МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ
ХАРКІВСЬКИЙ НАЦІОНАЛЬНИЙ ЕКОНОМІЧНИЙ УНІВЕРСИТЕТ

Guideline
for practical tasks in analytic geometry
of the educational discipline "Higher and applied mathematics"
for foreign and English-learning full-time students
of the preparatory direction "Management"

Харків, Вид. ХНЕУ, 2011

Ministry of Education and Science of Ukraine
Kharkiv National University of Economics

Guideline
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Methodical recommendations are intended for foreign and English-learning students of the preparatory direction "Management" for practical studies of “Analytic geometry” of the discipline "Higher and applied mathematics".

The sufficient theoretical material and the number of solved typical examples of each theme give students the possibility to master “Analytic geometry” and apply the obtained knowledge in practice on their own.

Practical tasks for self-work and the list of theoretical questions which promote improving and extending students’ knowledge of all the themes are given.

It is recommended for foreign and English-learning full-time students of the preparatory direction "Management".

Introduction

Methodical recommendations are intended for foreign and English-learning students of the preparatory direction "Management" for practical studies of “Analytic geometry” of the discipline "Higher and applied mathematics".

Its aim is the practical application of mathematic apparatus to the problem solutions in “Analytic geometry”, which consists of the following themes: Cartesian coordinates of a vector and a point, linear operations with vectors, scalar, cross and mixed products and their properties, ways to define a straight line on a plane and in space and mutual arrangement of lines, ways to define a plane in space and mutual arrangement of planes, second-order curves (circle, ellipse, hyperbola, parabola) and investigation of their forms.

The sufficient number of solved typical examples of each theme gives students the possibility to master “Analytic geometry” and apply the obtained knowledge in practice on their own. At the end of methodical recommendations there are tasks for self-work and the list of theoretical questions which promote improving and extending students’ knowledge of all the themes.

1. Vectors

1.1. Cartesian Coordinate System on the Plane and in Space

If a one-to-one correspondence between points on the plane and numbers (pairs of numbers) is specified, then one says that a *coordinate system* is introduced on the plane.

A *rectangular Cartesian coordinate system on the plane* is determined by a scale segment for measuring lengths and two mutually perpendicular axes. The point of intersection of the axes is usually denoted by the letter O and is called the *origin*, while the axes themselves are called the *coordinate axes*. As a rule, one of the coordinate axes is horizontal and the right sense is positive. This axis is called the *abscissa axis* and is denoted by the letter X or by OX. On the vertical axis, which is called the *ordinate axis* and is denoted by Y or OY, the upward sense is usually positive (Fig. 1.1). The coordinate system introduced above is often denoted by XY or OXY.

The abscissa axis divides the plane into the *upper* and *lower* half-planes, while the ordinate axis divides the plane into the *right* and *left* half-planes. The two coordinate axes divide the plane into four parts, which are called *quadrants* and numbered as shown in Fig. 1.1.

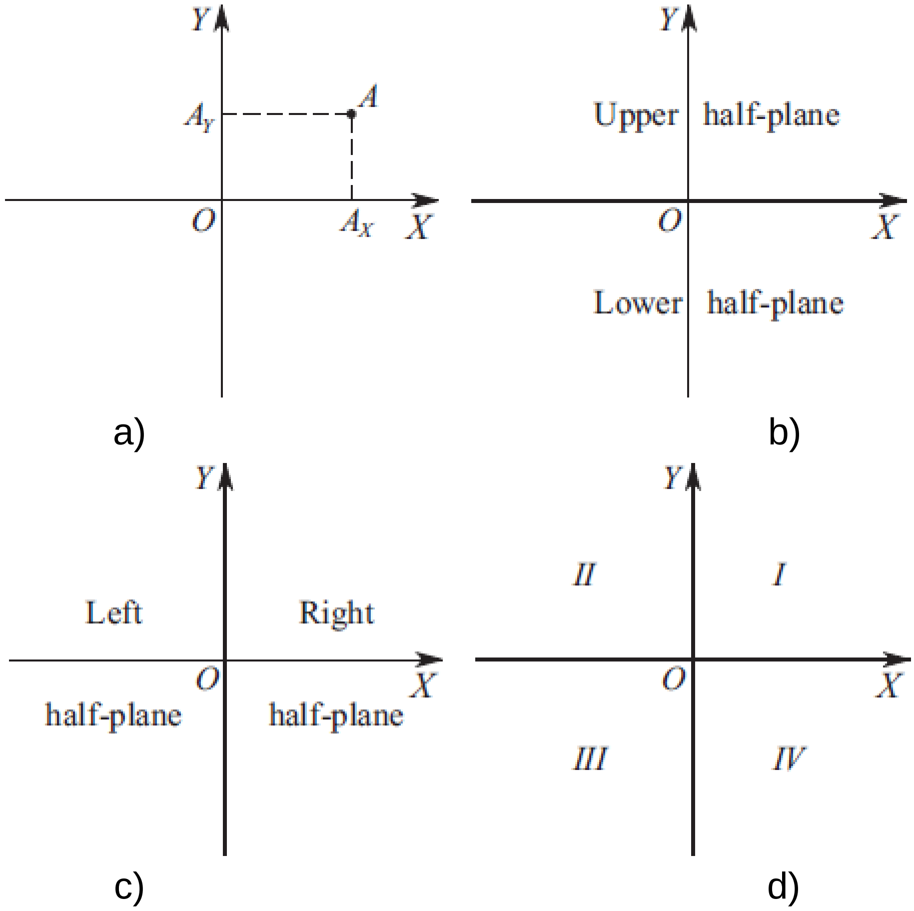


Fig. 1.1. A rectangular Cartesian coordinate system

Let's take an arbitrary point A on the plane and project it onto the coordinate axes, i.e., draw perpendiculars to the axes OX and OY through A. The points of intersection of the perpendiculars with the axes are denoted by A_X and A_Y, respectively (Fig. 1.1). The numbers $x = OA_X$ and $y = OA_Y$, where OA_X and OA_Y are the respective values of the segments $\overrightarrow{OA_X}$ and $\overrightarrow{OA_Y}$ on the abscissa and ordinate axes, are called the *coordinates of the point* A in the rectangular Cartesian coordinate system.

The number x is the first coordinate, or the *abscissa*, of the point *A*, and y is the second coordinate, or the *ordinate*, of the point A. One says that the point A has the coordinates (x, y) and uses the notation $A(x, y)$.

A *rectangular Cartesian coordinate system in space* is determined by a scale segment for measuring lengths and three pairwise perpendicular directed straight lines OX, OY and OZ (the *coordinate axes*) concurrent at a single point O (the *origin*). The three coordinate axes divide the space into eight parts called *octants*.

We choose an arbitrary point M in space and project it onto the coordinate axes, i.e., draw the perpendiculars to the axes OX, OY and OZ through M. We denote the points of intersection of the perpendiculars with the axes by x, y and z, respectively. These numbers (Fig. 1.2) are the signed lengths of the segments of the axes OX, OY and OZ, respectively, are called the *coordinates of the point M* in the rectangular Cartesian coordinate system.

The number x is called the first coordinate or the *abscissa* of the point M, the number y is called the second coordinate or the *ordinate* of the point M, and the number z is called the third coordinate or the *applicate* of the point M. Usually one says that the point M has the coordinates (x, y, z), and the notation $M(x, y, z)$ is used.

1.2. Formulas of division of a segment in the given ratio

Let's assume that the point $M(x_M, y_M, z_M)$ divides a segment between the points $M_1(x_{M_1}, y_{M_1}, z_{M_1})$ and $M_2(x_{M_2}, y_{M_2}, z_{M_2})$ in the ratio λ, that is $\lambda = \dfrac{|M_1M|}{|MM_2|}$.

In this case the following formulas should be used to find the coordinates of the point M:

$$x_M = \frac{x_{M_1} + \lambda \cdot x_{M_2}}{1+\lambda},\ y_M = \frac{y_{M_1} + \lambda \cdot y_{M_2}}{1+\lambda},\ z_M = \frac{z_{M_1} + \lambda \cdot z_{M_2}}{1+\lambda}.$$

In the particular case, if the point M bisects the segment M_1M_2 $(\lambda = 1)$ then

$$x_M = \frac{x_{M_1} + x_{M_2}}{2},\ y_M = \frac{y_{M_1} + y_{M_2}}{2},\ z_M = \frac{z_{M_1} + z_{M_2}}{2}.$$

1.3. Vectors and basic vector operations

A segment bounded by points A and B is called a *directed segment* if its initial point and endpoint are chosen. Such a segment with initial point A and endpoint B is denoted by $\overrightarrow{AB}$ (Fig. 1.3).

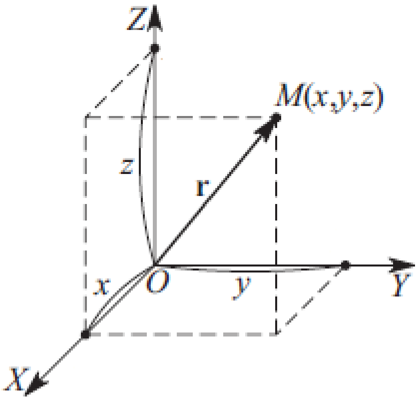


Fig. 1.2. Point in rectangular Cartesian coordinate system in space

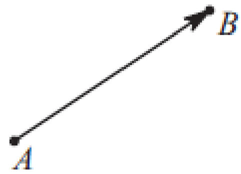


Fig. 1.3. Vector

A directed segment with initial point A and endpoint B is called the *vector* $\overrightarrow{AB}$ or $\vec{a}$. A nonnegative number equal to the length of the segment AB joining the points A and B is called the *length* $\left|\overrightarrow{AB}\right|$ of the *vector* $\overrightarrow{AB}$. The vector $\overrightarrow{BA}$ is said to be *opposite to the vector* $\overrightarrow{AB}$, i.e. $\overrightarrow{AB} = -\overrightarrow{BA}$.

Let the point $A(x_A, y_A, z_A)$ be an initial point of the vector $\overrightarrow{AB}$ and the point $B(x_B, y_B, z_B)$ be its endpoint. Coordinates of the vector $\overrightarrow{AB}$ are defined as $\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A)$.

To each point M of three-dimensional space one can assign its position vector. The directed segment $\overrightarrow{OM}$ is called the *position vector of the point* M. The position vector determines the vector $\vec{r}$ $\left(\vec{r} = \overrightarrow{OM}\right)$ whose coordinates are its projections on the axes OX, OY and OZ, respectively.

An arbitrary vector $\vec{a} = (a_x, a_y, a_z)$ can be represented as

$$\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j} + a_z \cdot \vec{k},$$

where a_x, a_y, a_z are projections of the vector $\vec{a}$ on the axes OX, OY and OZ, respectively; $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$ are the unit vectors with the same directions as the coordinate axes OX, OY and OZ.

Every of the vectors $\vec{i}, \vec{j}, \vec{k}$ is perpendicular (orthogonal) to the both others. These vectors form a so called orthonormalized basis. The projections a_x, a_y, a_z are coordinates of the vector in the orthonormalized basis.

The distance between an initial point and an endpoint of a vector is called its length or module and designated by $|\vec{a}|$ or $|\overrightarrow{AB}|$.

The module of a vector $\vec{a}$ is calculated according to the following formula:

$$|\vec{a}| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}\,.$$

The module of a vector $\overrightarrow{AB}$ is calculated according to the following formula:

$$|\overrightarrow{AB}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}\,.$$

Direction cosines are called cosines of the angles between the vector $\vec{a}$ and positive directions of the corresponding coordinate axes and defined as follows:

$$\cos\alpha = \frac{a_x}{|\vec{a}|} = \frac{a_x}{\sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}}; \qquad \cos\beta = \frac{a_y}{|\vec{a}|} = \frac{a_y}{\sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}};$$

$$\cos\gamma = \frac{a_z}{|\vec{a}|} = \frac{a_z}{\sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}}\,.$$

They are related to the equality

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Example 1.1. Find the direction cosines of the vector $\overrightarrow{AB}$ if the points

$A(1,2,0)$ and $B(3,1,-2)$ are given.

