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Theme:
General theory of systems of linear algebraic equations. Linear model of international trade

Lecture plan

1. Definition of system of linear algebraic equations

2. Solving system of equations using Cramer’s method

3. Solving system of equations using an inverse matrix

4. Solving system of equations using Jordan–Gauss method

5. Notion of a matrix rank and its calculation

6. Investigation of the system compatibility with the help of Kronecker-Capelli theorem

Application of System of linear equations to economics

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

Applications of System of Linear Equations

The following examples can be used to illustrate the common methods of solving systems of linear equations that result from applied business and economic problems.

Illustration 6 – Mr. X invested a par of his investment in 10% bond A and a part in 15% bond B. His interest income during the first year is Rs 4,000. If he invests 20% more in 10% bond A and 10 % more in 15% bond B, his income during the second year increases by Rs 500. Find his initial investment and the new investment in bonds A and B using matrix method.

Solution –
Let initial investment be x in 10% bond A and y in 15% bond B. Then, according to given information, we have

$$0.10x+0.15y=4{,}000 \quad \text{or} \quad 2x+3y=80{,}000$$

$$0.12x+0.165y=4{,}500 \quad \text{or} \quad 8x+11y=3{,}00{,}000$$

Illustration *7* – An automobile company uses three types of steel s_1, s_2 and s_3 for producing three types of cars c_1, c_2 and c_3. The steel requirement (in tons) for each type of car is given below:

		Cars		
		c_1	c_2	c_3
	s_1	2	3	4
Steel	s_2	1	1	2
	s_3	3	2	1

Determine the number of cars of each type which can be produced using 29, 13 and 16 tons of steel of the three types respectively.

Solution –

Let x, y and z denote the number of cars that can be produced of each type. Then we have

$$2x+3y+4z=29$$
$$x+y+2z=13$$
$$3x+2y+z=16$$

Illustration 8 – A company produces three products everyday. Their total production on a certain day is 45 tons. It is found that the production of the third product exceeds the production of the first product by 8 tons while the total combined production of the first and third product is twice that of the second product. Determine the production level of each product using Cramer's rule.

Solution –

Let the production level of the three products be x, y and z respectively. Therefore, we will have the following equations

$$x + y + z = 45 \qquad -(1)$$

$$z = x + 8$$

$$\text{i.e.} \quad -x + 0y + z = 8 \qquad -(2)$$

$$x + z = 2y$$

$$\text{i.e.} \quad x - 2y + z = 0 \qquad -(3)$$

Definition

- Equation / рівняння
- Unknown / невідома
- Solution / розв’язок

1. Definition of system of linear algebraic equations

*A system of **m** linear algebraic equations with **n** unknowns* has the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases}$$

where $a_{11}, a_{12}, ..., a_{mn}$ are *the coefficients of the system*;

$b_1, b_2, ..., b_m$ are *its free terms* and

$x_1, x_2, ..., x_n$ are *the unknowns*.

Definition. *A solution* of the system is a set of ***n*** numbers $x_1, x_2, ..., x_n$ satisfying every equation of the system.

Every system can have either no solution, one solution or infinitely many solutions.

Definition

- *Compatible* / сумісний
- *Incompatible* / несумісний
- *Definite* / визначений
- *Indefinite* / невизначений

Definition. A system is *compatible* if there exists at least one solution, otherwise it is *incompatible*.

Definition. A compatible system is called *definite* if it has the only solution.

Definition. A compatible system is called *indefinite* if it has more than one various solutions.

A system can be written in a matrix form as $A \cdot X = B$

where the matrix of coefficients or the basic matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

the matrix-column of free terms is

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

the unknown matrix-column is

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

A system can be written in a matrix form as $$A \cdot X = B$$

where the matrix of coefficients or the basic matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

the matrix-column of free terms is

$$B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

the unknown matrix-column is

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

BASIC METHODS

1) **Cramer's method**
2) **Inverse matrix method**
3) **Jordan–Gauss method**
4) **Gauss method**

2. Solving system of equations using Cramer's method

Let a system consist of ***n*** linear equations with ***n*** unknowns and its determinant $\det A \neq 0$ then unknowns can be found accordingly to the formulas by Cramer:

$$x_i = \frac{\Delta_i}{\Delta} \qquad i = \overline{1, n}$$

where

$\Delta = \det A$ is the determinant of the system;

Δ_i is a determinant obtained from the determinant of the system by substituting the column ***i*** by the matrix-column ***B***.

