25}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{2x-16}{\left(x-8\right)(x+2)\left(\sqrt{9+2x}+5\right)}=\lim_{x\to 8}\frac{2(x-8)}{\left(x-8\right)(x+2)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{2}{(x+2)\left(\sqrt{9+2x}+5\right)}=$$

$$=\frac{2}{\left(8+2\right)\cdot\left(\sqrt{9+2\cdot 8}+5\right)}=\frac{2}{10\cdot 10}=\frac{1}{50}$$

3) While disclosing indeterminations like $\left\| \infty - \infty \right\|$ we have to carry out identical transformations allowing to reduce this indetermination to

$\left\| \frac{0}{0} \right\|$ or $\left\| \frac{\infty}{\infty} \right\|$.

Example 4. Let's calculate $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + 3x - 2} - x \right)$

Solution. $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + 3x - 2} - x \right) = \left\| \infty - \infty \right\| =$

3) While disclosing indeterminations like $\left\| \infty - \infty \right\|$ we have to carry out identical transformations allowing to reduce this indetermination to $\left\| \frac{0}{0} \right\|$ or $\left\| \frac{\infty}{\infty} \right\|$.

Example 4. Let's calculate $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + 3x - 2} - x \right)$

Solution. $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + 3x - 2} - x \right) = \left\| \infty - \infty \right\| =$

We multiply and divide the given expression by the «conjugate» expression.

$$= \lim_{x \to \infty} \frac{\left(\sqrt{x^2 + 3x - 2} - x \right)\left(\sqrt{x^2 + 3x - 2} + x \right)}{\left(\sqrt{x^2 + 3x - 2} + x \right)} =$$

$$= \lim_{x\to\infty} \frac{x^2 + 3x - 2 - x^2}{\sqrt{x^2 + 3x - 2} + x} = \lim_{x\to\infty} \frac{3x - 2}{\sqrt{x^2 + 3x - 2} + x} = \left\| \frac{\infty}{\infty} \right\| =$$

$$= \lim_{x\to\infty} \frac{x^2 + 3x - 2 - x^2}{\sqrt{x^2 + 3x - 2} + x} = \lim_{x\to\infty} \frac{3x - 2}{\sqrt{x^2 + 3x - 2} + x} = \left\| \frac{\infty}{\infty} \right\| =$$

$$= \lim_{x\to\infty} \frac{\frac{3x}{x} - \frac{2}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{3x}{x^2} - \frac{2}{x^2}} + \frac{x}{x}} = \lim_{x\to\infty} \frac{3 - \frac{2}{x}}{\sqrt{1 + \frac{3}{x} - \frac{2}{x^2}} + 1} =$$

$$= \frac{3 - 0}{\sqrt{1 + 0 - 0} + 1} = \frac{3}{1 + 1} = \frac{3}{2}$$

Calculating limits of functions using the 1-st and the 2-nd remarkable limits and their consequences

1) The 1-st remarkable limit:

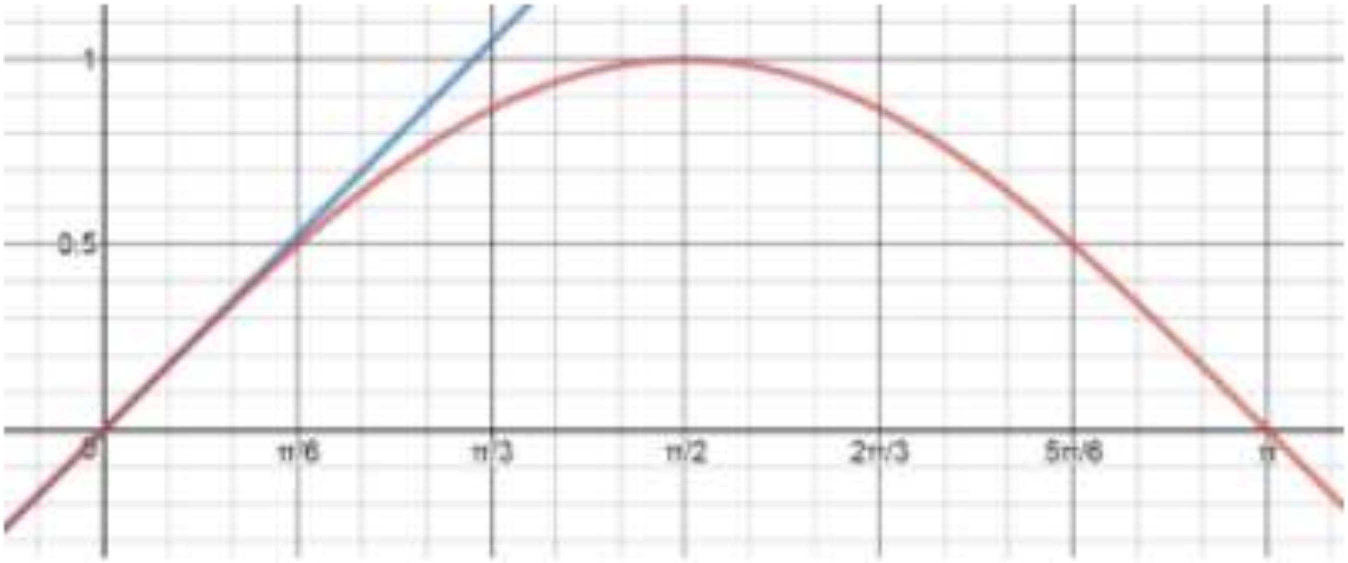
$$\lim_{x\to 0}\frac{\sin x}{x}=\left\|\frac{0}{0}\right\|=1$$

To disclose indeterminations like $\left\|\frac{0}{0}\right\|$ we should use the 1-st remarkable limit, carry out the elementary transformations with the numerator and the denominator and apply the trigonometric formulas.

Calculating limits of functions using the 1-st and the 2-nd remarkable limits and their consequences

1) The 1-st remarkable limit:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



Consequences from the first remarkable limit:

$$\lim_{x\to 0}\frac{x}{\sin x}=1$$

$$f(x)\to 0$$

$$\lim_{x\to 0}\frac{\sin f(x)}{f(x)}=\left\|\frac{0}{0}\right\|=1$$

$$\lim_{x\to 0}\frac{\arcsin x}{x}=1;$$

$$\lim_{x\to 0}\frac{\arcsin f(x)}{f(x)}=\left\|\frac{0}{0}\right\|=1$$

$$\lim_{x\to 0}\frac{\operatorname{tg} x}{x}=1;$$

$$\lim_{x\to 0}\frac{tg\ f(x)}{f(x)}=\left\|\frac{0}{0}\right\|=1$$

$$\lim_{x\to 0}\frac{\operatorname{arctg} x}{x}=1.$$

$$\lim_{x\to 0}\frac{1-\cos x}{\frac{x^2}{2}}=1$$

$$\lim_{x\to 0}\frac{arctg\ f(x)}{f(x)}=\left\|\frac{0}{0}\right\|=1$$

Example 5. Let's calculate $\lim\limits_{x\to 0}\frac{\sin 5x}{x}$

Solution. $\lim\limits_{x\to 0}\frac{\sin 5x}{x}=\left\|\frac{0}{0}\right\|=$

We apply the consequence $\lim\limits_{x\to 0}\frac{\sin\ f(x)}{f(x)}=\left\|\frac{0}{0}\right\|=1$

Here $f(x)=5x$.