Solution. The coordinates of the vector $\overrightarrow{AB}$ are calculated in this way:

$$\overrightarrow{AB}=(3-1,1-2,-2-0)=(2,-1,-2).$$

Its length is

$$\left|\overrightarrow{AB}\right|=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}=\sqrt{2^2+(-1)^2+(-2)^2}=$$

$$=\sqrt{9}=3.$$

The direction cosines are

$$\cos\alpha=\frac{a_x}{\left|\vec{a}\right|}=\frac{2}{3},\qquad \cos\beta=\frac{a_y}{\left|\vec{a}\right|}=-\frac{1}{3},\qquad \cos\gamma=\frac{a_z}{\left|\vec{a}\right|}=-\frac{2}{3}.$$

Two vectors are said to be *collinear (parallel)* if they lie on the same straight line or on parallel lines. Three vectors are said to be *coplanar* if they lie in the same plane or in parallel planes. Two vectors should be considered *equal* if they are *collinear,* equally directed and have equal lengths.

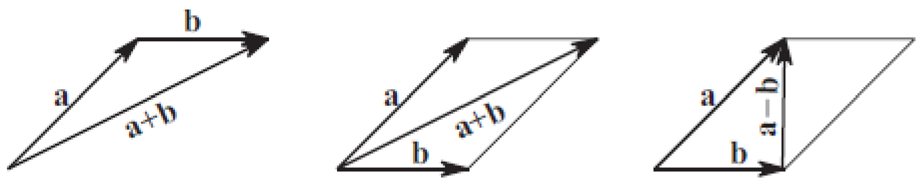
A vector $\vec{0}=(0,0,0)$, whose initial point and endpoint coincide is called the *zero vector* (*the null vector*).

The length of the zero vector is equal to zero $\left(\left|\vec{0}\right|=0\right)$, and the direction of the zero vector is assumed to be arbitrary. A vector $\vec{e}$ of unit length is called a *unit vector*.

Basic vector operations:

1. The *sum* $\vec{a}+\vec{b}$ *of vectors* $\vec{a}$ *and* $\vec{b}$ is defined as the vector directed from the initial point of $\vec{a}$ to the endpoint of $\vec{b}$ under the condition that $\vec{b}$ is applied at the endpoint of $\vec{a}$.

The rule for addition of vectors, which is contained in this definition, is called the *triangle rule of vectors* (Fig. 1.4, a). The sum $\vec{a}+\vec{b}$ can also be found using the *parallelogram rule* (Fig. 1.4, b). The *difference* $\vec{a}-\vec{b}$ *of vectors* $\vec{a}$ *and* $\vec{b}$ is defined as follows: $\vec{b}+\left(\vec{a}-\vec{b}\right)=\vec{a}$ (Fig. 1.4, c).



a) b) c)

Fig. 1.4. The sum of vectors: triangle rule (*a*) and parallelogram rule (*b*). The difference of vectors (*c*)

A sum or difference of vectors are determined according to the formulas:

$$\vec{a} \pm \vec{b} = \left(a_x \pm b_x, a_y \pm b_y, a_z \pm b_z\right).$$

2. The *product* $\lambda\vec{a}$ *of a vector* $\vec{a}$ *by a number* λ is defined as the vector whose length is equal to $|\lambda\vec{a}| = |\lambda| \cdot |\vec{a}|$ and whose direction coincides with that of the vector $\vec{a}$ if $\lambda > 0$ or is opposite to the direction of the vector $\vec{a}$ if $\lambda < 0$.

A product (multiplication) of a vector by a number is determined according to the formula:

$$\alpha \cdot \vec{a} = \left(\alpha \cdot a_x, \alpha \cdot a_y, \alpha \cdot a_z\right).$$

Remark. If $\vec{a} = 0$ or $\lambda = 0$, then the absolute value of the product is zero, i.e., it is the zero vector. In this case, the direction of the product $\lambda\vec{a}$ is undetermined.

Main properties of operations with vectors:

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutativity).
2. $\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$ (associativity of addition).
3. $\vec{a} + \vec{0} = \vec{a}$ (existence of the zero vector).
4. $\vec{a} + \left(-\vec{a}\right) = \vec{0}$ (existence of the opposite vector).
5. $\lambda\left(\vec{a} \pm \vec{b}\right) = \lambda\vec{a} \pm \lambda\vec{b}$ (distributivity with respect to addition or difference of vectors).
6. $(\lambda \pm \mu)\vec{a} = \lambda\vec{a} \pm \mu\vec{a}$ (distributivity with respect to addition or difference of constants).

7. $\lambda\left(\mu\vec{a}\right)=(\lambda\mu)\vec{a}$ (associativity of product).

8. $1\cdot\vec{a}=\vec{a}$ (multiplication by unity).

Projection of vector onto axis. A straight line with a unit vector $\vec{e}$ lying on it determining the positive sense of the line is called an *axis*. The *projection* $pr_{\vec{e}}\vec{a}$ of a vector $\vec{a}$ onto the axis (Fig. 1.5) is defined as the directed segment on the axis whose signed length is equal to the scalar product of $\vec{a}$ by the unit vector $\vec{e}$, i.e., is determined by the formula

$$pr_{\vec{e}}\vec{a}=\vec{a}\cdot\cos\varphi,$$

where φ is the angle between the vectors $\vec{a}$ and $\vec{e}$.

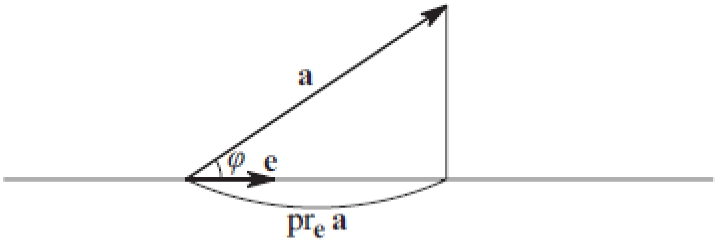


Fig. 1.5. Projection of a vector onto the axes

Example 1.2. Two vectors $\vec{a}=(3,-2,-6)$ and $\vec{b}=(-2,1,0)$ are given. Determine the projections on the coordinate axes of the following vectors:
1) $\vec{a}+\vec{b}$; 2) $\vec{a}-\vec{b}$; 3) $2\vec{a}$; 4) $2\vec{a}-3\vec{b}$.

Solution. By the rule of vector addition and vector multiplication by a number we have:

$$\vec{a}+\vec{b}=\left(a_x+b_x,a_y+b_y,a_z+b_z\right)=(3+(-2),-2+1,-6+0)=(1,-1,-6);$$

$$\vec{a}-\vec{b}=\left(a_x-b_x,a_y-b_y,a_z-b_z\right)=(3-(-2),-2-1,-6-0)=(5,-3,-6);$$

$$2\cdot\vec{a}=\left(2\cdot a_x,2\cdot a_y,2\cdot a_z\right)=(2\cdot 3,2\cdot(-2),2\cdot(-6))=(6,-4,-12);$$

$$2\vec{a}-3\vec{b}=\left(2a_x+3b_x,2a_y+3b_y,2a_z+3b_z\right)=$$

$$=(2\cdot 3-3\cdot(-2),2\cdot(-2)-3\cdot 1,2\cdot(-6)-3\cdot 0)=(12,-7,-12).$$

1.4. Scalar product of two vectors

The *scalar product* of two vectors is defined as the product of their absolute values times the cosine of the angle between the vectors (Fig. 1.6),

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi. \tag{1.1}$$

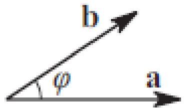


Fig. 1.6. Scalar product of two vectors

If the angle between vectors $\vec{a}$ and $\vec{b}$ is acute, then $\vec{a} \cdot \vec{b} > 0$; if the angle is obtuse, then $\vec{a} \cdot \vec{b} < 0$; if the angle is right, then $\vec{a} \cdot \vec{b} = 0$. Taking into account (1.1), we can write the scalar product as

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi = |\vec{a}| \cdot pr_{\vec{a}}\vec{b} = |\vec{b}| \cdot pr_{\vec{b}}\vec{a}.$$

Remark. The scalar product of a vector $\vec{a}$ by a vector $\vec{b}$ is also denoted by $(\vec{a}, \vec{b})$ or $\vec{a}\vec{b}$.

The *angle* φ *between vectors* is determined by the formula

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z}{\sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}\sqrt{b_x{}^2 + b_y{}^2 + b_z{}^2}}.$$

Properties of scalar product:

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutativity).
2. $\vec{a} \cdot (\vec{b} \pm \vec{c}) = \vec{a} \cdot \vec{b} \pm \vec{a} \cdot \vec{c}$ (distributivity with respect to addition of vectors). This property holds for any number of summands.
3. If vectors $\vec{a}$ and $\vec{b}$ are collinear, then $\vec{a} \cdot \vec{b} = \pm|\vec{a}| \cdot |\vec{b}|$. (The sign + is taken if the vectors $\vec{a}$ and $\vec{b}$ have the same sense, and the sign – is taken if the senses are opposite.)
4. $(\lambda\vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b})$ (associativity with respect to a scalar factor).

5. $\vec{a}\cdot\vec{a}=|\vec{a}|^2$. The scalar product $\vec{a}\cdot\vec{a}$ is denoted by $|\vec{a}|^2$ (the *scalar square* of the vector $\vec{a}$).

6. The length of a vector is expressed via the scalar product by

$$|\vec{a}|=\sqrt{\vec{a}\cdot\vec{a}}=\sqrt{|\vec{a}|^2}\,.$$

7. Two nonzero vectors $\vec{a}$ and $\vec{b}$ are perpendicular if and only if $\vec{a}\vec{b}=0$.

8. The scalar products of basis vectors are

$$\vec{i}\cdot\vec{j}=\vec{i}\cdot\vec{k}=\vec{j}\cdot\vec{k}=0,\ \ \vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1.$$

9. If vectors are given by their coordinates, $\vec{a}=\left(a_x,a_y,a_z\right)$ and $\vec{b}=\left(b_x,b_y,b_z\right)$, then

$$\vec{a}\cdot\vec{b}=\left(a_x\vec{i}+a_y\vec{j}+a_z\vec{k}\right)\cdot\left(b_x\vec{i}+b_y\vec{j}+b_z\vec{k}\right)=a_x\cdot b_x+a_y\cdot b_y+a_z\cdot b_z\,.$$

10. The *Cauchy – Schwarz inequality* and the *Minkowski inequality*

$$\left|\vec{a}\cdot\vec{b}\right|\le|\vec{a}|\cdot\left|\vec{b}\right| \quad \text{and} \quad \left|\vec{a}+\vec{b}\right|\le|\vec{a}|+\left|\vec{b}\right|.$$

1.5. Cross product of two vectors

The *cross product* of a vector $\vec{a}$ by a vector $\vec{b}$ is defined as the vector $\vec{c}$ (Fig. 1.7) satisfying the following three conditions:

1. Its absolute value is equal to the area of the parallelogram spanned by the vectors $\vec{a}$ and $\vec{b}$; i.e.,

$$|\vec{c}|=|\vec{a}|\cdot\left|\vec{b}\right|\cdot\sin\varphi\,.$$

2. It is perpendicular to the plane of the parallelogram; i.e., $\vec{c}\perp\vec{a}$ and $\vec{c}\perp\vec{b}$.

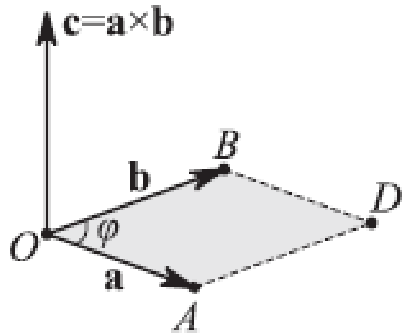


Fig. 1.7. Cross product of two vectors

3. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a *right-handed trihedral*; i.e., the vector $\vec{c}$ points to the side from which the sense of the shortest rotation from $\vec{a}$ to $\vec{b}$ is anticlockwise.

Remark 1. The cross product of a vector $\vec{a}$ by a vector $\vec{b}$ is also denoted by $\vec{c} = [\vec{a}, \vec{b}]$.

Remark 2. If vectors $\vec{a}$ and $\vec{b}$ are collinear, then the parallelogram $OABD$ is degenerate and should be assigned the zero area. Hence the cross product of collinear vectors is defined to be the zero vector whose direction is arbitrary.

Properties of cross product:

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (anticommutativity).

2. $\vec{a} \times \left(\vec{b} \pm \vec{c}\right) = \vec{a} \times \vec{b} \pm \vec{a} \times \vec{c}$ (distributivity with respect to the addition of vectors). This property holds for any number of summands.

3. Vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if $\vec{a} \times \vec{b} = 0$. In particular, $\vec{a} \times \vec{a} = 0$ and $\vec{a} \cdot \left(\vec{a} \times \vec{b}\right) = \vec{b} \cdot \left(\vec{a} \times \vec{b}\right) = 0$.

4. $\left(\lambda \vec{a}\right) \times \vec{b} = \vec{a} \times \left(\lambda \vec{b}\right) = \lambda \left(\vec{a} \times \vec{b}\right)$ (associativity with respect to a scalar factor).

5. The cross product of basis vectors is

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0 \text{ and } \vec{i} \times \vec{j} = \vec{k},\ \vec{j} \times \vec{k} = \vec{i},\ \vec{k} \times \vec{i} = \vec{j}.$$

6. If the vectors are given by their coordinates $\vec{a} = \left(a_x, a_y, a_z\right)$ and $\vec{b} = \left(b_x, b_y, b_z\right)$, then

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=\vec{i}\cdot\left(a_y\cdot b_z-a_z\cdot b_y\right)-\vec{j}\cdot\left(a_x\cdot b_z-a_z\cdot b_x\right)+$$

$$+\vec{k}\cdot\left(a_x\cdot b_y-a_y\cdot b_x\right).$$

7. The area of the parallelogram spanned by vectors $\vec{a}$ and $\vec{b}$ equals

$$S=\left|\vec{a}\times\vec{b}\right|=\sqrt{\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}^2+\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}^2+\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}^2}\ .$$

8. The area of the triangle spanned by vectors $\vec{a}$ and $\vec{b}$ is equal to

$$S=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|=\frac{1}{2}\sqrt{\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}^2+\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}^2+\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}^2}\ .$$

Conditions for vectors to be parallel or perpendicular. A vector $\vec{a}$ is collinear to a vector $\vec{b}$ if

$$\vec{b}=\lambda\vec{a} \quad \text{or} \quad \vec{a}\times\vec{b}=0.$$

The vector equality $\vec{b}=\alpha\vec{a}$ is equivalent to three numerical ones:

$$b_x=\alpha a_x, \qquad b_y=\alpha a_y, \qquad b_z=\alpha a_z,$$

from which it follows that

$$\frac{b_x}{a_x}=\alpha, \qquad \frac{b_y}{a_y}=\alpha, \qquad \frac{b_z}{a_z}=\alpha \quad \text{or} \quad \frac{b_x}{a_x}=\frac{b_y}{a_y}=\frac{b_z}{a_z}=\alpha.$$

Thus vectors are *collinear* if their *coordinates are proportional.*

A vector $\vec{a}$ is perpendicular to a vector $\vec{b}$ if

$$\vec{a}\cdot\vec{b}=0.$$

Remark. In general, the condition $\vec{a}\cdot\vec{b}=0$ implies that the vectors $\vec{a}$ and $\vec{b}$ are perpendicular or one of them is the zero vector. The zero vector can be viewed to be perpendicular to any other vector.

Example 1.3. Let three apexes of a parallelogram $A(1,-1,2)$, $B(5,-6,2)$ and $C(1,3,-1)$ be given. Calculate the area of the parallelogram.

Solution. The area is $S=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$.

Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$\overrightarrow{AB}=(5-1,-6-(-1),2-2)=(4,-5,0)$,

$\overrightarrow{AC}=(1-1,3-(-1),-1-2)=(0,4,-3)$.

The vector product of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is equal to:

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -5 & 0\\ 0 & 4 & -3\end{vmatrix}=\vec{i}\cdot\begin{vmatrix}-5 & 0\\ 4 & -3\end{vmatrix}-\vec{j}\cdot\begin{vmatrix}4 & 0\\ 0 & -3\end{vmatrix}+\vec{k}\cdot\begin{vmatrix}4 & -5\\ 0 & 4\end{vmatrix}=$$

$$=15\vec{i}+12\vec{j}+16\vec{k}.$$

Calculate the area of the parallelogram:

$$S=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{15^2+12^2+16^2}=\sqrt{625}=25 \text{ (sq.unit).}$$

Example 1.4. Determine values of α and β , when the vectors $\vec{a}=-2\vec{i}+3\vec{j}+\beta\vec{k}$ and $\vec{b}=\alpha\vec{i}-6\vec{j}+2\vec{k}$ are collinear.

Solution. From the condition of collinear vectors we get

$$\frac{-2}{\alpha}=\frac{3}{-6}=\frac{\beta}{2}.$$

From this it follows that $\dfrac{-2}{\alpha}=\dfrac{3}{-6}$ and $\dfrac{3}{-6}=\dfrac{\beta}{2}$. Thus $\alpha=\dfrac{-6}{3}\cdot(-2)=4$ and $\beta=\dfrac{3}{-6}\cdot 2=-1$.

1.6. Mixed product of three vectors

The *mixed product* of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as the scalar product of $\vec{a}$ by the cross product of $\vec{b}$ and $\vec{c}$:

$$\left(\vec{a},\vec{b},\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right).$$

Remark. The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is also denoted by $\vec{a}\vec{b}\vec{c}$.

Properties of scalar triple product:

1. $\left(\vec{a},\vec{b},\vec{c}\right)=\left(\vec{b},\vec{c},\vec{a}\right)=\left(\vec{c},\vec{a},\vec{b}\right)=-\left(\vec{b},\vec{a},\vec{c}\right)=-\left(\vec{c},\vec{b},\vec{a}\right)=-\left(\vec{a},\vec{c},\vec{b}\right)$.

2. $\left(\vec{a}+\vec{b}\right)\vec{c}\vec{d}=\vec{a}\vec{c}\vec{d}+\vec{b}\vec{c}\vec{d}$ (distributivity with respect to addition of vectors). This property holds for any number of summands.

3. $\left(\lambda\vec{a},\vec{b},\vec{c}\right)=\lambda\left(\vec{a},\vec{b},\vec{c}\right)$ (associativity with respect to a scalar factor).

4. If the vectors are given by their coordinates $\vec{a}=\left(a_x,a_y,a_z\right)$, $\vec{b}=\left(b_x,b_y,b_z\right)$ and $\vec{c}=\left(c_x,c_y,c_z\right)$, then

$$\left(\vec{a},\vec{b},\vec{c}\right)=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

5. The scalar triple product $\left(\vec{a},\vec{b},\vec{c}\right)$ is equal to the volume *V* of the parallelepiped spanned by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ taken with the sign + if the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a right-handed trihedral and the sign – if the vectors form a left-handed trihedral, or the module of the mixed product equals the volume of the parallelepiped constructed on the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$,

$$V=\pm\left(\vec{a},\vec{b},\vec{c}\right)=\left|\left(\vec{a},\vec{b},\vec{c}\right)\right|.$$

6. The volume of a tetrahedron constructed on the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is equal to

$$V=\frac{1}{6}\left|\left(\vec{a},\vec{b},\vec{c}\right)\right|.$$

7. Three nonzero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar if and only if $\left(\vec{a},\vec{b},\vec{c}\right)=0$ or $\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=0$. In this case, the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent; they satisfy a relation of the form $\vec{c}=\alpha\vec{a}+\beta\vec{b}$.

Example 1.5. Let the following apexes of a pyramid $A(2,3,1)$, $B(4,1,-2)$, $C(6,3,7)$, $D(-5,-4,2)$ be given. Calculate pyramid volume and the length of the altitude put down from the apex D.

Solution. The volume of the pyramid is equal to:

$$V=\frac{1}{6}\left|\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)\right| \qquad \text{or} \qquad V=\frac{1}{3}S_{\Delta ABC}\cdot DH,$$

thus $DH=\dfrac{3V}{S_{\Delta ABC}}$.

Let's find $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$: $\overrightarrow{AB}=(2,-2,-3)$, $\overrightarrow{AC}=(4,0,6)$, $\overrightarrow{AD}=(-7,-7,1)$.

Then $\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)=\begin{vmatrix} 2 & -2 & -3 \\ 4 & 0 & 6 \\ -7 & -7 & 1 \end{vmatrix}=260$.

Thus $V=\frac{1}{6}\left|\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)\right|=\frac{1}{6}\cdot 260=\frac{130}{3}$ (cubed unit).

Let's find the area of triangle ABC:

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -3 \\ 4 & 0 & 6 \end{vmatrix}=-12\vec{i}-24\vec{j}+8\vec{k},$$

$$S_{\Delta ABC} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{(-12)^2 + (-24)^2 + 8^2} =$$

$$= \frac{1}{2}\sqrt{784} = \frac{28}{2} = 14 \text{ (sq. units).}$$

Thus $DH = \dfrac{3 \cdot \dfrac{130}{3}}{14} = \dfrac{65}{7}$ (units of length).

2. Equations of Straight Lines on Plane

2.1. Slope-intercept equation of a straight line

The tangent of the angle of inclination of a straight line to the axis OX is called the *slope* of the straight line. The slope characterizes the direction of the line. For straight lines perpendicular to the OX -axis, slope does not make sense, although one often says that the slope of such straight lines is equal to infinity.

The *slope-intercept equation of a straight line* in the rectangular Cartesian coordinate system OXY has the form

$$y = kx + b, \tag{2.1}$$

where $k = tg\varphi = (y-b)/x$ is the slope of the line and *b* is the *y*-intercept of the line, i.e., the signed distance from the point of intersection of the line with the ordinate axis to the origin. Equation (2.1) is meaningful for any straight line that is not perpendicular to the abscissa axis (Fig. 2.1, a).

If a straight line is not perpendicular to the OX -axis, then its equation can be written as (2.1), but if a straight line is perpendicular to the OX -axis, then its equation can be written as

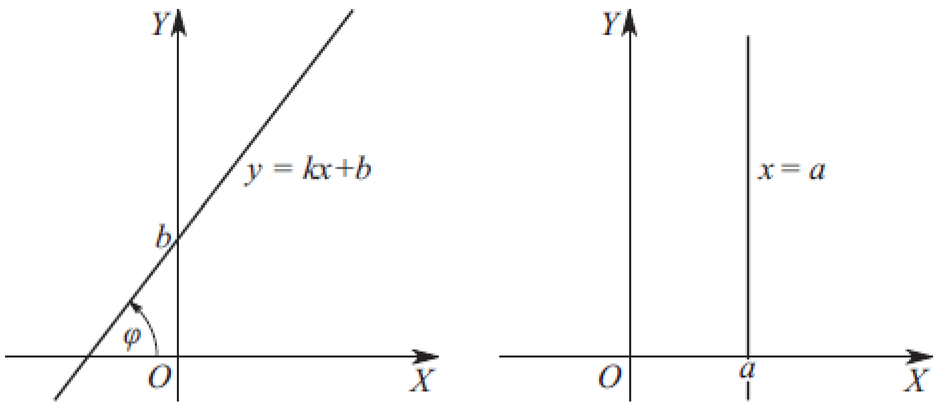
$$x = a, \tag{2.2}$$

where a is the abscissa of the point of intersection of this line with the OX -axis (Fig. 2.1, b).

For the slope of a straight line, we also have the formula

$$k = \frac{x_2 - x_1}{y_2 - y_1}, \tag{2.3}$$

where $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ are two arbitrary points of the line.



a) b)

Fig. 2.1. Straight lines on plane

In the rectangular Cartesian coordinate system OXY, the equation of a straight line with slope k passing through a point $M_0(x_0, y_0)$ has the form

$$y - y_0 = k(x - x_0). \tag{2.4}$$

Remark. If we set $x_0 = 0$ and $y_0 = b$ in equation (2.4), then we obtain equation (2.1).

2.2. Equation of a straight line passing through two given points

In the rectangular Cartesian coordinate system OXY, the equation of a straight line with slope k passing through points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ has the form (2.4), where k is given by the expression (2.3):

$$y - y_1 = \frac{x - x_1}{x_2 - x_1}(y_2 - y_1). \tag{2.5}$$

This equation is usually written as

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}. \qquad (2.6)$$

Equation (2.6) is also called the *canonical equation of the straight line passing through two given points on the plane*.

Sometimes one writes this equation in terms of a third-order determinant as follows:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0. \qquad (2.7)$$

Example 2.1. Let us derive the equation of the straight line passing through the points $M_1(5,1)$ and $M_2(7,3)$.