Let's consider a system which consists of 3 linear equations with 3 unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

Then unknowns can be found accordingly to Cramer's formulas:

$$x_1 = \frac{\Delta_1}{\Delta} \qquad x_2 = \frac{\Delta_2}{\Delta} \qquad x_3 = \frac{\Delta_3}{\Delta}$$

where:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let's consider a system which consists of 3 linear equations with 3 unknowns

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Then unknowns can be found accordingly to Cramer's formulas:

$$x_1 = \frac{\Delta_1}{\Delta} \qquad x_2 = \frac{\Delta_2}{\Delta} \qquad x_3 = \frac{\Delta_3}{\Delta}$$

where:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

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Then unknowns can be found accordingly to Cramer's formulas:

$$x_1 = \frac{\Delta_1}{\Delta} \qquad x_2 = \frac{\Delta_2}{\Delta} \qquad x_3 = \frac{\Delta_3}{\Delta}$$

where:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

Let's consider a system which consists of 3 linear equations with 3 unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

Then unknowns can be found accordingly to Cramer's formulas:

$$x_1 = \frac{\Delta_1}{\Delta} \qquad x_2 = \frac{\Delta_2}{\Delta} \qquad x_3 = \frac{\Delta_3}{\Delta}$$

where:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Let’s illustrate this method by example.

Example 1. Solve the given system of equations using Cramer’s method:

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases}$$

Solution. Find the determinant of the given system:

$$\Delta = \begin{vmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 5 \cdot 2 \cdot 2 + 4 \cdot 3 \cdot (-1) + 1 \cdot 3 \cdot \left(-1\right) - 4 \cdot 2 \cdot (-1) - 2 \cdot 1 \cdot (-1) -$$

$$-3 \cdot 3 \cdot 5 = 20 - 12 - 3 + 8 + 2 - 45 = -30 \neq 0$$

Its determinant is non-zero. Let’s apply the formulas by Cramer:

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases}$$

$$\Delta_1 = \begin{vmatrix} 0 & -1 & -1 \\ 14 & 2 & 3 \\ 16 & 3 & 2 \end{vmatrix} = 0 - 48 - 42 + 32 - 0 + 28 = -30$$

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{-30}{-30} = 1$$

Its determinant is non-zero. Let's apply the formulas by Cramer:

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases}$$

$$\Delta_2 = \begin{vmatrix} 5 & 0 & -1 \\ 1 & 14 & 3 \\ 4 & 16 & 2 \end{vmatrix} = 140 + 0 - 16 + 56 - 0 - 240 = -60$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{-60}{-30} = 2$$

Its determinant is non-zero. Let's apply the formulas by Cramer:

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases}$$

$$\Delta_3 = \begin{vmatrix} 5 & -1 & 0 \\ 1 & 2 & 14 \\ 4 & 3 & 16 \end{vmatrix} = 160 - 56 + 0 - 0 - 210 + 16 = -90$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{-90}{-30} = 3$$

Checking the answer using the substitution:

$$x_1 = 1, x_2 = 2, x_3 = 3$$

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 5 \cdot 1 - 2 - 3 = 0 \\ 1 + 2 \cdot 2 + 3 \cdot 3 = 14 \\ 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 = 16 \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ 14 = 14 \\ 16 = 16 \end{cases}$$

Economic EXAMPLE

Illustration 8 – A company produces three products everyday. Their total production on a certain day is 45 tons. It is found that the production of the third product exceeds the production of the first product by 8 tons while the total combined production of the first and third product is twice that of the second product. Determine the production level of each product using Cramer's rule.

Solution –

Let the production level of the three products be x, y and z respectively. Therefore, we will have the following equations

$$x + y + z = 45 \qquad -(1)$$

$$z = x + 8$$

$$i.e. \quad -x + 0y + z = 8 \qquad -(2)$$

$$x + z = 2y$$

$$i.e. \quad x - 2y + z = 0 \qquad -(3)$$

Solution –

Let the production level of the three products be x, y and z respectively. Therefore, we will have the following equations

$$x + y + z = 45 \qquad -(1)$$

$$z = x + 8$$

$$i.e. \quad -x + 0y + z = 8 \qquad -(2)$$

$$x + z = 2y$$

$$i.e. \quad x - 2y + z = 0 \qquad -(3)$$

Therefore, we have, using (1), (2) and (3)

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ 8 \\ 0 \end{bmatrix}$$

Therefore, we have, using (1), (2) and (3)

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ 8 \\ 0 \end{bmatrix}$$

Which gives us

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 6$$

Since $\Delta \neq 0$, there is a unique solution.

$$\Delta_1 = \begin{vmatrix} 45 & 1 & 1 \\ 8 & 0 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 66$$

Therefore, we have, using (1), (2) and (3)

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ 8 \\ 0 \end{bmatrix}$$

Which gives us

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 6$$

Since $\Delta \neq 0$, there is a unique solution.

$$\Delta_1 = \begin{vmatrix} 45 & 1 & 1 \\ 8 & 0 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 66$$

$$\Delta_2 = \begin{vmatrix} 1 & 45 & 1 \\ -1 & 8 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 90$$

Therefore, we have, using (1), (2) and (3)

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ 8 \\ 0 \end{bmatrix}$$

Which gives us

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 6$$

Since $\Delta \neq 0$, there is a unique solution.

$$\Delta_1 = \begin{vmatrix} 45 & 1 & 1 \\ 8 & 0 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 66 \qquad \Delta_2 = \begin{vmatrix} 1 & 45 & 1 \\ -1 & 8 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 90$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 45 \\ -1 & 0 & 8 \\ 1 & -2 & 0 \end{vmatrix} = 114$$

