PhD Misiura Ie.Iu. (доцент Місюра Є.Ю.)

Theme: Empirical and logical foundations of probability theory. Elements of combinatorics. Basic theorems of probability theory, their economic interpretation. Probability theory in international trade strategies (part 1)

BASIC NOTIONS OF

PROBABILITY THEORY

A subject of probability theory

- **Probability theory** is the branch of mathematics which studies properties, laws and the analysis of mass random phenomena. The basic objects of probability theory are random variables, stochastic process and random events. In practice we often deal with random events, i.e. with events which can occur or can’t occur under definite conditions which can’t be analyzed by direct computations. *Analysis of quantitative laws* which can be described by mass random phenomena is **the subject of probability theory.**
- **Probability theory** plays an important role in everyday life in economics, in business, in trade on financial markets, in risk assessment and many other areas where statistics is applied to the real world.

Let’s consider ***the fundamental concepts*** **of probability theory**.

- An ***experiment*** is a repeatable process that gives rise to a number of outcomes.
- An ***outcome*** is something that follows as a result or consequence.
- An ***event*** is a collection (or set) of *one or more* outcomes.

Example. The experiment is TOSSING A COIN once

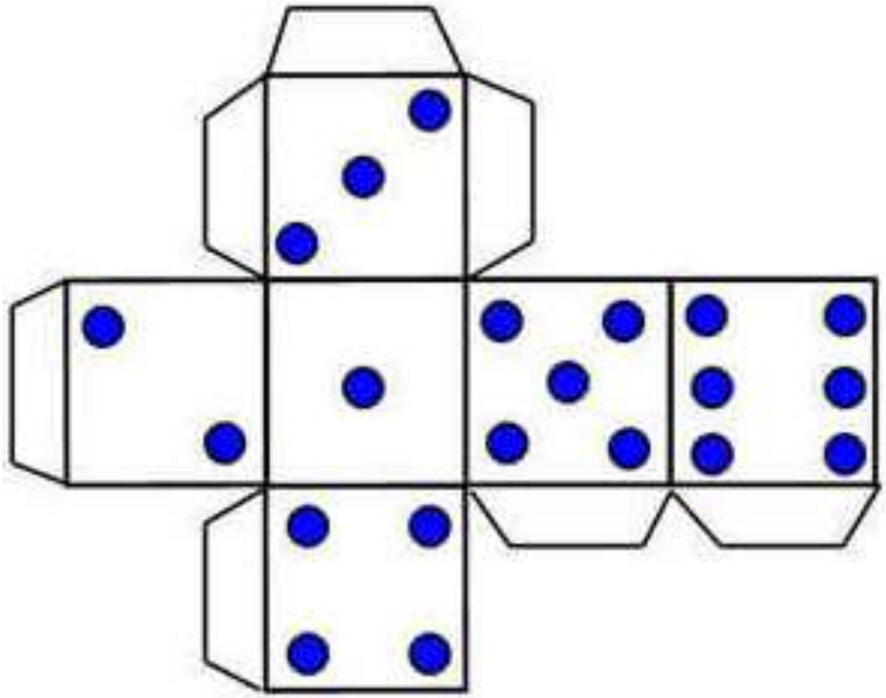
Example. The experiment is ROLLING A DIE once

Example. This experiment has 2 OUTCOMES:
HEAD (the first outcome) and

TAIL (the second outcome)

Example. This experiment has 6 OUTCOMES:
1 score, 2 scores, 3 scores, 4 scores, 5 scores and

6 scores.



Example.

The event is getting “HEAD”.

Question 1

Which of the following is an outcome?

1) Rolling a pair of dice.
2) Landing on red.
3) Choosing 2 marbles from a jar.
4) None of the above.

Events are sets and set notation is used to describe them. We use upper letters to denote events. They are denoted as