Solution. Substituting the coordinates of these points into formula (2.5), we obtain

$$\frac{y-1}{3-1}=\frac{x-5}{7-5}, \qquad \text{or} \qquad \frac{y-1}{2}=\frac{x-5}{2},$$

$$2(y-1)=2(x-5), \qquad \text{or} \qquad y=x-4.$$

2.3. General equation of a straight line on a plane

An equation of the form

$$Ax+By+C=0. \qquad (2.8)$$

is called the *general equation of a straight line* in the rectangular Cartesian coordinate system OXY. In rectangular Cartesian coordinates, each straight line is determined by an equation of degree 1, and, conversely, each equation of degree 1 determines a straight line.

If $B\neq 0$, then equation (2.8) can be written as (2.1), where $k=-A/B$ and $b=-C/B$.

If $B=0$, then equation (2.8) can be written as (2.2), where $a=-C/A$.

If $C = 0$, then the equation of a straight line becomes $Ax + By = 0$ and determines a straight line passing through the origin.

If $B = 0$ and $A \neq 0$, then the equation of a straight line becomes $Ax + C = 0$ and determines a straight line parallel to the axis OY.

If $A = 0$ and $B \neq 0$, then the equation of a straight line becomes $By + C = 0$ and determines a straight line parallel to the axis OX.

2.4. General equation of a straight line passing through given points on a plane

In the rectangular Cartesian coordinate system OXY, the general equation of a straight line passing through the point $M_0(x_0, y_0)$ on the plane has the form

$$A(x - x_0) + B(y - y_0) = 0. \tag{2.9}$$

If this equation is written in the form

$$\frac{x - x_0}{B} = \frac{y - y_0}{-A}, \tag{2.10}$$

then it is called the *canonical equation of a straight line passing through a given point on the plane*.

If $B = 0$, then one sets $x - x_0 = 0$, and if $A = 0$, then one sets $y - y_0 = 0$.

Remark. The general equation of the straight line passing through two given points on the plane has the form (2.6).

2.5. Parametric equations of a straight line on a plane

The *parametric equations of a straight line on the plane* through the point $M_0(x_0, y_0)$ in the rectangular Cartesian coordinate system OXY have the form

$$x = x_0 + Bt, \qquad y = y_0 - At. \tag{2.11}$$

where A and B are the coefficients of the general equation (2.8) or (2.9) of a straight line and t is a variable parameter.

In the rectangular Cartesian coordinate system OXY, the parametric equations of the straight line passing through two points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ on the plane can be written as

$$x = x_1(1-t) + x_2 t, \qquad y = y_1(1-t) - y_2 t. \tag{2.12}$$

Remark. Eliminating the parameter t from equations (2.11) and (2.12), we obtain equations (2.9) and (2.6), respectively.

2.6. Intercept-intercept equation of a straight line

The *intercept-intercept equation of a straight line* in the rectangular Cartesian coordinate system OXY has the form

$$\frac{x}{a} + \frac{y}{b} = 1, \tag{2.13}$$

where a and b are the x- and y-intercepts of the line, i.e., the signed distances from the points of intersection of the line with the coordinate axes to the origin (Fig. 2.2).

2.7. Normalized equation of a straight line

Let's suppose that a rectangular Cartesian coordinate system OXY and a straight line are given on the plane. We draw the perpendicular to the straight line through the origin.

This perpendicular is called the *normal* to the line. By P we denote the point of intersection of the normal with the line.

The equation

$$x\cos\alpha + y\sin\alpha - p = 0, \tag{2.14}$$

where α is the polar angle of the normal and p is the length of the segment OP (the distance from the origin to the straight line) (Fig. 2.3), is called the *normalized equation* of the straight line in the rectangular Cartesian coordinate system OXY.

In the normalized equation of a straight line, $p \geq 0$ and $\cos^2 \alpha + \sin^2 \alpha = 1$.

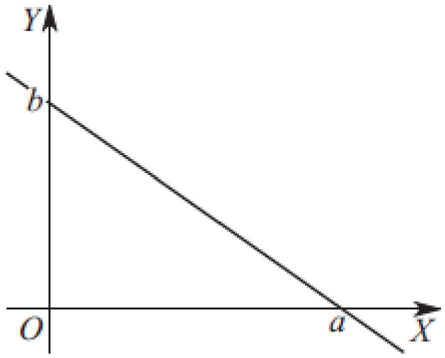


Fig. 2.2. A straight line with intercept-intercept equation

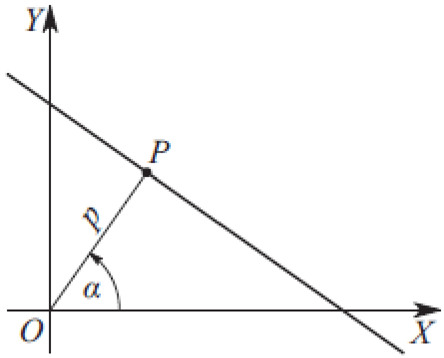


Fig. 2.3. A straight line with normalized equation

For all positions of the straight line with respect to the coordinate axes, its equation can always be written in normalized form.

The general equation of a straight line (2.8) can be reduced to a normalized form (2.14) by setting

$$\cos\alpha = \pm\frac{A}{\sqrt{A^2+B^2}}, \qquad \sin\alpha = \pm\frac{B}{\sqrt{A^2+B^2}},$$
$$p = \mp\frac{C}{\sqrt{A^2+B^2}}, \tag{2.15}$$

where the upper sign is taken for $C < 0$ and the lower sign for $C > 0$. For $C = 0$, either sign can be taken.

2.8. Condition for three points to be collinear

Let's suppose that points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ and $M_3(x_3, y_3)$ are given in the Cartesian coordinate system OXY on the planes. They are collinear (lie on the same straight line) if and only if

$$\frac{y_3 - y_1}{y_2 - y_1} = \frac{x_3 - x_1}{x_2 - x_1}. \tag{2.16}$$

2.9. Distance from a point to a straight line

The *distance d from a point to a straight line* is the absolute value of the deviation. It can be calculated by the formula

$$d = \left| x_0 \cos\alpha + y_0 \sin\alpha - p \right|. \qquad (2.17)$$

The distance from a point $M_0(x_0, y_0)$ to a straight line given by the general equation $Ax + By + C = 0$ can be calculated by the formula

$$d = \frac{\left| Ax_0 + By_0 + C \right|}{\sqrt{A^2 + B^2}}. \qquad (2.18)$$

Example 2.2. Let's find the distance from the point $A(2,1)$ to the straight line $3x + 4y + 5 = 0$.

Solution. We use formula (2.18) to obtain

$$d = \frac{\left| Ax_0 + By_0 + C \right|}{\sqrt{A^2 + B^2}} = \frac{\left| 3 \cdot 2 + 4 \cdot 1 + 5 \right|}{\sqrt{3^2 + 4^2}} = \frac{\left| 6 + 4 + 5 \right|}{\sqrt{9 + 16}} = \frac{15}{5} = 3.$$

2.10. Mutual Arrangement of Lines

2.10.1. Angle between two straight lines. We consider two straight lines given by the equations

$$y = k_1 x + b_1 \text{ and } y = k_2 x + b_2, \qquad (2.19)$$

where $k_1 = tg\varphi_1$ and $k_2 = tg\varphi_2$ are the slopes of the respective lines (Fig. 2.4). The angle α between these lines can be obtained by the formula

$$tg\alpha = \left| \frac{k_2 - k_1}{1 + k_1 k_2} \right|, \qquad (2.20)$$

where $k_1 k_2 \neq -1$. If $k_1 k_2 = -1$, then $\alpha = \frac{\pi}{2}$.

Remark. If at least one of the lines is perpendicular to the axis OX, then formula (2.20) does not make sense. In this case, the angle between the lines can be calculated by the formula

$$\alpha = \varphi_2 - \varphi_1. \tag{2.21}$$

The angle α between the two straight lines given by the general equations

$$A_1x + B_1y + C_1 = 0 \text{ and } A_2x + B_2y + C_2 = 0 \tag{2.22}$$

can be calculated using the expression

$$tg\alpha = \frac{A_1B_2 - A_2B_1}{A_1B_2 + A_2B_1}, \tag{2.23}$$

where $A_1B_2 + A_2B_1 \neq 0$. If $A_1B_2 + A_2B_1 = 0$, then $\alpha = \frac{\pi}{2}$.

Remark. If one needs to find the angle between straight lines and the order in which they are considered is not defined, then this order can be chosen arbitrarily. Obviously, a change in the order results in a change in the sign of the tangent of the angle.

2.10.2. Point of intersection of straight lines. Let's suppose that two straight lines are defined by general equations in the form (2.22).

Let's consider the system of two first-order algebraic equations (2.22). Each common solution of equations (2.22) determines a common point of the tow lines. If the determinant of system (2.22) is not zero, i.e.,

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = A_1B_2 - B_1A_2 \neq 0, \tag{2.24}$$

then the system is consistent and has a unique solution; hence these straight lines are distinct and nonparallel and meet at the point $M_0(x_0, y_0)$, where

$$x_0 = \frac{B_1C_2 - B_2C_1}{B_1C_2 + B_2C_1} \qquad y_0 = \frac{C_1A_2 - C_2A_1}{C_1A_2 + C_2A_1}. \tag{2.25}$$

Condition (2.24) is often written as

$$\frac{A_1}{A_2} \neq \frac{B_1}{B_2}. \tag{2.26}$$

Example 2.3. Let's find the point of intersection of the straight lines $y = 2x - 1$ and $y = -4x + 5$.

Solution. We solve the system (2.22):

$$\begin{cases} y = 2x - 1 \\ y = -4x + 5 \end{cases},$$

and obtain $x = 1$, $y = 1$.

Thus the intersection point has the coordinates $(1,1)$.

2.10.3. Condition for straight lines to be perpendicular. For two straight lines determined by slope-intercept equations (2.19) to be perpendicular, it is necessary and sufficient that

$$k_1 k_2 = -1. \tag{2.27}$$

Relation (2.27) is usually written as

$$k_1 = -\frac{1}{k_2}, \tag{2.28}$$

and one also says that the slopes of perpendicular straight lines are inversely proportional in absolute value and opposite in sign.

If the straight lines are given by general equations (2.22), then a necessary and sufficient condition for them to be perpendicular can be written as (see Paragraph 2.10.1)

$$A_1 A_2 + B_1 B_2 = 0. \tag{2.29}$$

Example 2.4. Let's check the condition if the lines $3x + y - 3 = 0$ and $x - 3y + 8 = 0$ are perpendicular.

Solution. Let's check the condition of perpendicularity:

$$A_1 A_2 + B_1 B_2 = 3 \cdot 1 + 1 \cdot (-3) = 0.$$

These lines satisfy condition (2.29) and are perpendicular.

2.10.4. Condition for straight lines to be parallel. For two straight lines defined by slope-intercept equations (2.19) to be parallel and not to coincide, it is necessary and sufficient that

$$k_1 = k_2, \qquad b_1 \neq b_2. \tag{2.30}$$

If the straight lines are given by general equations (2.22), then a necessary and sufficient condition for them to be parallel can be written as

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}. \tag{2.31}$$

in this case, the straight lines do not coincide (Fig. 2.5).

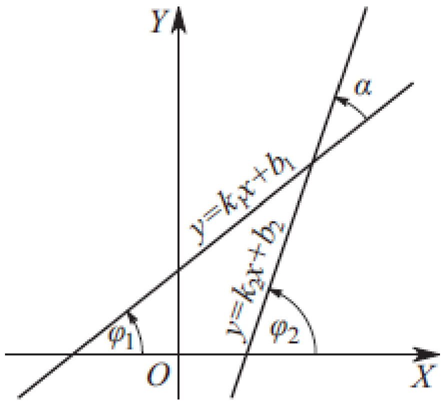


Fig. 2.4. Angle between two straight lines

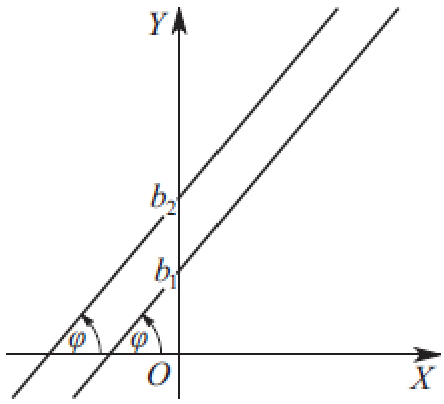


Fig. 2.5. Parallel straight lines

Example 2.5. Let's check the condition if the lines $3x + 4y + 5 = 0$ and $\frac{3}{2}x + 2y + 6 = 0$ are parallel.

Solution. Let's check the condition of parallelity:

$$\frac{3}{3/2} = \frac{4}{2} \neq \frac{5}{6}.$$

The following condition (2.31) is satisfied, therefore the lines are parallel.