Which gives us

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 6$$

Since $\Delta \neq 0$, there is a unique solution.

$$\Delta_1 = \begin{vmatrix} 45 & 1 & 1 \\ 8 & 0 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 66 \qquad \Delta_2 = \begin{vmatrix} 1 & 45 & 1 \\ -1 & 8 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 90$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 45 \\ -1 & 0 & 8 \\ 1 & -2 & 0 \end{vmatrix} = 114$$

Therefore,

$$x = \frac{66}{6} = 11$$

$$y = \frac{90}{6} = 15$$

$$z = \frac{114}{6} = 19$$

Hence, the production levels of the products are as follows:

First product - 11 tons
Second product - 15 tons
Third product - 19 tons

3. Solving system of equations using an inverse matrix method (matrix method)

Let the system consist of ***n*** linear equations with ***n*** unknowns and its determinant $\det A \neq 0$. Write this system in a matrix form as

$$A \cdot X = B$$

Let us multiply both parts a matrix form by the inverse matrix A^{-1} on the left. Then we obtain

$$\underbrace{A^{-1} \cdot A}_{E} \cdot X = A^{-1} \cdot B \qquad \underbrace{E \cdot X}_{X} = A^{-1} \cdot B$$

$$X = A^{-1} \cdot B$$

$$A \cdot X = B$$

$$A \cdot X = B$$

$$A^{-1} \cdot A \cdot X = A^{-1} \cdot B$$

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$$A \cdot X = B$$

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$$\underbrace{A^{-1} \cdot A}_{E} \cdot X = A^{-1} \cdot B$$

$$\underbrace{E \cdot X}_{X} = A^{-1} \cdot B$$

$$X = A^{-1} \cdot B$$

Let us illustrate this method by example.

Example 2. Let's find the solution of the system from **example 3** by the matrix method.

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases}$$

Solution. Here

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 14 \\ 16 \end{pmatrix} \qquad A = \begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$$

Let’s calculate the determinant of the given system:

$$\Delta = \begin{vmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 5\cdot 2\cdot 2 + 4\cdot 3\cdot(-1) + 1\cdot 3\cdot\left(-1\right) - 4\cdot 2\cdot(-1) - 2\cdot 1\cdot(-1) -$$

$$-3\cdot 3\cdot 5 = 20 - 12 - 3 + 8 + 2 - 45 = -30 \neq 0$$

Its determinant is non-zero. Find the inverse matrix by cofactors:

Its determinant is non-zero. Find the inverse matrix by cofactors:

$$A=\begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$$

$$A_{11}=(-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}=-5 \qquad A_{12}=(-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}=10$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}=-5$$

Its determinant is non-zero. Find the inverse matrix by cofactors:

$$A = \begin{pmatrix} 5^{+} & -1^{-} & -1^{+} \\ 1^{-} & 2^{+} & 3^{-} \\ 4^{+} & 3^{-} & 2^{+} \end{pmatrix}$$

$$A_{11} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5$$

$$A_{12} = -\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 10$$

$$A_{13} = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = -5$$

Its determinant is non-zero. Find the inverse matrix by cofactors:

$$A=\begin{pmatrix}5 & -1 & -1\\ 1 & 2 & 3\\ 4 & 3 & 2\end{pmatrix}$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}-1 & -1\\ 3 & 2\end{vmatrix}=-1 \qquad A_{22}=(-1)^{2+2}\begin{vmatrix}5 & -1\\ 4 & 2\end{vmatrix}=14$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}5 & -1\\ 4 & 3\end{vmatrix}=-19$$

Its determinant is non-zero. Find the inverse matrix by cofactors:

$$A=\begin{pmatrix} 5^{+} & -1^{-} & -1^{+} \\ 1^{-} & 2^{+} & 3^{-} \\ 4^{+} & 3^{-} & 2^{+} \end{pmatrix}$$

$$A_{21}=-\begin{vmatrix} -1 & -1 \\ 3 & 2 \end{vmatrix}=-1$$

$$A_{22}=\begin{vmatrix} 5 & -1 \\ 4 & 2 \end{vmatrix}=14$$

$$A_{23}=-\begin{vmatrix} 5 & -1 \\ 4 & 3 \end{vmatrix}=-19$$

Its determinant is non-zero. Find the inverse matrix by cofactors:

$$A = \begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} = -1 \qquad A_{32} = (-1)^{3+2} \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} = -16$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 11$$

Its determinant is non-zero. Find the inverse matrix by cofactors:

$$A = \begin{pmatrix} 5^{+} & -1^{-} & -1^{+} \\ 1^{-} & 2^{+} & 3^{-} \\ 4^{+} & 3^{-} & 2^{+} \end{pmatrix}$$

$$A_{31} = \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} = -1$$

$$A_{32} = -\begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} = -16$$

$$A_{33} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 11$$

Let’s substitute:

$$A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \frac{1}{|A|} \cdot \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{-30} \cdot \begin{pmatrix} -5 & -1 & -1 \\ 10 & 14 & -16 \\ -5 & -19 & 11 \end{pmatrix} = \begin{pmatrix} \frac{5}{30} & \frac{1}{30} & \frac{1}{30} \\ -\frac{10}{30} & -\frac{14}{30} & \frac{16}{30} \\ \frac{5}{30} & \frac{19}{30} & -\frac{11}{30} \end{pmatrix}$$