$$A, B, C, \quad , A_1, A_2,$$

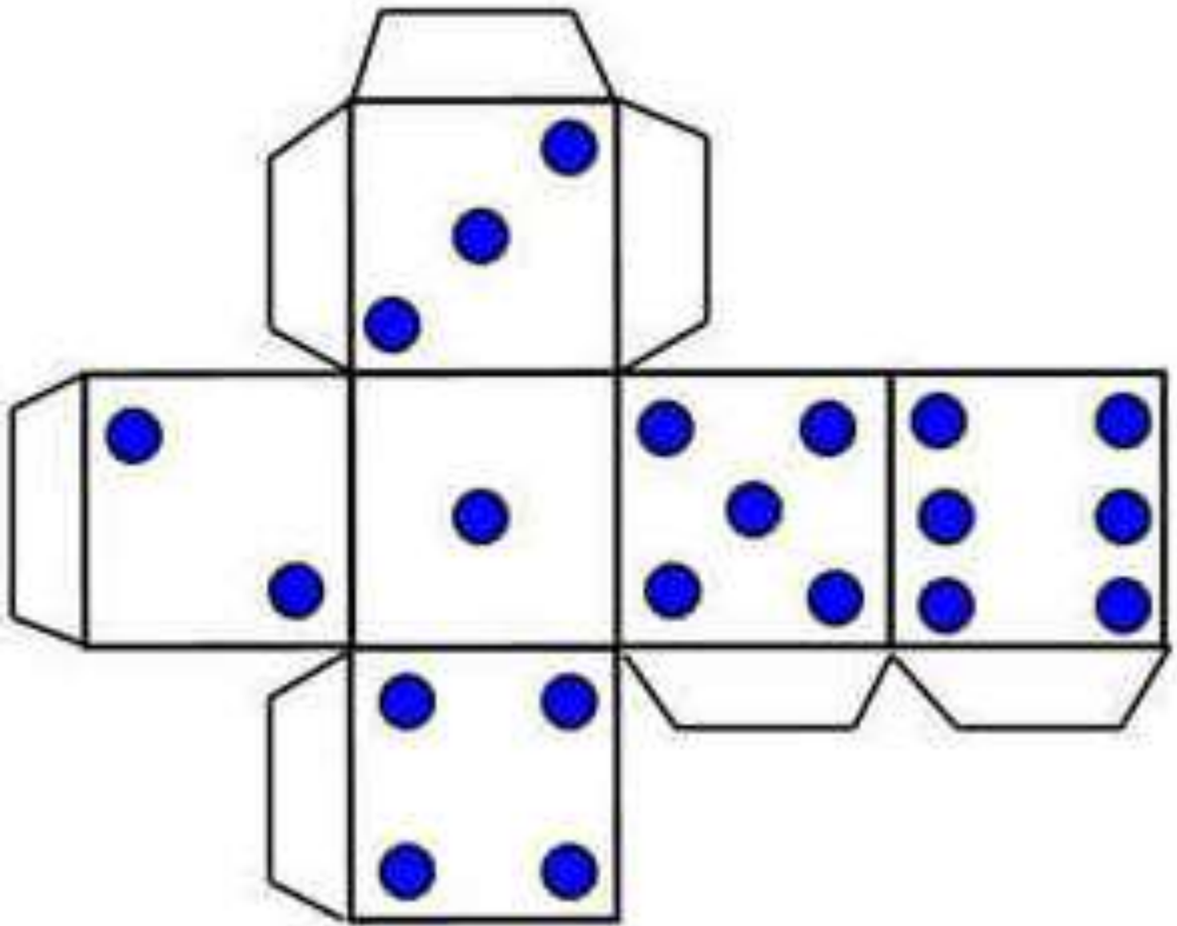
The simplest indivisible mutually exclusive outcomes of an experiment are called ***elementary events*** $\omega_1, \omega_2,$

A ***sample space*** or a ***space of elementary events*** is called the set of all possible elementary outcomes of an experiment, which we denote by the symbol Ω

Example. For this experiment (tossing a coin once) the sample space is

$$\Omega = \{head, tail\}$$

Example. For this experiment (rolling a die once) the sample space is



Question 2

Which of the following is the sample space when 2 coins are tossed?

(1) {H, T, H, T}

(2) {H, T}

(3) {HH, HT, TH, TT}

(4) None of the above

Example : If we toss a coin twice, we can observe 1 of 4 outcomes: (Heads, Heads), (Heads, Tails), (Tails, Heads), (Tails, Tails). In this case we could use the following outcome space

$$\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$$

and the symbol ω could be used to represent either (H,H), or (H,T), or (T,H), or (T,T). Note how in this case each basic outcome contains 2 elements. If we toss a coin n times each basic outcome ω would contain n elements.

- Any subset of Ω is called a ***random event*** A (or simply an ***event*** A).
- Elementary events that belong to A are said to ***favor*** A .
- An event is ***certain*** (or ***sure***) if it always happens.
- An event is **impossible** if it never happens.
- ***Equally likely events*** are such events that have the equal chance to happen at an experiment.

- The ***probability of an event*** is the chance that the event will occur as a result of an experiment.
- Where outcomes are *equally likely* the probability of an event is the number of outcomes in the event divided by the total number of possible outcomes in the sample space.
- An *impossible event* has probability 0 and an event that is *certain* has probability 1.