We obtain

$$=\lim_{x\to 0}\underbrace{\frac{\sin 5x}{5x}}_{\to 1}\cdot 5=5$$

Example

Find the limits:

$$\lim_{x\to 0}\frac{\sin 7x}{3x}=\frac{0}{0}=\lim_{x\to 0}\frac{\sin 7x}{3\cdot\frac{1}{7}\cdot 7x}=\frac{1}{\frac{3}{7}}=\frac{7}{3}$$

Example

Find the limits:

$$\lim_{x\to 0}\frac{5x^2}{\sin^2\frac{x}{2}}=\frac{0}{0}=\lim_{x\to 0}\frac{5x\cdot x}{\sin\frac{x}{2}\cdot\sin\frac{x}{2}}=\lim_{x\to 0}\frac{5\cdot 2\cdot 2\cdot\frac{x}{2}\cdot\frac{x}{2}}{\sin\frac{x}{2}\cdot\sin\frac{x}{2}}=5\cdot 2\cdot 2=20$$

Comparison of infinitesimals

Definition. A function $\beta = \beta(x)$ *is called infinitesimal if*

$$\lim_{x \to \infty} \beta(x) = 0$$

Comparison of infinitesimals

Equivalencies at $x \to 0$ **or** $f(x) \to 0$

$$\sin f(x) \sim f(x)$$

$$\arcsin f(x) \sim f(x)$$

$$tg\ f(x) \sim f(x)$$

$$arctg\ f(x) \sim f(x)$$

$$1 - \cos f(x) \sim \frac{1}{2} f^2(x)$$

Example

$$\lim_{x\to 0}\frac{arctg\,2x}{8x}=\left\|\frac{0}{0}\right\|=\lim_{x\to 0}\frac{2x}{8x}=\frac{2}{8}=\frac{1}{4}$$

$$arctg\,2x \underset{x\to 0}{\sim} 2x$$

Example

Example 19:

$$\lim_{x\to 0}\frac{\sin 5x}{tg\,8x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{5x}{8x}=\frac{5}{8}.$$

$$x\to 0,\ \sin 5x\ \sim 5x,\ tg8x\sim 8x,$$

Example

$$\lim_{x\to 0}\frac{\cos(x)-\cos(3x)}{x^2}=\left[\frac{1-1}{0}\right]=\lim_{x\to 0}\frac{2\sin(2x)\cdot\sin(x)}{x^2}=$$

$$=\lim_{x\to 0}\frac{2\boxed{\sin(2x)}\cdot\boxed{\sin(x)}}{\frac{1}{2}\cdot\boxed{2x}\cdot\boxed{x}}=4$$

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$$

Example

$$\lim_{x\to 0}\frac{\sin 7x - \sin 5x}{\sin x} = \lim_{x\to 0}\frac{2\sin\frac{7x-5x}{2}\cos\frac{7x+5x}{2}}{\sin x} = \lim_{x\to 0}\frac{2\sin x\cos 6x}{\sin x} = \lim_{x\to 0}(2\cos 6x) =$$

$$= \lim_{x\to 0}(2\cos 6x) = 2\lim_{x\to 0}\cos 6x =$$

$$= 2\cdot\cos(4\cdot 0) = 2\cdot 1 = 2.$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$$

Example

$$\lim_{x\to 0}\frac{1-\cos 4x}{5x}=\frac{0}{0}=\lim_{x\to 0}\frac{2\sin^2 2x}{5x}=\frac{2}{5}\lim_{x\to 0}\frac{\sin^2 2x}{x}=$$

$$=\frac{2}{5}\lim_{x\to 0}\frac{\sin 2x\cdot\sin 2x}{\frac{1}{2}\cdot 2x}=\frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{\frac{1}{2}}=\frac{2}{5}\cdot 2\lim_{x\to 0}(\sin 2x)^{\to 0}=\frac{4}{5}\cdot 0=0$$

2) The 2-nd remarkable limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \left\|1^{\infty}\right\| = e$$

where $e \approx 2{,}718$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \left\|1^{\infty}\right\| = e$$

To disclose indeterminations like $\left\|1^{\infty}\right\|$ we should use the 2-nd remarkable limit and carry out transformations with the base and the exponent.