Let's check the condition $A \cdot A^{-1} = E$

$$A \cdot A^{-1} = \begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{5}{30} & \frac{1}{30} & \frac{1}{30} \\ -\frac{10}{30} & -\frac{14}{30} & \frac{16}{30} \\ \frac{5}{30} & \frac{19}{30} & -\frac{11}{30} \end{pmatrix} =$$

$$= \frac{1}{30} \cdot \begin{pmatrix} 25+10-5 & 5+14-19 & 5-16+11 \\ 5-20+15 & 1-28+57 & 1+32-33 \\ 20-30+10 & 4-42+38 & 4+48-22 \end{pmatrix} =$$

$$= \frac{1}{30} \cdot \begin{pmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

The solution of the given system is $X = A^{-1} \cdot B$. Then

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \cdot B = \begin{pmatrix} \frac{5}{30} & \frac{1}{30} & \frac{1}{30} \\ -\frac{10}{30} & -\frac{14}{30} & \frac{16}{30} \\ \frac{5}{30} & \frac{19}{30} & -\frac{11}{30} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 14 \\ 16 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{5}{30} \cdot 0 + \frac{1}{30} \cdot 14 + \frac{1}{30} \cdot 16 \\ -\frac{10}{30} \cdot 0 - \frac{14}{30} \cdot 14 + \frac{16}{30} \cdot 16 \\ \frac{5}{30} \cdot 0 + \frac{19}{30} \cdot 14 - \frac{11}{30} \cdot 16 \end{pmatrix} = \begin{pmatrix} 0 + \frac{14}{30} + \frac{16}{30} \\ 0 - \frac{196}{30} + \frac{256}{30} \\ 0 + \frac{266}{30} - \frac{176}{30} \end{pmatrix} = \begin{pmatrix} \frac{30}{30} \\ \frac{60}{30} \\ \frac{90}{30} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Thus $x_1 = 1$ $x_2 = 2$ $x_3 = 3$

Economic EXAMPLE

Applications of System of Linear Equations

The following examples can be used to illustrate the common methods of solving systems of linear equations that result from applied business and economic problems.

Illustration 6 – Mr. X invested a par of his investment in 10% bond A and a part in 15% bond B. His interest income during the first year is Rs 4,000. If he invests 20% more in 10% bond A and 10 % more in 15% bond B, his income during the second year increases by Rs 500. Find his initial investment and the new investment in bonds A and B using matrix method.

Solution –

Let initial investment be x in 10% bond A and y in 15% bond B. Then, according to given information, we have

$$0.10x + 0.15y = 4{,}000 \quad \text{or} \quad 2x + 3y = 80{,}000$$

$$0.12x + 0.165y = 4{,}500 \quad \text{or} \quad 8x + 11y = 3{,}00{,}000$$

Solution –

Let initial investment be x in 10% bond A and y in 15% bond B. Then, according to given information, we have

$$0.10x+0.15y=4{,}000 \quad \text{or} \quad 2x+3y=80{,}000$$

$$0.12x+0.165y=4{,}500 \quad \text{or} \quad 8x+11y=3{,}00{,}000$$

Expressing the above equations in matrix form, we obtain

$$\underset{\mathbf{A}}{\begin{bmatrix} 2 & 3 \\ 8 & 11 \end{bmatrix}} \underset{\mathbf{X}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{\mathbf{B}}{\begin{bmatrix} 80{,}000 \\ 3{,}00{,}000 \end{bmatrix}}$$

This can be written in the form $AX = B$ or $X = A^{-1}B$

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This can be written in the form $AX = B$ or $X = A^{-1}B$

Since $|A| = -2 \neq 0$, A^{-1} exists and the solution can be given by:

$$X = A^{-1}B$$

$$= -\frac{1}{2}\begin{bmatrix} 11 & -3 \\ -8 & 2 \end{bmatrix}\begin{bmatrix} 80{,}000 \\ 3{,}00{,}000 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -20{,}000 \\ -40{,}000 \end{bmatrix}$$

$$= \begin{bmatrix} 10{,}000 \\ 20{,}000 \end{bmatrix}$$

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$$= \begin{bmatrix} 10{,}000 \\ 20{,}000 \end{bmatrix}$$

Hence x = Rs 10,000, y = Rs 20,000, and new investments would be Rs 12,000 and Rs 22,000 respectively.

4. Solving system of equations using Jordan–Gauss method

Jordan–Gauss method is used to solve the system, which consists of ***m*** linear equations with ***n*** unknowns. This method includes sequential elimination of unknowns to following scheme.