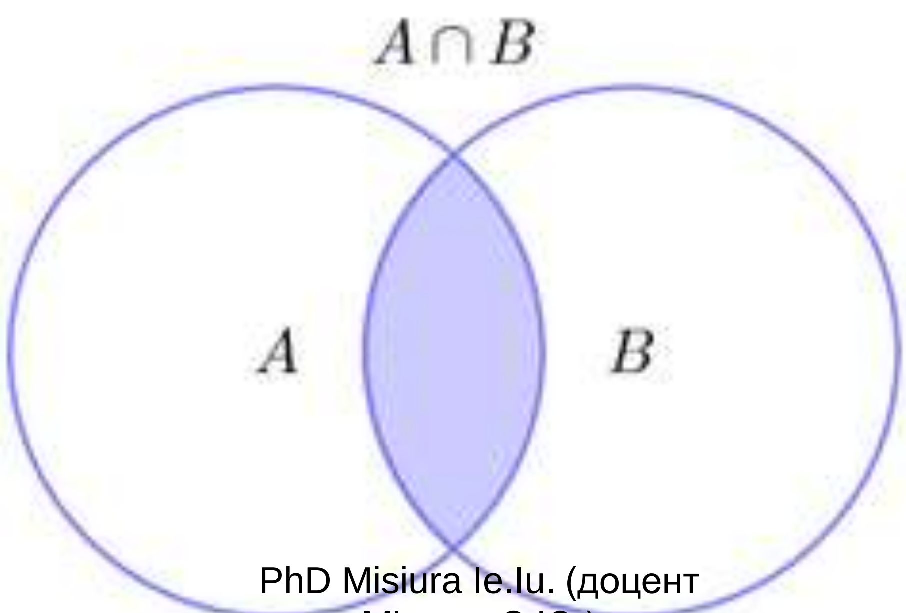
An algebra of random events

The mathematics of probability is expressed most naturally in terms of sets, therefore, let's consider *basic operations with events*.

BASES

- **Element inclusion:** We use the symbol $\in$ to represent element inclusion. The expression $\omega \in A$ tells us that ω is an element of the set A. The expression $\omega \notin A$ tells us that ω is **not** an element of the set A. For example, $1 \in \{1, 2\}$ is true since 1 is an element of the set $\{1, 2\}$.

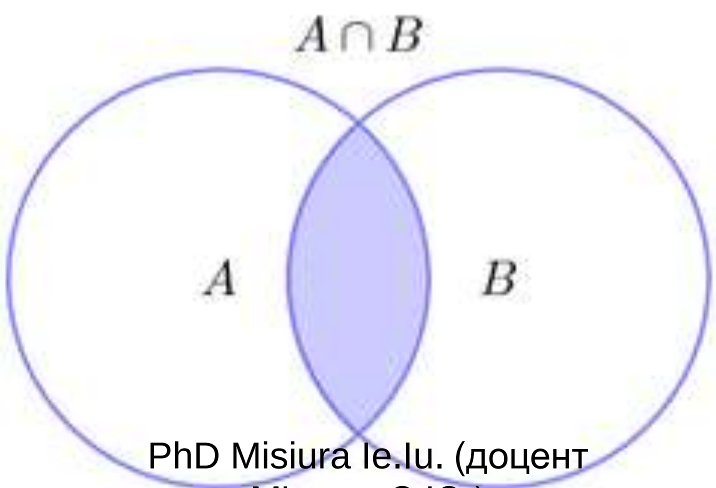
The ***intersection*** $C = A \cap B = A \cdot B$ of events ***A*** and ***B*** is the event that both ***A*** and ***B*** occur. The elementary outcomes of the intersection $A \cdot B$ are the elementary outcomes that simultaneously belong to ***A*** and ***B***.



Example. If ***A*** and ***B*** are given, then

$$A = \{1, 2, 3\} \qquad B = \{1, 3, 5\}$$

$$C = A \cap B = \{1, 3\}$$



Example. If ***A*** and ***B*** are given, then

$$A = \{1, 2, 3\} \qquad B = \{1, 3, 5\}$$

$$C = A \cap B = \{1, 3\}$$

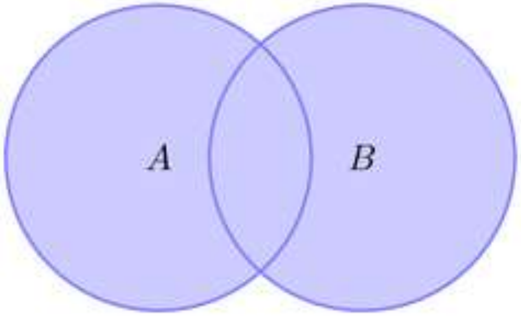
When events ***A*** and ***B*** have common outcomes ($A \cap B \neq \varnothing$), they are (***compatible events***).

When events A and B have no outcomes in common ($A \cap B = \varnothing$)
(this symbol Ø is called the empty set), they are ***incompatible events***.

Example 6. If events $A = \{1, 2\}$ and $B = \{3, 5\}$ are given, then

$C = A \cap B =$ Ø, because events ***A*** and ***B*** have no outcomes in common.

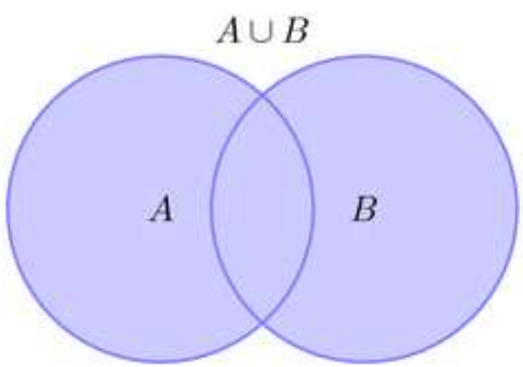
The ***union*** $C = A \cup B = A + B$ of events ***A*** and ***B*** is the event that at least one of the events ***A*** or ***B*** occurs. The elementary outcomes of the union $A + B$ are the elementary outcomes that belong to at least one of the events ***A*** and ***B***.