EXAMPLE

Let’s calculate the limit:

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}=\left[1^{\infty}\right]=\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{\frac{x}{2}\cdot 2}=\underbrace{\lim_{x\to\infty}\left(\left(1+\frac{2}{x}\right)^{\frac{x}{2}}\right)^{2}}_{e}=e^{2}$$

EXAMPLE

Let’s calculate the limit:

$$\lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^{4x}=1^{\infty}=\lim_{x\to\infty}\left(\left(1+\frac{1}{3x}\right)^{3x}\right)^{\frac{1}{3x}\cdot 4x}=e^{\lim\limits_{x\to\infty}\frac{4x}{3x}}=e^{\frac{4}{3}}$$

e

EXAMPLE

Let's calculate the limit:

$$\lim_{x\to\infty}\left(1-\frac{6}{x}\right)^{\frac{x}{3}}=\left[1^{\infty}\right]=\lim_{x\to\infty}\left(1+\frac{-6}{x}\right)^{\frac{-6\cdot x}{-6\cdot 3}}=\underbrace{\lim_{x\to\infty}\left(\left(1+\frac{-6}{x}\right)^{-\frac{x}{6}}\right)}_{e}{}^{-2}=e^{-2}$$

EXAMPLE

Let’s calculate the limit:

$$\lim_{x\to\infty}\left(1+\frac{2}{x+1}\right)^{3x}=\left[1^{\infty}\right]=\lim_{x\to\infty}\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}\cdot\frac{2}{x+1}\cdot 3x}=$$

$$=\underbrace{\lim_{x\to\infty}\left(\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}}\right)}_{e}^{\frac{6x}{x+1}}=e^{\lim\limits_{x\to\infty}\frac{6x}{x+1}}=e^{6}.$$

EXAMPLE

Let's calculate the limit:

$$\lim_{x\to\infty}\left(\frac{x+4}{x}\right)^{x}=(1^{\infty})=\lim_{x\to\infty}\left\{\left(1+\frac{4}{x}\right)^{x/4}\right\}^{4}=e^{4}$$

Case 2

$$\lim_{u \to 0}(1+u)^{\frac{1}{u}} = \begin{bmatrix} u = 1/x \\ x \to \infty \end{bmatrix} = \lim_{x \to \infty}\left(1+\frac{1}{x}\right)^{x} = e$$

Example 6. Let's calculate $\lim_{x\to\infty}\left(\frac{x+2}{x-4}\right)^x$

Solution. Let's find the limit of the base:

$$\lim_{x\to\infty}\frac{x+2}{x-4}=\left\|\frac{\infty}{\infty}\right\|=\lim_{x\to\infty}\frac{\frac{x}{x}+\frac{2}{x}}{\frac{x}{x}-\frac{4}{x}}=\lim_{x\to\infty}\frac{1+\frac{2}{x}}{1-\frac{4}{x}}=\frac{1+0}{1-0}=1$$

Example 6. Let’s calculate $$\lim_{x\to\infty}\left(\frac{x+2}{x-4}\right)^{x}$$

Solution.

We obtain

$$\lim_{x\to\infty}\left(\frac{x+2}{x-4}\right)^{x}=\left\|1^{\infty}\right\|=$$

If in the given example we add and subtract from the base of the power, then the expressions will remain unchanged.

$$= \lim_{x\to\infty}\left(1+\frac{x+2}{x-4}-1\right)^{x} = \lim_{x\to\infty}\left(1+\frac{6}{x-4}\right)^{x} =$$

Let us carry out the identical transformation:

$$= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-4}{6}}\right)^{x} = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-4}{6}}\right)^{\frac{x-4}{6}\frac{6}{x-4}x} =$$