1. Create an augmented matrix of the given system:

$$A|B = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right)$$

The augmented matrix is called an array with the matrix ***A*** on the left and the matrix-column ***B*** of free terms on the right and denoted by $A|B$. The vertical line separates the matrix-column ***B***.

The leading row and the leading element in **A|B** that corresponds to the choice of the leading equation and the leading unknown in the system are chosen. The system should be transformed in order to let the leading equation be the first one.

2. The leading unknown by means of the leading equation is eliminated from the other equations. For this the certain *elementary row operations* of the matrix $A|B$ are performed it is possible:

1) to change the order of rows (that corresponds to change of the order of the equations' sequence);

2) to multiply rows by any non-zero numbers (that corresponds to multiplying the corresponding equations by these numbers);

3) to add to any row of the matrix its any other row multiplied by any number (that corresponds to addition to one equation of the system another equation multiplied by this number).

Elementary row operations (or elementary transformations) are:

1) interchanging (exchanging) two different rows;
2) adding a multiple of one row to another row;
3) multiplying one row by a non-zero constant;
4) crossing out one of the same row;
5) crossing out zero row.

Due to such transformations we obtain an augmented matrix, *equivalent to the initial one* (i. e. having the same solutions).

On the second step a new leading unknown and a corresponding leading equation are chosen and then this variable is eliminated from all the other equations. The leading row in the matrix remains without change. After such actions the initial matrix *A* will be reduced to the unit matrix with the elements of the main diagonal equal to 1.

Let’s illustrate this method by example.

According to the method by Jordan–Gauss the leading unknown by means of the leading equation on the current step is eliminated not only from equations of the corresponding subsystem but also from the leading equations on previous steps and on any step the leading unknown has the coefficient equal to 1.

Example 3. Let's find the solution of the system from **example 1** by Jordan-Gauss method:

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases}$$

$$A|B = \left(\begin{array}{ccc|c} 5 & -1 & -1 & 0 \\ 1 & 2 & 3 & 14 \\ 4 & 3 & 2 & 16 \end{array}\right)$$

Solution. Let's apply this method. Let's exchange equations:

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 14 \\ 5x_1 - x_2 - x_3 = 0 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases} \Rightarrow$$

Let's consider the matrix form:

$$A|B = \left(\begin{array}{ccc|c} 5 & -1 & -1 & 0 \\ 1 & 2 & 3 & 14 \\ 4 & 3 & 2 & 16 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 5 & -1 & -1 & 0 \\ 4 & 3 & 2 & 16 \end{array}\right) \sim$$

Let’s eliminate the 1-st unknown from the other equations:

$$\Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 14 \\ 5x_1 - x_2 - x_3 = 0 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases} \Rightarrow \begin{bmatrix} \\ [r_2] + [r_1]\cdot(-5) \\ [r_3] + [r_1]\cdot(-4) \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 14 \\ -5x_2 - 10x_3 = -40 \\ -11x_2 - 16x_3 = -70 \end{cases} \Rightarrow$$

Let’s consider the matrix form:

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 5 & -1 & -1 & 0 \\ 4 & 3 & 2 & 16 \end{array}\right) \sim \begin{bmatrix} \\ [r_2] + [r_1]\cdot(-5) \\ [r_3] + [r_1]\cdot(-4) \end{bmatrix} \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -5 & -10 & -40 \\ 0 & -11 & -16 & -70 \end{array}\right) \sim$$

Let's obtain the coefficient 1 at the 2-nd unknown:

$$\Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 14 \\ -5x_2 - 10x_3 = -40 \\ -11x_2 - 16x_3 = -70 \end{cases} \Rightarrow \left[[r_2] : (-5) \right] \Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 14 \\ x_2 + 2x_3 = 8 \\ -11x_2 - 16x_3 = -70 \end{cases}$$

Let's consider the matrix form:

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -5 & -10 & -40 \\ 0 & -11 & -16 & -70 \end{array} \right) \sim \left[[r_2] : (-5) \right] \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & 2 & 8 \\ 0 & -11 & -16 & -70 \end{array} \right) \sim$$

Let's eliminate the 2-nd unknown:

$$\Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 14 \\ x_2 + 2x_3 = 8 \\ -11x_2 - 16x_3 = -70 \end{cases} \Rightarrow \begin{bmatrix} [r_1] + [r_2] \cdot (-2) \\ \\ [r_3] + [r_2] \cdot 11 \end{bmatrix} \Rightarrow \begin{cases} x_1 - x_3 = -2 \\ x_2 + 2x_3 = 8 \\ 6x_3 = 18 \end{cases} \Rightarrow$$

Let's consider the matrix form:

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & 2 & 8 \\ 0 & -11 & -16 & -70 \end{array} \right) \sim \begin{bmatrix} [r_1] + [r_2] \cdot (-2) \\ \\ [r_3] + [r_2] \cdot 11 \end{bmatrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 6 & 18 \end{array} \right) \sim$$