2.10.5. Condition for straight lines to coincide. For two straight lines given by slope-intercept equations (2.19) to coincide, it is necessary and sufficient that

$$k_1 = k_2, \qquad b_1 \neq b_2. \tag{2.32}$$

If the straight lines are given by general equations (2.22), then a necessary and sufficient condition for them to coincide has the form

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}. \tag{2.33}$$

Remark. Sometimes the case of coinciding straight lines is considered as a special case of parallel straight lines and it not distinguished as an exception.

2.10.6. Distance between parallel lines. The distance between the parallel lines given by equations (see Paragraph 2.10.4)

$$A_1x + B_1y + C_1 = 0 \text{ and } A_1x + B_1y + C_2 = 0 \tag{2.34}$$

can be found using the formula (see Paragraph 2.9)

$$d = \frac{|C_1 - C_2|}{\sqrt{A_1^2 + B_1^2}}. \tag{2.35}$$

Example 2.6. The coordinates apexes of a triangle ABC $A(-2,-2)$, $B(4,1)$, $C(0,4)$ are given (Fig. 2.6). Using methods of the analytical geometry do the following:

1) find the distance between point A and point B;
2) form equation of the sides AB, AC;
3) form equation of the altitude dropped from the apex C;
4) find the inner angle of the triangle at the apex A;
5) calculate length of the altitude dropped from the apex C;
6) find area of the ΔABC;
7) form equation of the median dropped from the apex C.

Solution. 1. Let's find the distance between point A and point B:

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} = \sqrt{(4-(-2))^2 + (1-(-2))^2} =$$

$$= \sqrt{(4+2)^2 + (1+2)^2} = \sqrt{6^2 + 3^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}.$$

2. Let's form equation of the sides $AB,\ AC$:

$$AB: \quad \frac{y-y_A}{y_B-y_A}=\frac{x-x_A}{x_B-x_A},$$

$$\frac{y-(-2)}{1-(-2)}=\frac{x-(-2)}{4-(-2)},$$

$$\frac{y+2}{3}=\frac{x+2}{6},$$

$$x-2y-2=0,$$

$$y=\frac{1}{2}x-1,\ k_{AB}=\frac{1}{2};$$

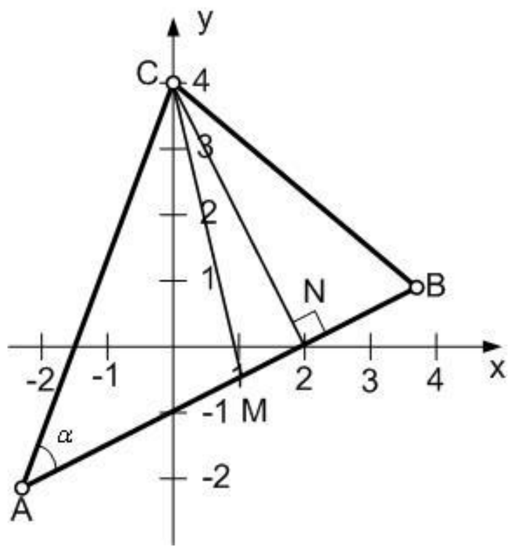


Fig. 2.6. Triangle ΔABC

$$AC: \quad \frac{y-y_A}{y_C-y_A}=\frac{x-x_A}{x_C-x_A},$$

$$\frac{y-(-2)}{4-(-2)}=\frac{x-(-2)}{0-(-2)},$$

$$\frac{y+2}{6}=\frac{x+2}{2},$$

$$3x-y+4=0,$$

$$y=3x+4,\ k_{AC}=3.$$

3. Let's form equation of the altitude dropped from the apex C:

$$y-y_C=k_{CN}(x-x_C),$$

$$k_{CN}\cdot k_{AB}=-1,\ k_{CN}=-\frac{1}{k_{AB}},$$

$$y-y_C=-\frac{1}{k_{AB}}(x-x_C),$$

$$y-4=-\frac{1}{1/2}(x-0),$$

$$y-4=-2x,$$

$$y=-2x+4.$$

4. Let's find the inner angle of the triangle at the apex A;

$$tg\alpha=\left|\frac{k_{AC}-k_{AB}}{1+k_{AC}k_{AB}}\right|=\left|\frac{3-1/2}{1+3\cdot 1/2}\right|=\left|\frac{5/2}{5/2}\right|=1,\ \alpha=arctg1=\frac{\pi}{4}.$$

5. Let's calculate length of the altitude dropped from the apex C:

$$CN=\frac{|1\cdot x_C-2\cdot y_C-2|}{\sqrt{1^2+(-2)^2}}=\frac{|1\cdot 0-2\cdot 4-2|}{\sqrt{1+4}}=\frac{|0-8-2|}{\sqrt{5}}=\frac{10}{\sqrt{5}}\ \text{(units of length).}$$

6. Let's find area of the ΔABC:

$$S_{\Delta ABC}=\frac{1}{2}AB\cdot CN=\frac{1}{2}\cdot 3\sqrt{5}\cdot\frac{10}{\sqrt{5}}=15\ \text{(sq. units).}$$

7. Let’s find the coordinates of the middle M of the segment AB:

$$x_M = \frac{x_A + x_B}{2} = \frac{-2+4}{2} = \frac{2}{2} = 1, \; y_M = \frac{y_A + y_B}{2} = \frac{-2+1}{2} = -\frac{1}{2}.$$

Let’s form equation of the median CM dropped from the apex C:

$$CM: \frac{y - y_M}{y_C - y_M} = \frac{x - x_M}{x_C - x_M},$$

$$\frac{y - \left(-\frac{1}{2}\right)}{4 - \left(-\frac{1}{2}\right)} = \frac{x-1}{0-1},$$

$$\frac{y + \frac{1}{2}}{\frac{9}{2}} = \frac{x-1}{-1},$$

$$\frac{9}{2}x + y - 4 = 0,$$

$$y = -\frac{9}{2}x + 4.$$

3. Plane and Line in Space

3.1. Plane in Space

3.1.1. Equation of a plane passing through point M_0 and perpendicular to vector $\overrightarrow{N}$. A *plane* is a first-order algebraic surface. In a Cartesian coordinate system, a plane is given by a first-order equation.

The equation of the plane passing through a point $M_0(x_0, y_0, z_0)$ and perpendicularly to a vector $\overrightarrow{N} = (A, B, C)$ has the form

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \text{ or } (\vec{r}-\vec{r}_0)\cdot\vec{N}=0 \tag{3.1}$$

where $\vec{r}$ and $\vec{r}_0$ are the position vectors of the point $M(x, y, z)$ and $M_0(x_0, y_0, z_0)$, respectively (Fig. 3.1). The vector $\overrightarrow{N}$ is called a *normal vector*. Its direction cosines are

$$\cos\alpha=\frac{A}{\sqrt{A^2+B^2+C^2}}, \qquad \cos\beta=\frac{B}{\sqrt{A^2+B^2+C^2}},$$
$$\cos\gamma=\frac{C}{\sqrt{A^2+B^2+C^2}}. \tag{3.2}$$

Example 3.1. Let's write out the equation of the plane that passes through the point $M_0(1,2,1)$ and is perpendicular to the vector $\overrightarrow{N}=(3,2,3)$.

Solution. According to (3.1), the desired equation is

$$3\cdot(x-1)+2\cdot(y-2)+3\cdot(z-1)=0 \qquad \text{or} \qquad 3x+2y+3z-10=0.$$

3.1.2. General equation of a plane. The *general (complete) equation of a plane* has the form

$$Ax+By+Cz+D=0 \text{ or } \vec{r}\cdot\vec{N}+D=0. \tag{3.3}$$

It follows from (3.1) that $D=-Ax_0-By_0-Cz_0$. If one of the coefficients in the equation of a plane is zero, then the equation is said to be *incomplete*:

1. For $D=0$, the equation has the form $Ax+By+Cz=0$ and defines a plane passing through the origin.

2. For $A=0$ (respectively, $B=0$ or $C=0$), the equation has the form $By+Cz+D=0$ and defines a plane parallel to the axis OX (respectively, OY or OZ).

3. For $A=D=0$ (respectively, $B=D=0$ or $C=D=0$), the equation has the form $By+Cz=0$ and defines a plane passing through the axis OX (respectively, OY or OZ).

4. For $A=B=0$ (respectively, $A=C=0$ or $B=C=0$), the equation has the form $Cz+D=0$ and defines a plane parallel to the plane OXY (respectively, OXZ or OYZ).

3.1.3. Parametric equation of a plane. Each vector $\overrightarrow{M_0M} = \vec{r} - \overrightarrow{r_0}$ lying in a plane (where $\vec{r}$ and $\overrightarrow{r_0}$ are the position vectors of the points M and M_0, respectively) can be represented as (Fig. 3.2)

$$\overrightarrow{M_0M} = t\overrightarrow{R_1} + s\overrightarrow{R_2}, \tag{3.4}$$

where $\overrightarrow{R_1} = (l_1, m_1, n_1)$ and $\overrightarrow{R_2} = (l_2, m_2, n_2)$ are two arbitrary noncollinear vectors lying in the plane. Obviously, these two vectors form a basis in this plane.

The *parametric equation of a plane* passing through the point $M_0 = (x_0, y_0, z_0)$ has the form

$$\vec{r} = \overrightarrow{r_0} + t\overrightarrow{R_1} + s\overrightarrow{R_2} \quad \text{or} \quad \begin{aligned} x &= x_0 + tl_1 + sl_2 \\ y &= y_0 + tm_1 + sm_2 \\ z &= z_0 + tn_1 + sn_2 \end{aligned}. \tag{3.5}$$

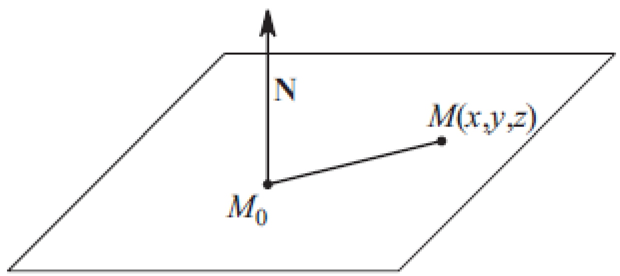


Fig. 3.1. Plane passing through point M_0 and perpendicularly to vector $\overrightarrow{N}$

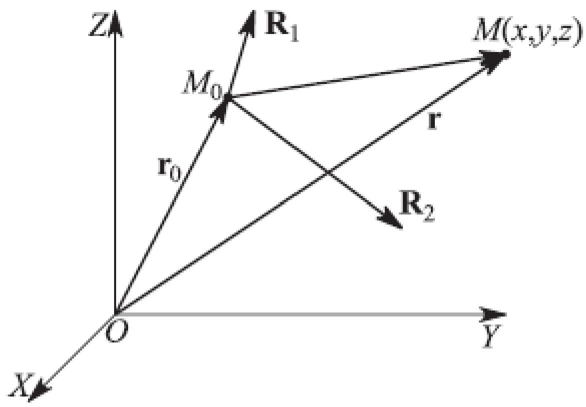


Fig. 3.2. Basis in plane

3.1.4. Intercept equation of a plane. A plane $Ax + By + Cz + D = 0$ that is not parallel to the axis OX (i.e., $A \neq 0$) meets this axis at a (signed) distance $a = -D/A$ from the origin (Fig. 3.3). The number a is called the x-intercept of the plane. Similarly, one defines the y-intercepts $b = -D/B$ (for $B \neq 0$) and the z-intercept $c = -D/C$ (for $C \neq 0$). Then such a plane can be defined by the equation

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1, \qquad (3.6)$$

which is called the *intercept equation of the plane*.

Remark 1. Equation (3.6) can be obtained as the equation of the plane passing through three given points.

Remark 2. A plane parallel to the axis OX but nonparallel to the other two axes is defined by the equation $\frac{y}{b}+\frac{z}{c}=1$, where b and c are the y- and z-intercepts of the plane. A plane simultaneously parallel to the axes OY and OZ can be represented in the form $\frac{z}{c}=1$.

Example 3.2. Let's consider the plane given by the general equation $2x+3y-z+6=0$ and rewrite it in intercept form.