The limit of the base in the obtained expression is the value *e*:

$$= \lim_{x\to\infty}\left(\underbrace{\left(1+\frac{1}{\frac{x-4}{6}}\right)^{\frac{x-4}{6}}}_{\to e}\right)^{\frac{6}{x-4}\cdot x} = \lim_{x\to\infty} e^{\frac{6}{x-4}\cdot x} =$$

The limit of the power is calculated as follows:

$$= e^{\lim\limits_{x\to\infty}\frac{6x}{x-4}} = \left\|\frac{\infty}{\infty}\right\| = e^{\lim\limits_{x\to\infty}\frac{6\,x/x}{x/x-4/x}} =$$

$$= e^{\lim\limits_{x\to\infty}\frac{6}{1-\frac{4}{x}}} = e^{\frac{6}{1-0}} = e^{6}$$

TASKS

Find the limit:

$$\lim_{x\to\infty}\left(\frac{x+3}{3x+4}\right)^{x+1}=\left(\frac{1}{3}\right)^{\infty}=0$$

TASKS

Find the limit:

$$\lim_{x\to\infty}\left(\frac{3x+4}{x+3}\right)^{x-5} = 3^{\infty} = \infty$$

Consequences of the second remarkable limit

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{x} = e^{k}$$

Comparison of infinitesimals

Definition. A function $\alpha = \alpha(x)$ is called *infinitude* (infinitely large variables) if .

$$\lim_{x \to \infty} \alpha(x)_{n} = \infty$$

Definition. A function $\beta = \beta(x)$ *is called infinitesimal if* .

$$\lim_{x \to \infty} \beta(x)_{n} = 0$$

Comparison of infinitesimals

Equivalencies at $x \to 0$ **or** $f(x) \to 0$

$$\log_a(1+f(x)) \sim \frac{f(x)}{\ln a}$$

$$\ln(1+f(x)) \sim f(x)$$

$$a^{f(x)} - 1 \sim f(x) \cdot \ln a, \text{ if } a > 0,\ a \neq 1$$

$$e^{f(x)} - 1 \sim f(x),$$

$$(1+x)^m - 1 \sim m \cdot x$$

Example

$$\lim_{x\to 0}\frac{e^{3x^2}-1}{5x^2}=\left\|\frac{0}{0}\right\|=\lim_{x\to 0}\frac{3x^2}{5x^2}=\frac{3}{5}$$

$$e^{3x^2}-1 \sim 3x^2$$
$$x\to 0$$

Continuity of a function. Types of breakpoints

Continuity of a function (definition 1)

A function $y = f(x)$ is called ***continuous*** at a point x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Example

Investigate a continuity of the function:

$$f(x) = x^3 + 1 \qquad x_0 = 2$$

Continuity of a function (definition 2)

Let's introduce the second definition of the continuity of a function which means the necessary and sufficient condition of the continuity at the point $x = x_0$, namely

$$\lim_{x \to x_0 - 0} f(x) = \lim_{x \to x_0} f(x) = \lim_{x \to x_0 + 0} f(x)$$

$$f(x_0 - 0) = f(x_0) = f(x_0 + 0)$$

Continuity of a function (definition 2)

$$\lim_{x\to x_0-0} f(x) = \lim_{x\to x_0} f(x) = \lim_{x\to x_0+0} f(x)$$

In the case if there is no the double equality it is said that the point x_0 is the point of the ***discontinuity*** of a function $y = f(x)$

Arithmetic operations on continuous functions

We can perform the arithmetic operations on continuous functions. It is established by the following theorems:

Theorem 1. The algebraic sum of a finite number of functions continuous at the point x_0 is a continuous function at that point;

Arithmetic operations on continuous functions

We can perform the arithmetic operations on continuous functions. It is established by the following theorems:

Theorem 2. The product of a finite number of functions continuous at the point x_0 is a continuous function at that point;

Arithmetic operations on continuous functions

We can perform the arithmetic operations on continuous functions. It is established by the following theorems:

Theorem 3. The quotient of two functions continuous at the point x_0 is a continuous function at the point x_0 provided that the denominator does not turn into zero at that point;

Arithmetic operations on continuous functions

We can perform the arithmetic operations on continuous functions. It is established by the following theorems:

Theorem 4. A function of a function composed of a finite number of continuous functions is a continuous function.