Let's obtain the coefficient 1 at the 3-rd unknown:

$$\Rightarrow \begin{cases} x_1 - x_3 = -2 \\ x_2 + 2x_3 = 8 \\ 6x_3 = 18 \end{cases} \Rightarrow \begin{bmatrix} \\ \\ [r_3]:6 \end{bmatrix} \Rightarrow \begin{cases} x_1 - x_3 = -2 \\ x_2 + 2x_3 = 8 \\ x_3 = 3 \end{cases} \Rightarrow$$

Let's consider the matrix form:

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 6 & 18 \end{array}\right) \sim \begin{bmatrix} \\ \\ [r_3]:6 \end{bmatrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right) \sim$$

Economic EXAMPLE. GAUSS METHOD

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Let’s eliminate the 3-rd unknown:

$$\Rightarrow \begin{cases} x_1 - x_3 = -2 \\ x_2 + 2x_3 = 8 \\ x_3 = 3 \end{cases} \Rightarrow \begin{bmatrix} [r_1]+[r_3] \\ [r_2]+[r_3]\cdot(-2) \\ \end{bmatrix} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

Let’s consider the matrix form:

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right) \sim \begin{bmatrix} [r_1]+[r_3] \\ [r_2]+[r_3]\cdot(-2) \\ \end{bmatrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Then write down the received augmented matrix as the system of equations:

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

Illustration 7 – An automobile company uses three types of steel s_1, s_2 and s_3 for producing three types of cars c_1, c_2 and c_3. The steel requirement (in tons) for each type of car is given below:

		Cars		
		c_1	c_2	c_3
	s_1	2	3	4
Steel	s_2	1	1	2
	s_3	3	2	1

Determine the number of cars of each type which can be produced using 29, 13 and 16 tons of steel of the three types respectively.

Solution –

Let x, y and z denote the number of cars that can be produced of each type. Then we have

$$2x+3y+4z=29$$
$$x+y+2z=13$$
$$3x+2y+z=16$$

Solution –

Let x, y and z denote the number of cars that can be produced of each type. Then we have

$$2x+3y+4z=29$$
$$x+y+2z=13$$
$$3x+2y+z=16$$

The above information can be represented using the matrix method, as under.

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 29 \\ 13 \\ 16 \end{bmatrix}$$

The above equation can be solved using Gauss Jordan elimination method.
By applying the operation $R_1 \leftrightarrow R_2$ the given system is equivalent to

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 13 \\ 29 \\ 16 \end{bmatrix}$$

The above equation can be solved using Gauss Jordan elimination method.
By applying the operation $R_1 \leftrightarrow R_2$ the given system is equivalent to

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 29 \\ 16 \end{bmatrix}$$

Now applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, the above system is equivalent to

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \\ -23 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2$ the above system is equivalent to

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \\ -23 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2$ the above system is equivalent to

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \\ -20 \end{bmatrix}$$

$$x + y + 2z = 13 \qquad -(1)$$

$$y = 3 \qquad -(2)$$

$$-5z = -20 \qquad -(3)$$

Therefore, $z = 4$. Substituting $y = 3$ and $z = 4$ in (1), we get $x = 2$. Hence the solution is

$x = 2$, $y = 3$ and $z = 4$

Definition

- *rank* / ранг
- elementary row operations / елементарні операції над рядками

5. Notion of a matrix rank and its calculation

The ***rank*** of the ***m×n***-matrix ***A*** is the number of non-zero rows of the transformed matrix denoted by $r(A)$ or $rank\,A$.

To find a rank of a matrix we can use elementary row operations to reduce the given matrix to the triangular form:

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ 0 & 0 & ... & a_{nn} \end{pmatrix}$$

or to the truncated-triangular form:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} & a_{1\,k+1} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2k} & a_{2\,k+1} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{kk} & a_{k\,k+1} & \dots & a_{kn} \end{pmatrix}$$

then the number of non-zero rows of the transformed matrix defines the rank of the initial matrix.

Elementary row operations (or elementary transformations) are:

1) interchanging (exchanging) two different rows;
2) adding a multiple of one row to another row;
3) multiplying one row by a non-zero constant;
4) crossing out one of the same row;
5) crossing out zero row.

Example 4. Calculate the rank of the matrix: $A=\begin{pmatrix}2 & -2 & 3\\ 1 & -3 & 2\\ 3 & 3 & 9\end{pmatrix}$

Solution. Let us exchange the 1-st and the 2-nd rows:

$$A\sim\begin{pmatrix}1 & -3 & 2\\ 2 & -2 & 3\\ 3 & 3 & 9\end{pmatrix}\sim\begin{bmatrix} \\ [r_1]\cdot(-2)+[r_2]\\ [r_1]\cdot(-3)+[r_3]\end{bmatrix}\sim\begin{pmatrix}1 & -3 & 2\\ 0 & -4 & -1\\ 0 & 12 & 3\end{pmatrix}\sim\begin{bmatrix} \\ [r_2]:(-1)\\ [r_3]:3\end{bmatrix}\sim$$

$$\sim\begin{pmatrix}1 & -3 & 2\\ 0 & 4 & 1\\ 0 & 4 & 1\end{pmatrix}\sim\begin{pmatrix}1 & -3 & 2\\ 0 & 4 & 1\end{pmatrix}$$