Solution. The x-, y-, and z-intercepts of this plane are

$$a=-\frac{D}{A}=-\frac{6}{2}=-3,\ b=-\frac{D}{B}=-\frac{6}{3}=-2 \text{ and } c=-\frac{D}{C}=-\frac{6}{-1}=6.$$

Thus the intercept equation of the plane reads

$$\frac{x}{-3}+\frac{y}{-2}+\frac{z}{6}=1.$$

3.1.5. Normalized equation of a plane. The *normalized equation of a plane* has the form

$$\vec{r}\cdot\overrightarrow{N^0}-p=0, \text{ or } x\cos\alpha+y\cos\beta+z\cos\gamma-p=0 \qquad (3.7)$$

where $\overrightarrow{N^0}=(\cos\alpha,\cos\beta,\cos\gamma)$ is a unit vector and p is the distance from the plane to the origin; here $\cos\alpha,\cos\beta$ and $\cos\gamma$ are the direction cosines of the normal to the plane (Fig. 3.4).

The numbers $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and p can be expressed via the coefficients A,B,C as follows:

$$\cos\alpha = \pm\frac{A}{\sqrt{A^2+B^2+C^2}},\ \cos\beta = \pm\frac{B}{\sqrt{A^2+B^2+C^2}},$$
$$\cos\gamma = \pm\frac{C}{\sqrt{A^2+B^2+C^2}},\ p = \mp\frac{D}{\sqrt{A^2+B^2+C^2}}. \tag{3.8}$$

where the upper sign is taken if $D < 0$ and the lower sign is taken if $D > 0$. or $D = 0$, either sign can be taken.

The normalized equation (3.7) can be obtained from a general equation (3.3) by multiplication by the normalizing factor

$$\mu = \pm\frac{C}{\sqrt{A^2+B^2+C^2}}, \tag{3.9}$$

where the sign of μ must be opposite to that of D.

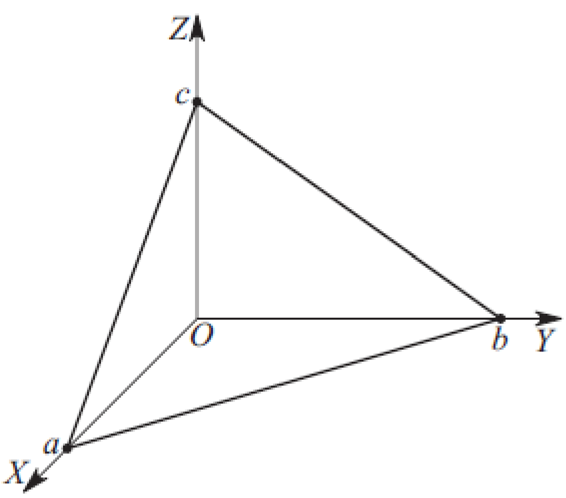


Fig. 3.3. A plane with intercept equation

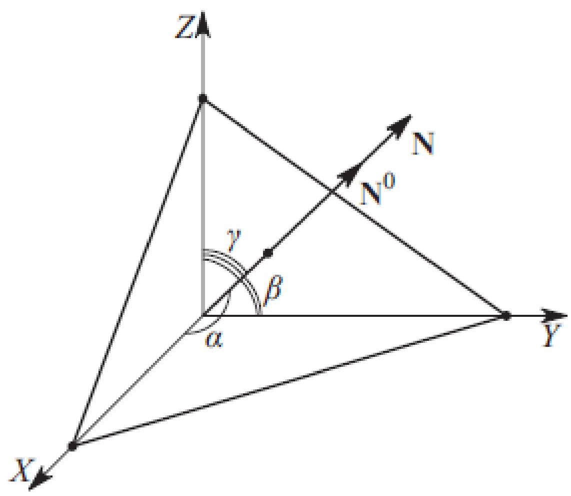


Fig. 3.4. A plane with normalized equation

Example 3.3. Let's reduce the equation of the plane $-2x+2y-z-6=0$ to normalized form.

Solution. Since $D = -6 < 0$, we see that the normalizing factor is

$$\mu = +\frac{1}{\sqrt{(-2)^2+2^2+(-1)^2}} = \frac{1}{\sqrt{9}} = \frac{1}{3}.$$

We multiply the equation by this factor and obtain

$$-\frac{2}{3}x+\frac{2}{3}y-\frac{1}{3}z-\frac{6}{3}=0.$$

Hence for this plane we have

$$\cos\alpha=-\frac{2}{3},\ \cos\beta=\frac{2}{3},\ \cos\gamma=-\frac{1}{3},\ p=2.$$

Remark. The numbers $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and p are also called the *polar parameters of a plane*.

3.1.6. Equation of a plane passing through a point and parallel to another plane. The plane that passes through a point $M_0(x_0, y_0, z_0)$ and is parallel to a plane $Ax+By+Cz+D=0$ is given by the equation

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0, \qquad (3.10)$$

Example 3.4. Let us derive the equation of the plane that passes through the point $M_0(1,2,-1)$ and is parallel to the plane $x+2y+z+2=0$.

Solution. According to (3.10), the desired equation is

$$(x-1)+2(y-2)+(z+1)=0,$$

$$x+2y+z-1-4+1=0,$$

$$x+2y+z-4=0.$$

3.1.7. The equation of a plane passing through three points. The plane passing through three points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ and $M_3(x_3, y_3, z_3)$ (Fig. 3.5) is described by the equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0, \text{ or } (\vec{r}-\vec{r}_1)\cdot(\vec{r}_2-\vec{r}_1)\cdot(\vec{r}_3-\vec{r}_1)=0, \qquad (3.11)$$

where $\vec{r}, \vec{r}_1, \vec{r}_2$ and $\vec{r}_3$ are the position vectors of the points $M(x, y, z)$, $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ and $M_3(x_3, y_3, z_3)$, respectively.

Remark 1. Equation (3.11) means that the vectors $\overrightarrow{M_1M}$, $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_1M_3}$ are coplanar.

Example 3.5. Let us construct an equation of the plane passing through the three points $M_1(1,1,1)$, $M_2(2,2,1)$ and $M_3(1,2,2)$.

Solution. Obviously, the points M_1, M_2 and M_3 are not collinear, since the vectors

$$\overrightarrow{M_1M} = (x - x_1, y - y_1, z - z_1) = (x - 1, y - 1, z - 1),$$

$$\overrightarrow{M_1M_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) = (2 - 1, 2 - 1, 1 - 1) = (1,1,0),$$

$$\overrightarrow{M_1M_3} = (x_3 - x_1, y_3 - y_1, z_3 - z_1) = (1 - 1, 2 - 1, 2 - 1) = (0,1,1)$$

are not collinear.

According to (3.11), the desired equation is

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0,$$

whence

$$x - y + z - 1 = 0.$$

3.1.8. The equation of the plane passing through two points and parallel to a line. The equation of the plane passing through two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ and parallel to a straight line with direction vector $\vec{R} = (l, m, n)$ (Fig. 3.6) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0, \text{ or } (\vec{r} - \vec{r}_1) \cdot (\vec{r}_2 - \vec{r}_1) \cdot \vec{R} = 0, \qquad (3.12)$$

where $\vec{r}, \vec{r}_1$ and $\vec{r}_2$ are the position vectors of the points $M(x, y, z)$, $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$, respectively.

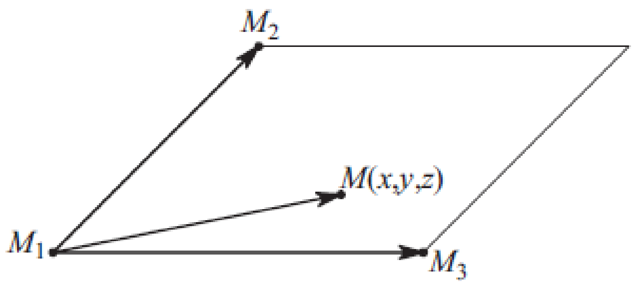


Fig. 3.5. Plane passing through three points

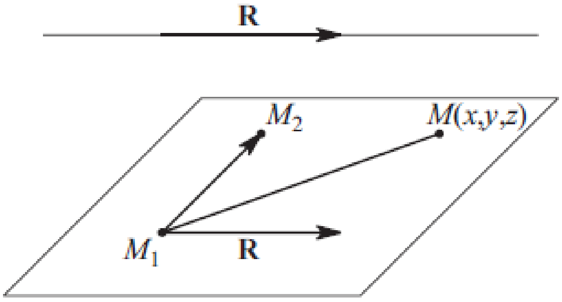


Fig. 3.6. Plane passing through two points and parallel to line

Remark. If the vectors $\overrightarrow{M_1 M_2}$ and $\vec{R}$ are collinear, then equations (3.12) become identities.

Example 3.6. Let's construct an equation of the plane passing through the points $M_1(0,1,0)$ and $M_2(1,1,1)$ and parallel to the straight line with direction vector $\vec{R} = (0,1,1)$.

Solution. According to (3.12), the desired equation is

$$\begin{vmatrix} x-0 & y-1 & z-0 \\ 1-0 & 1-1 & 1-0 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

whence
$$-x - y + z + 1 = 0.$$

3.1.9. Equation of plane passing through point and parallel to two straight lines. The equation of the plane passing through a point $M_1(x_1, y_1, z_1)$ and parallel to two straight lines with direction vectors $\overrightarrow{R_1} = (l_1, m_1, n_1)$ and $\overrightarrow{R_2} = (l_2, m_2, n_2)$ (Fig. 3.7) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0, \text{ or } (\vec{r} - \vec{r}_1) \cdot \overrightarrow{R_1} \cdot \overrightarrow{R_2} = 0, \qquad (3.13)$$

where $\vec{r}$ and $\vec{r}_1$ are the position vectors of the points $M(x, y, z)$ and $M_1(x_1, y_1, z_1)$, respectively.

Example 3.7. Let us find the equation of the plane passing through the point $M_1(0,1,0)$ and parallel to the straight lines with direction vectors $\overrightarrow{R_1} = (1,0,1)$ and $\overrightarrow{R_2} = (0,1,2)$.

Solution. According to (3.13), the desired equation is

$$\begin{vmatrix} x-0 & y-1 & z-0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 0,$$

whence

$$-x - 2y + z + 2 = 0.$$

3.1.10. The plane passing through two points and perpendicular to a given plane. The plane (Fig. 3.8) passing through two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ and perpendicular to the plane given by the equation $Ax + By + Cz + D = 0$ is determined by the equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ A & B & C \end{vmatrix} = 0, \text{ or } (\vec{r}-\vec{r}_1)\cdot(\vec{r}_2-\vec{r}_1)\cdot\overrightarrow{N} = 0, \qquad (3.14)$$

where $\vec{r}$, $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the points $M(x, y, z)$, $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$, respectively.

Remark. If the straight line passing through points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ is perpendicular to the original plane, then the desired plane is undetermined and equations (3.14) become identities.

Example 3.8. Let us find an equation of the plane passing through the points $M_1(0,1,2)$ and $M_2(2,2,3)$ and perpendicular to the plane $x - y + z + 5 = 0$.

Solution. According to (3.14), the desired equation is

$$\begin{vmatrix} x-0 & y-1 & z-2 \\ 2-0 & 2-1 & 3-2 \\ 1 & -1 & 1 \end{vmatrix} = 0$$

whence

$$2x - y - 3z + 7 = 0.$$

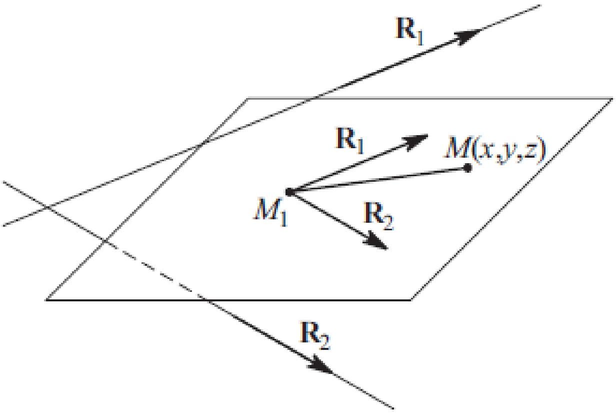


Fig. 3.7. Plane passing through a point and parallel to two straight lines

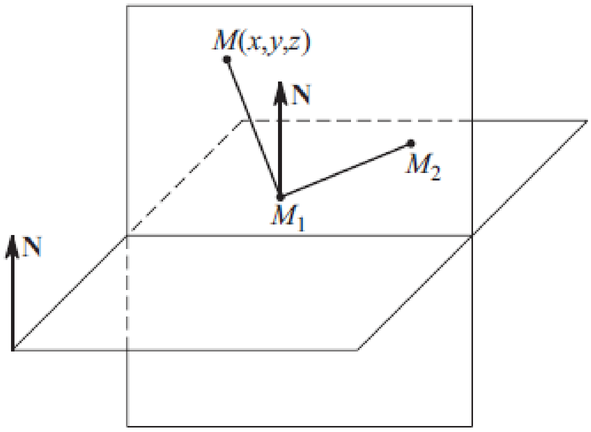


Fig. 3.8. Plane passing through two points and perpendicular to given plane

3.1.11. The plane passing through a point and perpendicular to two planes. The plane (Fig. 3.9) passing through a point $M_1(x_1, y_1, z_1)$ and perpendicular to two (nonparallel) planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is given by the equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = 0, \text{ or } (\vec{r} - \vec{r}_1) \cdot \overrightarrow{N_1} \cdot \overrightarrow{N_2} = 0, \qquad (3.15)$$

where $\overrightarrow{N_1} = (A_1, B_1, C_1)$ and $\overrightarrow{N_2} = (A_2, B_2, C_2)$ are the normals to the given planes and $\vec{r}$ and $\vec{r}_1$ are the position vectors of the points $M(x, y, z)$ and $M_1(x_1, y_1, z_1)$, respectively.