Example. If events $A = \{1, 2, 3, 4, 5\}$

and $B = \{2, 4, 6\}$ are given, then

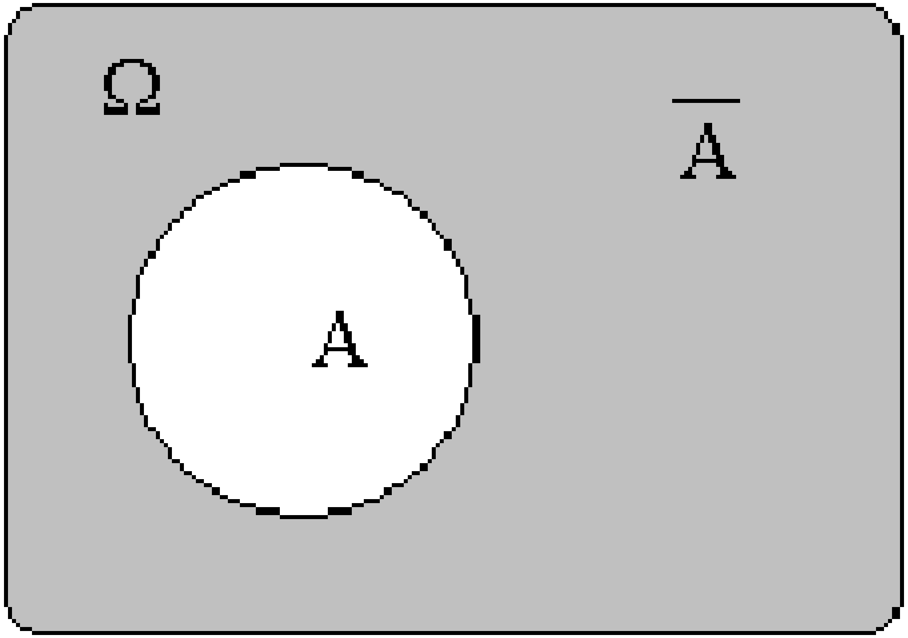
$$C = A \cup B = \{1, 2, 3, 4, 5, 6\}$$



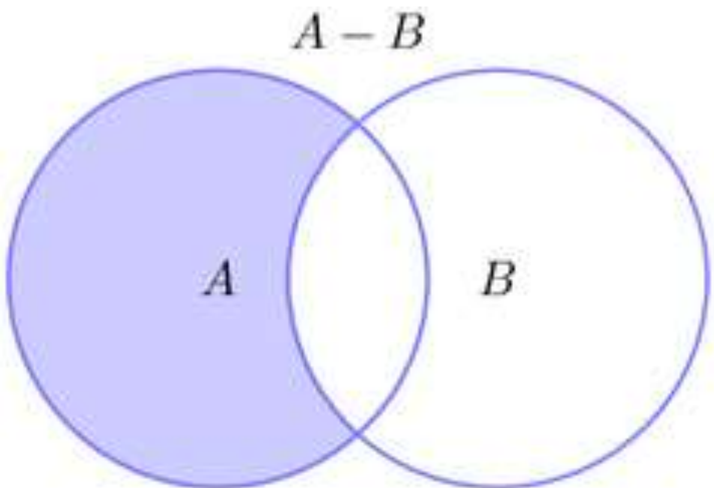
Two events A and $\overline{A}$ are said to be ***opposite*** (***complementary***)

if they simultaneously satisfy the following conditions: $A \cup \overline{A} = \Omega$

and $A \cap \overline{A} = \varnothing$.

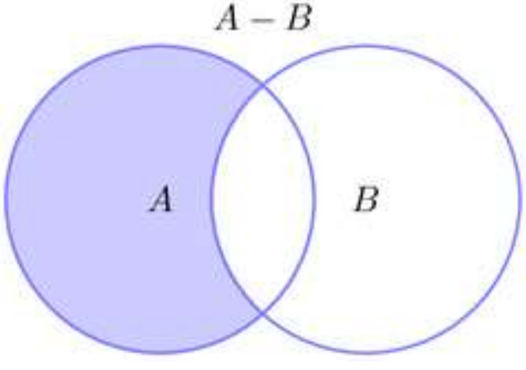


The ***difference*** $C = A \setminus B = A - B$ of events ***A*** and ***B*** is the event that ***A*** occurs and ***B*** does not occur. The elementary outcomes of the difference $A \setminus B$ are the elementary outcomes of ***A*** that do not belong to ***B***.