The number of non-zero rows of the transformed matrix equivalent to the initial one is 2. Therefore $rank\,A=2$

6. Investigation of the system compatibility with the help of Kronecker-Capelli theorem

Let's consider ***Kronecker–Capelli theorem:***

1. A system of linear equations is compatible if its basic matrix and its augmented matrix have the same rank, i. e.

$$rank\ A = rank\ A|B$$

Let’s consider ***Kronecker–Capelli theorem:***

A compatible system is determined if the ranks are equal to the unknowns number, i. e.

$$rank\, A = rank\, A|B = n$$

2. A compatible system is undetermined if the ranks are less than the unknowns number, i.e.

$$rank\, A = rank\, A|B < n$$

3. A linear system is incompatible if its basic matrix and its augmented matrix have the different rank, i. e.

$$rank\, A \neq rank\, A|B$$

Practical homework: investigate the compatibility of the given system:

$$\begin{cases} x_1 + 2x_2 - 3x_3 + 4x_4 = 7 \\ 2x_1 + 4x_2 + 5x_3 - x_4 = 2 \\ 5x_1 + 10x_2 + 7x_3 + 2x_4 = 11 \end{cases}$$

Theoretical homework (self-work):

1) to learn Theme 1;

2) to learn Theme 2.

Practical homework: investigate the compatibility of the given system:

$$\begin{cases} x_1 + 2x_2 - 3x_3 + 4x_4 = 7 \\ 2x_1 + 4x_2 + 5x_3 - x_4 = 2 \\ 5x_1 + 10x_2 + 7x_3 + 2x_4 = 11 \end{cases}$$

Let's solve it:

$$A|B = \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 2 & 4 & 5 & -1 & 2 \\ 5 & 10 & 7 & 2 & 11 \end{array}\right) \sim \left[\begin{array}{c} \\ [r_2]+[r_1]\cdot(-2) \\ [r_3]+[r_1]\cdot(-5) \end{array}\right] \sim \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 11 & -9 & -12 \\ 0 & 0 & 22 & -18 & -24 \end{array}\right) \sim$$

$$A|B=\left(\begin{array}{cccc|c}1&2&-3&4&7\\2&4&5&-1&2\\5&10&7&2&11\end{array}\right)\sim\begin{bmatrix}\\ [r_2]+[r_1]\cdot(-2)\\ [r_3]+[r_1]\cdot(-5)\end{bmatrix}\sim\left(\begin{array}{cccc|c}1&2&-3&4&7\\0&0&11&-9&-12\\0&0&22&-18&-24\end{array}\right)\sim$$

$$\sim\begin{bmatrix}\\ [r_2]:11\\ \\ \end{bmatrix}\sim\left(\begin{array}{cccc|c}1&2&-3&4&7\\0&0&1&-9/11&-12/11\\0&0&22&-18&-24\end{array}\right)\sim\begin{bmatrix}\\ \\ [r_3]+[r_2]\cdot(-22)\end{bmatrix}\sim$$

$$A|B=\left(\begin{array}{cccc|c}1 & 2 & -3 & 4 & 7\\ 2 & 4 & 5 & -1 & 2\\ 5 & 10 & 7 & 2 & 11\end{array}\right)\sim\left[\begin{array}{c} \\ [r_2]+[r_1]\cdot(-2)\\ [r_3]+[r_1]\cdot(-5)\end{array}\right]\sim\left(\begin{array}{cccc|c}1 & 2 & -3 & 4 & 7\\ 0 & 0 & 11 & -9 & -12\\ 0 & 0 & 22 & -18 & -24\end{array}\right)\sim$$

$$\sim\left[\begin{array}{c} \\ [r_2]:11\\ \\ \end{array}\right]\sim\left(\begin{array}{cccc|c}1 & 2 & -3 & 4 & 7\\ 0 & 0 & 1 & -9/11 & -12/11\\ 0 & 0 & 22 & -18 & -24\end{array}\right)\sim\left[\begin{array}{c} \\ \\ [r_3]+[r_2]\cdot(-22)\end{array}\right]\sim$$

$$\sim\left(\begin{array}{cccc|c}1 & 2 & -3 & 4 & 7\\ 0 & 0 & 1 & -9/11 & -12/11\\ 0 & 0 & 0 & 0 & 0\end{array}\right)\sim\left(\begin{array}{cccc|c}1 & 2 & -3 & 4 & 7\\ 0 & 0 & 1 & -9/11 & -12/11\end{array}\right)\sim\left[\begin{array}{c}[r_1]+[r_2]\cdot 3\\ \\ \end{array}\right]\sim$$