Remark 1. Equations (3.15) mean that the vectors $\overrightarrow{M_1M_2}$, $\overrightarrow{N_1}$ and $\overrightarrow{N_2}$ are coplanar.

Remark 2. If the original planes are parallel, then the desired plane is undetermined. In this case, equations (3.15) become identities.

Example 3.9. Let's find an equation of the plane passing through the point $M_1(0,1,2)$ and perpendicular to the planes $x-y+z+5=0$ and $-x+y+z-1=0$.

Solution. According to (3.15), the desired equation is

$$\begin{vmatrix} x-0 & y-1 & z-2 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 0,$$

whence

$$x+y-1=0.$$

3.2. Line in Space

3.2.1. Parametric equation of a straight line. The *parametric equation of the line* that passes through a point $M_1(x_1, y_1, z_1)$ and is parallel to a direction vector $\vec{R}=(l,m,n)$ (Fig. 3.10) is

$$x=x_1+lt, \quad y=y_1+mt, \quad z=z_1+nt, \text{ or } \vec{r}=\vec{r}_1+t\vec{R}, \tag{3.16}$$

where $\vec{r}=\overrightarrow{OM}$ and $\vec{r}_1=\overrightarrow{M_1M}$. As the parameter t varies from $-\infty$ to $+\infty$, the point M with position vector $\vec{r}=(x,y,z)$ determined by formula (3.2.1) runs over the entire straight line in question. It is convenient to use the parametric equation (3.16) if one needs to find the point of intersection of a straight line with a plane.

The numbers l, m and n characterize the direction of the straight line in space; they are called the *direction coefficients* of the straight line. For a unit vector $\vec{R}=\vec{R_0}$, the coefficients l, m, n are the cosines of the angles α,

β and γ formed by this straight line (the direction of the vector $\overrightarrow{R_0}$) with the coordinate axes OX, OY and OZ. These cosines can be expressed via the coordinates of the direction vector $\vec{R}$ as

$$\cos\alpha = \frac{l}{\sqrt{l^2+m^2+n^2}}, \qquad \cos\beta = \frac{m}{\sqrt{l^2+m^2+n^2}},$$
$$\cos\gamma = \frac{n}{\sqrt{l^2+m^2+n^2}}, \tag{3.17}$$

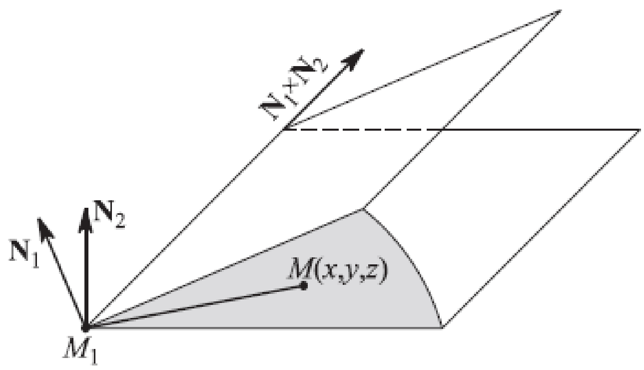


Fig. 3.9. Plane passing through a point and perpendicular to two planes

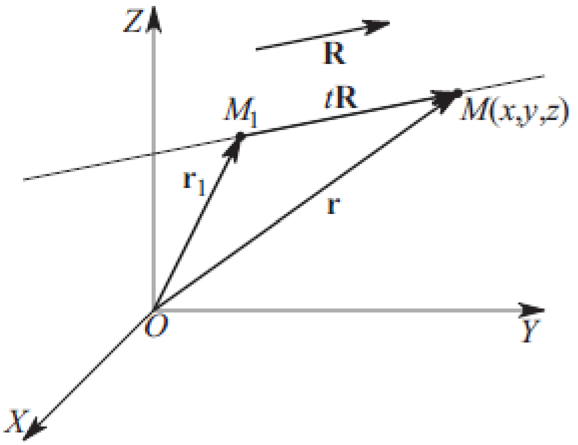


Fig. 3.10. Straight line passing through a point and parallel to direction vector

Example 3.10. Let's find the equation of the straight line that passes through the point $M_1(2,-3,1)$ and is parallel to the direction vector $\vec{R}=(1,2,-3)$.

Solution. According to (3.16), the desired equation is

$$x = 2+t, \quad y = -3+2t, \quad z = 1-3t.$$

3. 2. 2. Canonical equation of straight line. The equation

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}, \text{ or } (\vec{r}-\vec{r}_1)\times\vec{R} = 0, \tag{3.18}$$

is called the *canonical equation of the straight line* passing through the point $M_1(x_1, y_1, z_1)$ with position vector $\vec{r_1} = (x_1, y_1, z_1)$ and parallel to the direction vector $\vec{R} = (l, m, n)$.

Remark 1. One can obtain the canonical equation (3.18) from the parametric equations (3.16) by eliminating the parameter t.

Remark 2. In the canonical equation, all coefficients l, m and n cannot be zero simultaneously, since $|\vec{R}| \neq 0$. But some of them may be zero. If one of the denominators in equations (3.18) is zero, this means that the corresponding numerator is also zero.

Example 3.11. The equations $\dfrac{x-1}{1} = \dfrac{y-3}{4} = \dfrac{z-3}{0}$ determine the straight line passing through the point $M_1(1,3,3)$ and perpendicular to the axis OZ. This means that the line lies in the plane $z = 3$, and hence $z - 3 = 0$ for all points of the line.

Example 3.12. Let us find the equation of the straight line passing through the point $M_1(2,-3,1)$ and parallel to the direction vector $\vec{R} = (1, 2, -3)$.

Solution. According to (3.18), the desired equation is

$$\frac{x-2}{1} = \frac{y+3}{2} = \frac{z-1}{-3}.$$

3.2.3. General equation of a straight line. The *general equation of a straight line in space* defines it as the line of intersection of two planes (Fig. 3.11) and is given analytically by a system of two linear equations

$$\begin{matrix} A_1x + B_1y + C_1z + D_1 = 0, & & \vec{r} \cdot \overrightarrow{N_1} + D_1 = 0, \\ & \text{or} & \\ A_2x + B_2y + C_2z + D_2 = 0, & & \vec{r} \cdot \overrightarrow{N_2} + D_2 = 0. \end{matrix} \qquad (3.19)$$

where $\overrightarrow{N_1} = (A_1, B_1, C_1)$ and $\overrightarrow{N_2} = (A_2, B_2, C_2)$ are the normals to the planes and $\vec{r}$ is the position vector of the point $M(x, y, z)$.

The direction vector $\vec{R}$ is equal to the cross product of the normals $\overrightarrow{N_1}$

and $\overrightarrow{N_2}$, i.e.,

$$\vec{R} = \overrightarrow{N_1} \times \overrightarrow{N_2}, \tag{3.20}$$

and its coordinates l, m and n can be obtained by the formulas

$$l = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \ m = \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \ n = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}. \tag{3.21}$$

Remark 1. Simultaneous equations of the form (3.19) define a straight line if and only if the coefficients A_1, B_1, and C_1 in one of them are not proportional to the respective coefficients A_2, B_2, and C_2 in the other.

Remark 2. For $D_1 = D_2 = 0$ (and only in this case), the line passes through the origin.

Example 3.13. Let's reduce the equation of the straight line $x + 2y - z + 1 = 0$, $x - y + z + 3 = 0$ to canonical form.

Solution. We choose one of the coordinates arbitrarily; say, $x = 0$. Then $2y - z + 1 = 0$, $-y + z + 3 = 0$, and hence $y = -4$, $z = -7$.

Thus the desired line contains the point $M(0, -4, -7)$. We find the cross product of the vectors $\overrightarrow{N_1} = (1, 2, -1)$ and $\overrightarrow{N_2} = (1, -1, 1)$ and, according to (3.20), obtain the direction vector $\vec{R} = (1, -2, -3)$ of the desired line.

Therefore, with (3.18) taken into account, the equation of the line becomes

$$\frac{x - 0}{1} = \frac{y + 4}{-2} = \frac{z + 7}{-3}.$$

3.2.4. Equation of the straight line passing through two points. The canonical equation of the straight line (Fig. 3.12) passing through two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}, \text{ or } (\vec{r} - \vec{r}_1) \times (\vec{r}_2 - \vec{r}_1) = 0, \tag{3.22}$$

where $\vec{r}$, $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the points $M(x, y, z)$, $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$, respectively.

The parametric equations of the straight line passing through two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ in the rectangular Cartesian coordinate system $OXYZ$ can be written as

$$\begin{aligned} x &= x_1(1-t) + x_2 t \\ y &= y_1(1-t) + y_2 t \\ z &= z_1(1-t) + z_2 t \end{aligned}, \text{ or } \vec{r} = (1-t)\vec{r}_1 + t\vec{r}_2, \tag{3.23}$$

Remark. Eliminating the parameter *t* from equations (3.23), we obtain equations (3.22).

3.2.5. Equation of the straight line passing through a point and perpendicular to a plane. The equation of the straight line passing through a point $M_0(x_0, y_0, z_0)$ and perpendicular to the plane given by the equation $Ax + By + Cz + D = 0$, or $\vec{r} \cdot \overrightarrow{N} + D = 0$ (Fig. 3.13), is

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}, \tag{3.24}$$

where $\overrightarrow{N} = (A, B, C)$ is the normal to the plane.

3.3. Mutual Arrangement of Points, Lines and Planes

3.3.1. Angles between lines in space. Let's consider two straight lines determined by vector parametric equations $\vec{r} = \vec{r}_1 + t\overrightarrow{R_1}$ and $\vec{r} = \vec{r}_2 + t\overrightarrow{R_2}$. The angle φ between these lines (Fig. 3.14) can be obtained from the formulas

$$\cos\varphi = \frac{\overrightarrow{R_1} \cdot \overrightarrow{R_2}}{\left|\overrightarrow{R_1}\right| \cdot \left|\overrightarrow{R_2}\right|}, \qquad \sin\varphi = \frac{\overrightarrow{R_1} \times \overrightarrow{R_2}}{\left|\overrightarrow{R_1}\right| \cdot \left|\overrightarrow{R_2}\right|}.$$

If the lines are given by the canonical equations

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \text{ and } \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2},$$

then the angle φ between the lines can be found from the formulas

$$\cos\varphi = \frac{\overrightarrow{R_1}\cdot\overrightarrow{R_2}}{\left|\overrightarrow{R_1}\right|\cdot\left|\overrightarrow{R_2}\right|} = \frac{l_1\cdot l_2 + m_1\cdot m_2 + n_1\cdot n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}},$$

$$\sin\varphi = \frac{\left|\overrightarrow{R_1}\times\overrightarrow{R_2}\right|}{\left|\overrightarrow{R_1}\right|\cdot\left|\overrightarrow{R_2}\right|} = \frac{\sqrt{\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2 + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2}}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}, \tag{3.25}$$

which coincide with formulas (3.25) written in coordinate form.