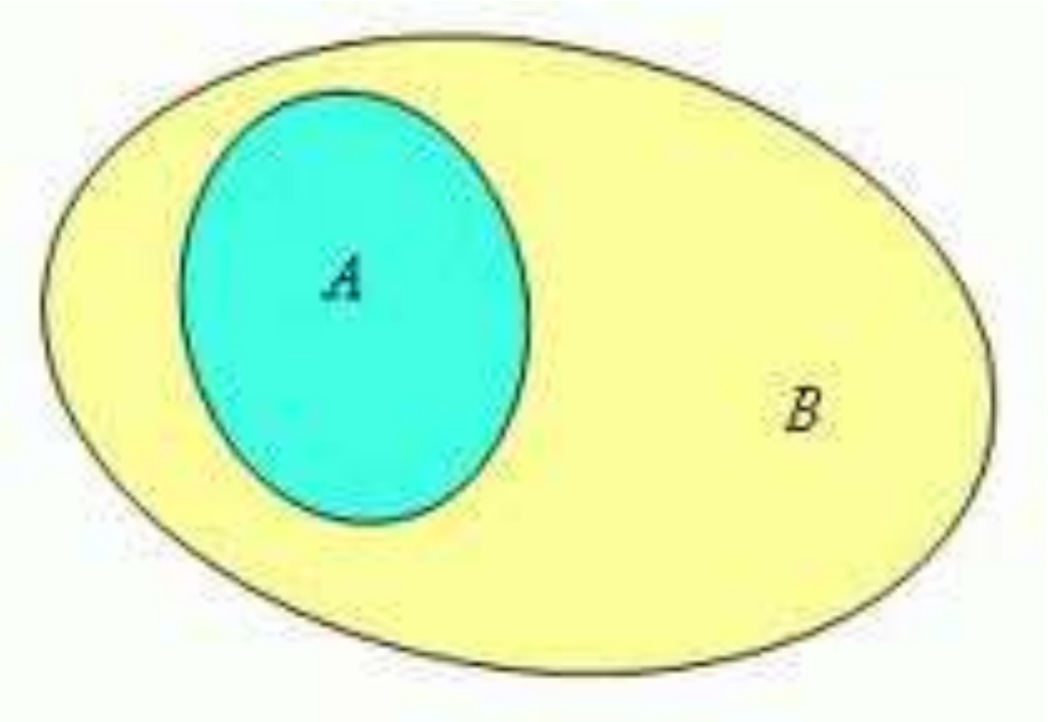


Example. If events $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5\}$ are given, then

$$C = A \setminus B = \{2, 4\}$$

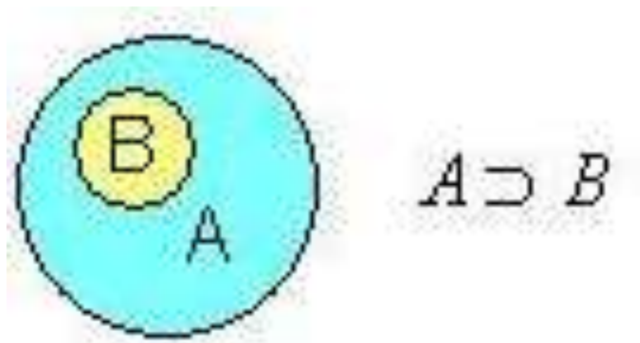


An event ***A*** ***implies*** an event ***B*** ($A \subset B$) if ***B*** occurs in each realization of an experiment for which ***A*** occurs.



Example. If events $A = \{1, 2, 3, 4, 5\}$

and $B = \{1, 3, 5\}$, then the event ***A*** implies the event ***B*** or $A \subset B$.



Events ***A*** and ***B*** are said to be ***equivalent*** ($A = B$) if ***A*** implies ***B*** ($A \subset B$) and ***A*** implies ***B*** ($B \subset A$), i.e., if, for each realization of an experiment, both events ***A*** and ***B*** occur or do not occur simulta-neously.

Example. If events $A = \{1,\ 2,\ 3\}$

and $B = \{3,\ 2,\ 1\}$ are given, then events ***A*** and ***B*** are equivalent or $A = B$

A BASIC NOTION OF A COMBINATORIAL ANALYSIS

COLLECTION OF FORMULAS OF COMBINATORICS WITHOUT REPETITIONS

Take ***n*** different elements. We’ll permute them in all possible ways, saving their quantity and changing only their order. Each of combinations, received so, is called a *permutation without repetitions.* A total quantity of *permutations of* ***n*** *elements* is signed as P_n . This number is equal to a product of all integer numbers from 1 to *n*:

$$P_n = 1 \cdot 2 \cdot \ldots \cdot n = n!$$

The symbol *n*! (it is called a *factorial*)

Permutations without repetitions

Example.

Let's consider the set {1, 2, 3} of ***n* = 3** elements.

The elements of this set give $P_3 = 3! = 6$ permutations:

(1, 2, 3), (1, 3, 2),

(2, 1, 3), (2, 3, 1),

(3, 1, 2), (3, 2, 1).

Let's compose groups of ***k*** different elements, taken from a set of ***n*** elements, placing these ***k*** taken elements in a different order.