$$\sim\begin{bmatrix} \\ [r_2]:11 \\ \\ \end{bmatrix}\sim\left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \\ 0 & 0 & 22 & -18 & -24 \end{array}\right)\sim\begin{bmatrix} \\ \\ [r_3]+[r_2]\cdot(-22) \end{bmatrix}\sim$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)\sim\left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \end{array}\right)\sim\begin{bmatrix} [r_1]+[r_2]\cdot 3 \\ \\ \end{bmatrix}\sim$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \end{array}\right)\sim\begin{bmatrix} [r_1]+[r_2]\cdot 3 \\ \\ \end{bmatrix}\sim\left(\begin{array}{cccc|c} 1 & 2 & 0 & 17/11 & 41/11 \\ 0 & 0 & 1 & -9/11 & -12/11 \end{array}\right)$$

$$\sim\left(\begin{array}{cccc|c}1 & 2 & 0 & 4 & 7\\ 0 & 0 & 1 & -9/11 & -12/11\end{array}\right)\sim\left[\begin{array}{c}[r_1]+[r_2]\cdot 3\\ \end{array}\right]\sim\left(\begin{array}{cccc|c}1 & 2 & 0 & 17/11 & 41/11\\ 0 & 0 & 1 & -9/11 & -12/11\end{array}\right)$$

The initial system is equivalent to the following system of equations:

$$\begin{cases} x_1 + 2x_2 + \dfrac{17}{11}x_4 = \dfrac{41}{11} \\ x_3 - \dfrac{9}{11}x_4 = -\dfrac{12}{11} \end{cases}$$

The initial system is equivalent to the following system of equations:

$$\begin{cases} x_1 + 2x_2 + \dfrac{17}{11}x_4 = \dfrac{41}{11} \\ x_3 - \dfrac{9}{11}x_4 = -\dfrac{12}{11} \end{cases}$$

Let's obtain the general solution:

$$\begin{cases} x_1 = \dfrac{41}{11} - 2x_2 - \dfrac{17}{11}x_4 \\ x_3 = -\dfrac{12}{11} + \dfrac{9}{11}x_4 \end{cases}$$

Let's obtain the general solution:

$$\begin{cases} x_1 = \frac{41}{11} - 2x_2 - \frac{17}{11}x_4 \\ x_3 = -\frac{12}{11} + \frac{9}{11}x_4 \end{cases}$$

where x_1, x_3 are basic unknowns,

x_2, x_4 are free ones.

Let's obtain the general solution:

$$\begin{cases} x_1 = \frac{41}{11} - 2x_2 - \frac{17}{11} x_4 \\ x_3 = -\frac{12}{11} + \frac{9}{11} x_4 \end{cases}$$

:

where x_1, x_3 are basic unknowns,

x_2, x_4 are free ones.

For example, obtain the particular solution, if

$$x_2 = 1, x_4 = -1$$

Let's obtain the general solution:

$$\begin{cases} x_1 = \frac{41}{11} - 2x_2 - \frac{17}{11} x_4 \\ x_3 = -\frac{12}{11} + \frac{9}{11} x_4 \end{cases}$$

For example, obtain the particular solution, if

$$x_2 = 1, x_4 = -1$$

$$x_1 = \frac{41}{11} - 2 + \frac{17}{11} = \frac{36}{11} \qquad x_3 = -\frac{12}{11} - \frac{9}{11} = -\frac{21}{11}$$

Let's obtain the general solution:

$$\begin{cases} x_1 = \frac{41}{11} - 2x_2 - \frac{17}{11}x_4 \\ x_3 = -\frac{12}{11} + \frac{9}{11}x_4 \end{cases}$$

For example, obtain the particular solution, if

$$x_2 = 0, x_4 = 0$$

$$x_1 = \frac{41}{11} \qquad x_3 = -\frac{12}{11}$$

Thus $x_1 = \frac{41}{11} \quad x_2 = 0 \quad x_3 = -\frac{12}{11} \quad x_4 = 0$

are the basic solution.

In this example the basic solution is not the supporting one, because

$$x_3 = -\frac{12}{11} < 0$$

Homework. Solve the system:

$$\begin{cases} 2x_1 + x_2 + x_3 = 2 \\ 5x_1 + x_2 + 3x_3 = 4 \\ 7x_1 + 2x_2 + 4x_3 = 1 \end{cases}$$

Homework. Solve the system:

$$\begin{cases} 2x_1 + 2x_2 - 2x_3 = -2 \\ -x_1 + x_2 - 3x_3 = -5 \\ x_1 + 3x_2 - 5x_3 = -7 \end{cases}$$

EXAMPLE

Solve this system and find unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

EXAMPLE

Solve this system and find unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

Solution:

$$x_1 = \frac{b_1 \cdot a_{22} - a_{12} \cdot b_2}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}}$$

EXAMPLE

Solve this system and find unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

Solution:

$$x_1 = \frac{b_1 \cdot a_{22} - a_{12} \cdot b_2}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

EXAMPLE

Solve this system and find unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

Solution:

$$x_1 = \frac{b_1 \cdot a_{22} - a_{12} \cdot b_2}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{\Delta_1}{\Delta}$$

EXAMPLE

Solve this system and find unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

Solution:

$$x_2 = \frac{a_{11} \cdot b_2 - a_{21} \cdot b_1}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{\Delta_2}{\Delta}$$