Continuity of a function (definition 2)

$$\lim_{x \to x_0 - 0} f(x) = \lim_{x \to x_0} f(x) = \lim_{x \to x_0 + 0} f(x)$$

In the case if there is no the double equality it is said that the point x_0 is the point of the ***discontinuity*** of a function $y = f(x)$

Discontinuity of a function

Let us present the ***classification*** of points of the ***discontinuity*** of a function by following definition.

Discontinuity of a function. The first kind

Both limits are finite

$$f(x_0 - 0) \neq f(x_0 + 0)$$

Discontinuity of a function. The first kind

Such a case occurs when there exist the limits on the right and on the left and they are finite, i.e. when the second condition of continuity is fulfilled and the rest of the condition for at least one of them is not fulfilled.

Example

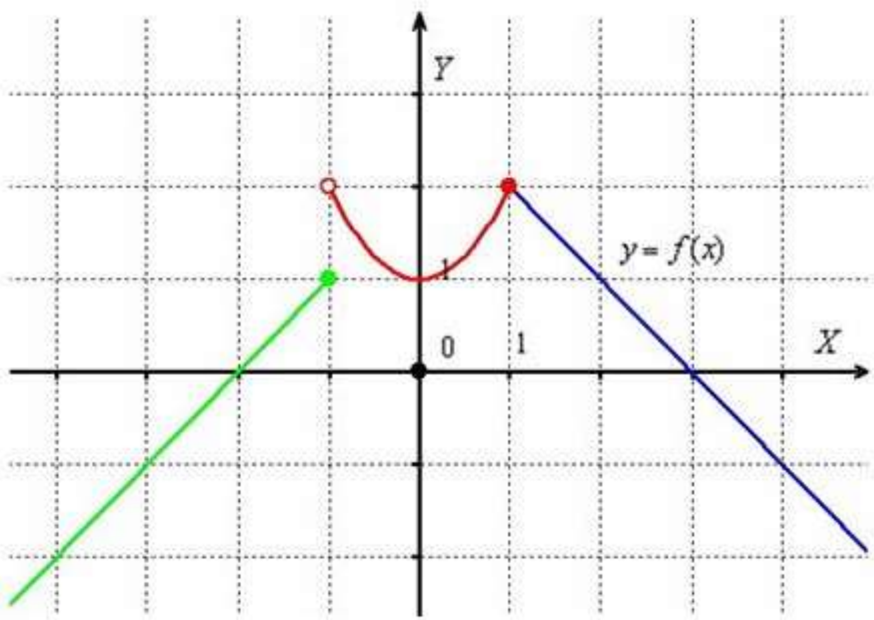
Investigate this function:

$$f(x)=\begin{cases} x+2, & x\le -1 \\ x^2+1, & -1<x\le 1 \\ -x+3, & x>1 \end{cases}$$

Example

Investigate this function:

$$f(x)=\begin{cases} x+2, & x\le -1 \\ x^2+1, & -1<x\le 1 \\ -x+3, & x>1 \end{cases}$$