The received combinations are called *arrangements without repetitions of* ***n*** *elements taken* ***k*** *at a time.*

Their total quantity is signed as A_n^k and equal to the product:

$$A_n^k = \frac{n!}{(n-k)!}$$

*Arrangements without repetitions of **n** elements taken **k** at a time*

Example. Let’s consider the set {1, 2, 3} of ***n* = 3** elements and take ***k* = 2** elements. The elements of this set give arrangements without repetitions (1, 2), (1, 3), (2, 3), (2, 1), (3, 2), (3, 1).

$$A_3^2 = \frac{3!}{(3-2)!} = 6$$

Let's compose groups of ***k*** different elements, taken from a set of ***n*** elements, *not taking into consideration an order of these* ***k*** *taken elements*. So, we received *combinations without repetitions of* ***n*** *elements taken* ***k*** *at a time.* Their total quantity is signed as C_n^k and can be calculated by the formula:

$$C_n^k = \frac{n!}{k! \cdot (n-k)!}$$

*Combinations without repetitions of **n** elements taken **k** at a time*

Example. Let's consider the set {1, 2, 3} of ***n* = 3** elements and take ***k* = 2** elements. The elements of this set give combinations without repetitions

$$C_3^2 = \frac{3!}{2!\cdot(3-2)!} = 3$$

(1, 2), (1, 3), (2, 3).

COLLECTION

OF FORMULAS

OF COMBINATORICS

WITH REPETITIONS

If, for different elements ***k*** out of ***n*** elements with replacement, no subsequent ordering is performed (i.e., each of the ***n*** elements can occur **0**, **1**, *. . .*, or ***k*** times in any combination), then one speaks of *combinations with repetitions*. The number $\overline{C}_n^k$ of all distinct combinations with repetitions of ***n*** elements taken ***k*** at a time is given by the formula:

$$\overline{C}_n^k = C_{n+k-1}^k$$

*Combinations with repetitions of **n** elements taken **k** at a time*

Example. Let’s consider the set {1, 2, 3} of ***n* = 3** elements and take ***k* = 2** elements. The elements of this set give combinations

$$\overline{C}_3^2 = C_{3+2-1}^2 = C_4^2 = \frac{4!}{2!\cdot(4-2)!} = 6$$

with repetitions
(1, 2), (1, 3), (2, 3), (1, 1), (2, 2), (3, 3).

If, for different elements ***k*** out of ***n*** elements with replacement, the chosen elements are ordered in some way, then one speaks of *arrangements with repetitions*. The number of distinct arrangements with repetitions of ***n*** elements taken ***k*** at a time is given by $\overline{A}_n^k$ the formula:

$$\overline{A}_n^k = n^k$$

Arrangements with repetitions of ***n*** *elements taken* ***k*** *at a time*

Example. Let’s consider the set {1, 2, 3} of ***n* = 3** elements and take ***k* = 2** elements.

The elements of this set give arrangements with repetitions

(1, 2), (1, 3), (2, 3),

(2, 1), (3, 2), (3, 1),

(1, 1), (2, 2), (3, 3).

$$\overline{A}_3^2 = 3^2 = 9$$

Let's suppose that a set of ***n*** elements contains ***k*** distinct elements, of which the first occurs n_1 times, the second occurs n_2 times, ..., and the ***k***-th occurs n_k times, $n_1 + n_2 + \quad + n_k = n$ Permutations of ***n*** elements of this set are called *permutations with repetitions on* ***n*** *elements*. The number $P_n(n_1, n_2, \quad, n_k)$ of permutations with repetitions on ***n*** elements is given by the formula:

$$P_n(n_1, n_2, \quad, n_k) = \frac{n!}{n_1! \cdot n_2! \cdot \quad \cdot n_k}$$

Permutations with repetitions

Example. If there are two letters ***a*** and one letter ***b***. The number of permutations with repetitions out of 3 elements and composition of letters 2, 1 equals (a, a, b), (a, b, a), (b, a, a).

$$P_3(2,1)=\frac{3!}{2!\cdot 1!}=3$$

Question 3

How to open a combination lock?
How many ways do you have?

Question 3

How to open a combination lock?
a) If each digit can be used only once

Question 3

How to open a combination lock?
b) If each digit can be used with repetitions

TASKS

1)How many three-digit numbers can be formed from the digits 1, 2, 3, 4, 5, if each digit can be used only once (with repetitions)?

2)A committe including 3 boys and 4 girls is to be formed from a group of 10 boys and 12 girls. How many different committee can be formed from the group?

3)How many different rearrangements of the letters in the word (a) EDUCATION, (b) MISSISSIPPI are there?

4)If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary. (a) How many variants to select the dictionary and 2 novel? (b) How many variants to select 1 novel and 2 books of poems?

HOMEWORK

Combinatorics:

1.How many five-digit numbers can be formed from the digits 1, 2, 3, 4, 5, if each digit can be used only once and five-digit number is divided by: (a) 5? (b) 3?

2. How many different unique combinations of letters can be created by rearranging the letters in *mathematics*?

3. In how many ways can you select a committee of 3 students out of 10 students?

Rules of a sum of incompatible events

The ***rule of sum*** is an intuitive principle stating that if there are ***a*** possible outcomes for an event ***A*** (or ways to do something) and ***b*** possible outcomes for another event ***B*** (or ways to do another thing) and **two events can't both occur** (or the two things can't be done) (***A*** and ***B*** are mutually exclusive or incompatible events) then there are ***a+b*** total possible outcomes for the events ***A*** and ***B*** (or total ways to do one of the things);

Rules of a sum of incompatible events

formally, the sum of sizes of two incompatible sets is equal to the size of their union, i.e.

$$\boxed{|A| + |B| = |A \cup B|}$$

Rules of a sum of incompatible events

Example. A woman has decided to shop at one store today, either in the north part of town or the south part of town. If she visits the north part of town, she will either shop at a mall, a furniture store, or a jewelry store (3 ways). If she visits the south part of town then she will either shop at a clothing store or a shoe store (2 ways).

Rules of a sum of incompatible events

Example. A woman has decided to shop at one store today, either in the north part of town or the south part of town. If she visits the north part of town, she will either shop at a mall, a furniture store, or a jewelry store (3 ways). If she visits the south part of town then she will either shop at a clothing store or a shoe store (2 ways). Let ***A*** be the woman visiting the north part of town and ***B*** be the woman visiting the south part of town, i.e.

$$|A| = 3 \quad \text{and} \quad |B| = 2$$

Rules of a sum of incompatible events

Let ***A*** be the woman visiting the north part of town and ***B*** be the woman visiting the south part of town, i.e.

$$|A| = 3 \quad \text{and} \quad |B| = 2$$

Thus there are $|A \cup B| = |A| + |B| = 3 + 2 = 5$

possible shops the woman could end up shopping at today

The rule of product of incompatible events

The ***rule of product*** is another intuitive principle stating that if there are ***a*** possible outcomes for an event ***A*** (or ways of doing something) and ***b*** possible outcomes for another event ***B*** (or ways of doing another thing) and **two events can both occur** (or the two things can be done) (***A*** and ***B*** are not mutually exclusive or compatible events) then there are $a \cdot b$ total ways of performing both things, i.e.

$$|A \cap B| = |A| \cdot |B|$$

The rule of product of incompatible events

When we decide to order pizza, we must first choose the type of crust: thin or deep dish (2 choices or $|A| = 2$). Next, we choose the topping: cheese, pepperoni, or sausage (3 choices or $|B| = 3$). Using the rule of product, you know that there are

$$|A \cap B| = |A| \cdot |B| = 2 \cdot 3 = 6$$

possible combinations of ordering a pizza.

Question 4

A large basket of fruit contains 3 oranges and 2 apples. How many ways of getting an orange **or** an apple?

Question 5

A team including 3 boys **and** 4 girls is to be formed from a group of 10 boys and 12 girls. How many different teams can be formed from the group?

The rule of inclusion and exclusion

The inclusion-exclusion principle relates to the size of the union of multiple sets, the size of each set and the size of each possible intersection of the sets. The smallest example is when there are two sets: the number of elements in the union of the events ***A*** and ***B*** is equal to the sum of the elements in the events and minus the number of elements in their intersection, i.e.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The rule of inclusion and exclusion

Example. 35 voters were queried about their opinions regarding two referendums. 14 supported referendum 1 and 26 supported referendum 2. How many voters supported both, assuming that every voter supported either referendum 1 or referendum 2 or both?

The rule of inclusion and exclusion

Example. 35 voters were queried about their opinions regarding two referendums. 14 supported referendum 1 and 26 supported referendum 2. How many voters supported both, assuming that every voter supported either referendum 1 or referendum 2 or both?

Solution. Let ***A*** be voters who supported referendum 1 and ***B*** be voters who supported referendum 2. Then we have

$$|A| = 14 \qquad |B| = 26 \qquad |A \cup B| = 35$$

The rule of inclusion and exclusion

How many voters supported both, assuming that every voter supported either referendum 1 or referendum 2 or both?

Solution. Let ***A*** be voters who supported referendum 1 and ***B*** be voters who supported referendum 2. Then we have

$$|A| = 14 \qquad |B| = 26 \qquad |A \cup B| = 35$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cap B| = 14 + 26 - 35 = 5$$

A CLASSICAL DEFINITION OF A PROBABILITY

A classical definition of a probability

Let a space of elementary events Ω be given and this space consists of ***n*** equally likely elementary outcomes (i.e. total number of outcomes) of the experiment, among which there are ***m*** outcomes, favorable for an event A (i.e. number of outcomes an event A can happen), and $\Omega \subset A$. Then the number:

$$P(A) = \frac{m}{n}$$

is called the ***probability of an event*** A

- As all events have probabilities between impossible (0) and certain (1), then probabilities are usually written as a fraction, a decimal or sometimes as a percentage. We will write probabilities fractions or decimals.
- The probability is the non-dimensional quantity. It can be measured in percent from 0 to 100. For example,

$$P(A) = \frac{4}{10} = 0.4 = 40\,\%$$

Example 11. Let's suppose the event A we are going to consider is *rolling a die* once and obtaining a 3. The die could land in a total of six different ways. We say that the total number ***n*** of outcomes of rolling the die is six, which means there are six ways it could land. The number ***m*** of ways of obtaining the particular outcome of A is one. We can apply the formula and find:

$$P(A) = \frac{m}{n} = \frac{1}{6}$$

When we roll a die it has an equal chance of landing on any of the six numbers 1, 2, 3, 4, 5, or 6. These events are called equally likely events.