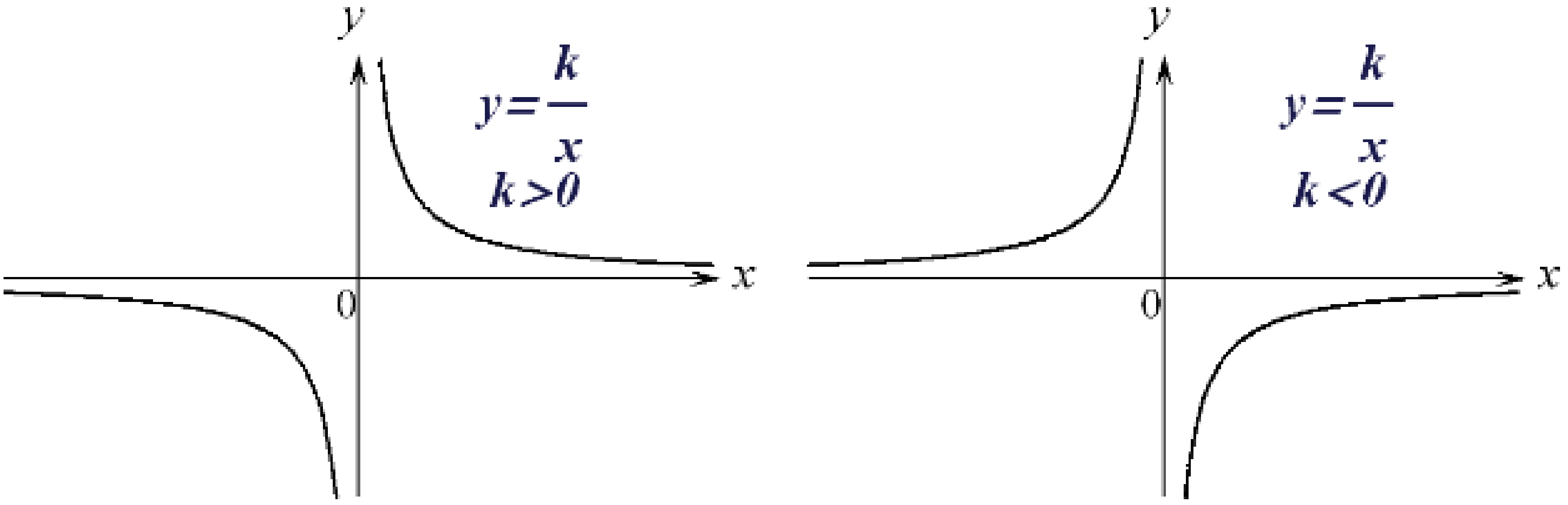
Discontinuity of a function.
The second kind

Either left limit or right limit is equal to infinity or

$$f\left(x_0-0\right)=\infty \quad or \quad f\left(x_0+0\right)=\infty$$

Example

The breakpoint is x=0



Example

Investigate a continuity of the function:

$$y = f(x) = 3^{\frac{1}{x}}.$$

$$\lim_{x\to 0-0} 3^{\frac{1}{x}} = 0, \quad \text{since} \quad \lim_{x\to 0-0} \frac{1}{x} = -\infty.$$

$$\lim_{x\to 0+0} 3^{\frac{1}{x}} = +\infty, \quad \text{it follows from} \quad \lim_{x\to 0+0} \frac{1}{x} = +\infty.$$

Removable discontinuity

$$f\left(x_0 - 0\right) = f\left(x_0 + 0\right) = A$$

$$f\left(x_0\right) \neq A$$ or $$f\left(x_0\right)$$ doesn't exist

Example

Investigate this function:

$$f(x) = \frac{x^3 - x^2}{x-1}$$

Example

Investigate this function:

$$f(x)=\frac{x^3-x^2}{x-1}$$

$$\lim_{x\to 1-0}\frac{x^3-x^2}{x-1}=\frac{0}{0}=\lim_{x\to 1-0}\frac{x^2(x-1)}{x-1}=\lim_{x\to 1-0}(x^2)=1$$

$$\lim_{x\to 1+0}\frac{x^3-x^2}{x-1}=\frac{0}{0}=\lim_{x\to 1+0}\frac{x^2(x-1)}{x-1}=\lim_{x\to 1+0}(x^2)=1$$

Example

Investigate this function:

$$f(x)=\frac{x^3-x^2}{x-1}$$