Example. This experiment has 6 OUTCOMES:

1 score, 2 scores, 3 scores, 4 scores, 5 scores and 6 scores.

A geometric definition of a probability

A ***geometric definition of a probability*** of an event A. Let Ω be a set of a positive finite measure $\mu(\Omega)$ and consist of all measurable (i.e. having a measure) subsets $A \supset \Omega$*.* The ***geometric probability*** of an event A is defined to be ratio of the measure of A to that of Ω, i.e.

$$P(A) = \frac{\mu(A)}{\mu(\Omega)}$$

As measures $\mu(A)$ and $\mu(\Omega)$ we can use different geometric measures, for example, lengths, areas or volumes.

Example . A point is randomly thrown into a disk of radius $R=1$. Find the probability of the event that the point lands in the disk of radius $r=1/2$ centered at the same point.

Solution. Let ***A*** be the event that the point lands in the smaller disk. We find the probability $P(A)$ as the ratio of the area of the smaller disk to that of the larger disk:

$$P(A)=\frac{\pi r^2}{\pi R^2}=\frac{r^2}{R^2}=\frac{(1/2)^2}{1^2}=\frac{1}{4}$$

DIFFERENT TYPES OF EVENTS

AND

PROPERTIES OF PROBABILITY

An event ***A*** is said to be ***impossible*** if it cannot occur for any realization of the experiment. Obviously, the impossible event does not contain any elementary outcome and hence should be denoted by the symbol Ø. Its probability is zero, i.e. $P(A)=0$

Example. Let's roll a die and obtain a score of 7 (the event ***A***). It's an impossible event, then

$$P(A)=0$$

Property 1. **The probability of an impossible event is 0, i.e.**

$$P(A) = 0$$

An event ***A*** is said to be ***sure*** (or ***certain***) if it is equivalent to the space of elementary events Ω , i.e. $A = \Omega$, or it happens with probability 1.

Example. Let's roll dice and obtain a score less than 13 (the event ***A***). It's a sure event or a space of elementary events Ω , because it consists of all possible outcomes of Ω . Then

$$P(A) = 1$$

Property 2. **The probability of a sure (certain) event is 1, i.e.**

$$P(A) = 1$$

Property 3. **The probability of a space of elementary events Ω is 1, i.e.**

$$P(\Omega) = 1$$

Property 4. **All probabilities that lie between zero and one are inclusive, i.e.**

$$0 \le P(A) \le 1$$

The event that ***A*** doesn't occur is called the ***complement*** of ***A***, or the ***complementary event***, and is denoted by $\overline{A}$. The elementary outcomes of $\overline{A}$ are the elementary outcomes that don't belong to the event ***A***.

Property 5. **The probability of the event** $\overline{A}$ **opposite to the event *A* is equal to**

$$P\left(\overline{A}\right) = 1 - P(A)$$

From this property we can obtain that

$$P(A) + P\left(\overline{A}\right) = 1$$

for complementary events A and $\overline{A}$.

Example. Helen rolls a die once. What is the probability she rolls an even number or an odd number?

Solution. The event of rolling an even number (***A***) and the event of rolling an odd number (***B***) are mutually exclusive events, because they both cannot happen at the same time, so we add the probabilities. In addition, these two events make up all the possible outcomes, so they are complementary events, i.e. ***B*** is $\overline{A}$. Let's write:

$$P(A)+P(B)=\frac{3}{6}+\frac{3}{6}=1$$

The events ***A*** and ***B*** are called ***equally likely events***, if

$$P(A) = P(B)$$

Example. When we roll a die it has an equal chance $\frac{1}{6}$ of landing on any of the six numbers 1, 2, 3, 4, 5, or 6. These events are called equally likely events.

Property 6. **Probabilities of equally likely events *A* and *B* are equal, i.e.**

$$P(A) = P(B)$$

Property 7. **Nonnegativity:**

$$P(A) \geq 0$$

for any $A \subset \Omega$

HOMEWORK

Classical definition of a probability:

What is the probability of choosing

(1) a vowel from the alphabet?

(2) a consonant from the alphabet?

(3) two vowels from the alphabet?

(4) three consonants from the alphabet?

(5) three vowels and four consonants from the alphabet?

Thank You for your time and attention! Hope this presentation was educational and helpful to you ☺