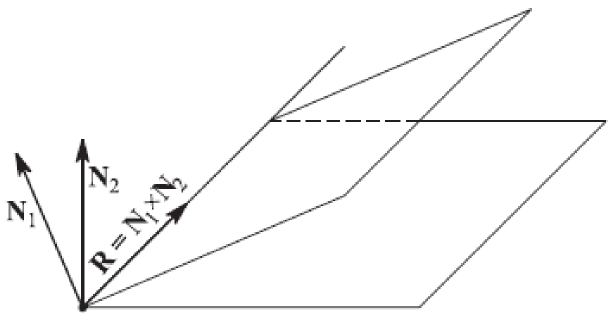


Fig. 3.11. Straight line as intersection of two planes

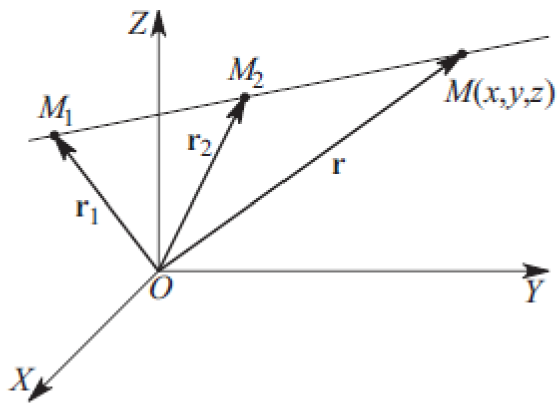


Fig. 3.12. Straight line passing through two points

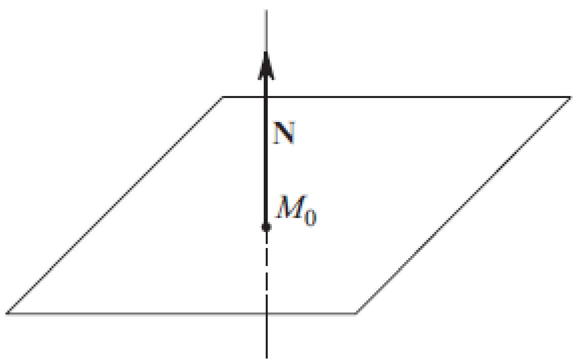


Fig. 4.47. Straight line passing through a point and perpendicular to the plane

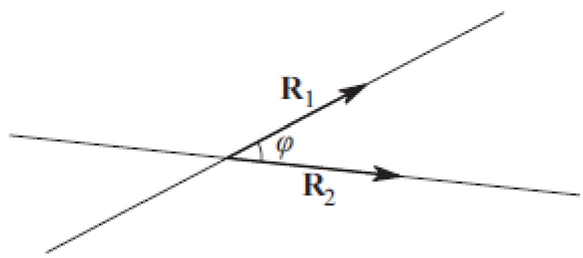


Fig. 4.48. Angles between lines in space

Example 3.14. Let's find the angle between the lines

$$\frac{x}{1}=\frac{y-2}{2}=\frac{z+1}{2} \text{ and } \frac{x}{0}=\frac{y-2}{3}=\frac{z+1}{4}.$$

Solution. Using the first formula in (3.25), we obtain

$$\cos\varphi=\frac{1\cdot 0+2\cdot 3+2\cdot 4}{\sqrt{1^2+2^2+2^2}\sqrt{0^2+3^2+4^2}}=\frac{14}{3\cdot 5}=\frac{14}{15},$$

and hence $\varphi \approx 0.3672$ rad.

3.3.2. Conditions for two lines to be parallel. Two straight lines given by vector parametric equations $\vec{r}=\vec{r}_1+t\overrightarrow{R_1}$ and $\vec{r}=\vec{r}_2+t\overrightarrow{R_2}$ are parallel if

$$\overrightarrow{R_2}=\lambda\overrightarrow{R_1}, \qquad\qquad \overrightarrow{R_1}\times\overrightarrow{R_2}=0.$$

i.e., if their direction vectors $\overrightarrow{R_1}$ and $\overrightarrow{R_2}$ are collinear. If the straight lines are given by canonical equations, then the condition that they are parallel can be written as

$$\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}, \tag{3.26}$$

Remark. If parallel lines have a common point (i.e., $\vec{r}_1=\vec{r}_2$ in parametric equations), then they coincide.

Example 3.15. Let’s show that the lines

$$\frac{x-1}{2}=\frac{y-3}{1}=\frac{z}{2} \text{ and } \frac{x-3}{4}=\frac{y+1}{2}=\frac{z}{4}$$

are parallel to each other.

Solution. Indeed, condition (3.26) is satisfied,

$$\frac{2}{4}=\frac{1}{2}=\frac{2}{4},$$

and hence the lines are parallel.

3.3.3. Conditions for two lines to be perpendicular. Two straight lines given by vector parametric equation $\vec{r}=\vec{r}_1+t\overrightarrow{R_1}$ and $\vec{r}=\vec{r}_2+t\overrightarrow{R_2}$ are perpendicular if

$$\overrightarrow{R_1} \cdot \overrightarrow{R_2} = 0, \tag{3.27}$$

If the lines are given by canonical equations, then the condition that they are perpendicular can be written as

$$l_1 \cdot l_2 + m_1 \cdot m_2 + n_1 \cdot n_2 = 0, \tag{3.28}$$

which coincides with formula (3.27) written in coordinate form.

Example 3.16. Let's show that the lines

$$\frac{x-1}{2} = \frac{y-3}{1} = \frac{z}{2} \qquad \text{and} \qquad \frac{x-2}{1} = \frac{y+1}{2} = \frac{z}{-2}$$

are perpendicular.

Solution. Indeed, condition (3.28) is satisfied

$$2 \cdot 1 + 1 \cdot 2 + 2 \cdot (-2) = 4 - 4 = 0,$$

and hence the lines are perpendicular.

3.3.4. Angles between planes. Let's consider two planes given by the general equations (3.19).

The angle between two planes (Fig. 3.15) is defined as any of the two adjacent dihedral angles formed by the planes (if the planes are parallel, then the angle between them is by definition equal to 0 or π). One of these dihedral angles is equal to the angle φ between the normal vectors $\overrightarrow{N_1} = (A_1, B_1, C_1)$ and $\overrightarrow{N_2} = (A_2, B_2, C_2)$ to the planes, which can be determined by the formula

$$\cos\varphi = \frac{\overrightarrow{N_1} \cdot \overrightarrow{N_2}}{\left|\overrightarrow{N_1}\right| \cdot \left|\overrightarrow{N_2}\right|} = \frac{A_1 \cdot A_2 + B_1 \cdot B_2 + C_1 \cdot C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}. \tag{3.29}$$

If the planes are given by vector parametric equations

$$\vec{r} = \vec{r}_1 + t\overrightarrow{R_1} + s\overrightarrow{R_2} \text{ and } \vec{r}' = \overrightarrow{r_1'} + t\overrightarrow{R_1'} + s\overrightarrow{R_2'}, \tag{3.30}$$

then the angle between the planes is given by the formula

$$\cos\varphi = \frac{\left(\overrightarrow{N_1}\times\overrightarrow{N_2}\right)\cdot}{\left|\overrightarrow{N_1}\times\overrightarrow{N_2}\right|\cdot\left|\overrightarrow{N_1'}\times\overrightarrow{N_2'}\right|}. \qquad (3.31)$$

3.3.5. Conditions for two planes to be parallel. Two planes given by the general equations (3.19) in coordinate form are parallel if and only if the following *condition for the planes to be parallel* is satisfied:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \neq \frac{D_1}{D_2}. \qquad (3.32)$$

in this case, the planes do not coincide. For planes given by the general equations (3.19) in vector form, the condition becomes

$$\overrightarrow{N_2} = \lambda\overrightarrow{N_1} \text{ or } \overrightarrow{N_1}\times\overrightarrow{N_2} = 0, \qquad (3.33)$$

i.e., the planes are parallel if their normals are parallel.

Example 3.17. Let‘s show that the planes $x - y + z = 0$ and $2x - 2y + 2z + 5 = 0$ are parallel.

Solution. Since condition (3.32) is satisfied,

$$\frac{1}{2} = \frac{-1}{-2} = \frac{1}{2},$$

we see that the planes are parallel to each other.

3.3.6. Conditions for planes to coincide. Two planes coincide if they are parallel and have a common point.

Two planes given by the general equations (3.19) coincide if and only if the following *condition for the planes to coincide* is satisfied:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}. \qquad (3.34)$$

Remark. Sometimes the case in which the planes coincide is treated as a special case of parallel straight lines and is not distinguished as an exceptional case.

3.3.7. Conditions for two planes to be perpendicular. Planes are perpendicular if their normals are perpendicular.

Two planes determined by the general equations (3.19) are perpendicular if and only if the following *condition for the planes to be perpendicular* is satisfied:

$$A_1 \cdot A_2 + B_1 \cdot B_2 + C_1 \cdot C_2 = 0 \text{ or } \overrightarrow{N_1} \cdot \overrightarrow{N_2} = 0 \tag{3.35}$$

where $\overrightarrow{N_1} = (A_1, B_1, C_1)$ and $\overrightarrow{N_2} = (A_2, B_2, C_2)$ are the normals to the planes.

Example 3.18. Let's show that the planes $x - y + z = 0$ and $x - y - 2z + 5 = 0$ are perpendicular.

Solution. Since condition (3.35) is satisfied,

$$1 \cdot 1 + (-1) \cdot (-1) + 1 \cdot (-2) = 2 - 2 = 0,$$

we see that the planes are perpendicular to each other.

3.3.8. The angle between a straight line and a plane. Let's consider a plane given by the general equation (3.3) and a line given by the canonical equation (3.18), where $\vec{N} = (A, B, C)$ is the normal to the plane, $\vec{r}$ and $\vec{r_1}$ are the respective position vectors of the points $M(x, y, z)$ and $M_1(x_1, y_1, z_1)$, and $\vec{R} = (l, m, n)$ is the direction vector of the line.

The angle between the line and the plane (Fig. 3.16) is defined as the complementary angle θ of the angle φ between the direction vector $\vec{R}$ of the line and the normal $\vec{N}$ to the plane. For this angle, one has the formula

$$\sin\theta = \cos\varphi = \frac{\vec{N} \cdot \vec{R}}{|\vec{N}| \cdot |\vec{R}|} = \frac{A \cdot l + B \cdot m + C \cdot n}{\sqrt{A^2 + B^2 + C^2}\sqrt{l^2 + m^2 + n^2}}. \tag{3.36}$$

3.3.9. Conditions for a line and a plane to be parallel. A plane given by the general equation (3.3) and a line given by the canonical equation (3.18) are parallel if

$$Al + Bm + Cn = 0, \quad \text{or} \quad \vec{N} \cdot \vec{R} = 0, \tag{3.37}$$

$$Ax_1 + By_1 + Cz_1 + D \neq 0, \qquad \vec{N} \cdot \vec{r_1} + D \neq 0.$$

i.e., a line is parallel to a plane if the direction vector of the line is perpendicular to the normal to the plane. Conditions (3.37) include the condition under which the line is not contained in the plane.

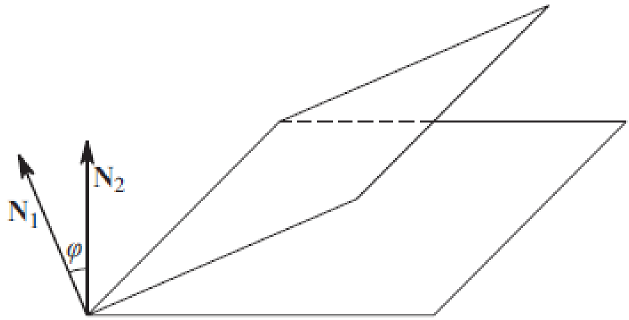


Fig. 3.15. Angles between planes

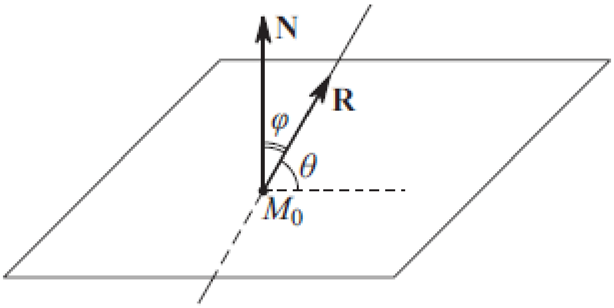


Fig. 3.16. The angle between the straight line and the plane

3.3.10. The condition for a line to be entirely contained in a plane. A straight line given by the canonical equation (3.18) is entirely contained in a plane given by the general equation (3.3) if

$$\begin{aligned} &Al + Bm + Cn = 0, && \vec{N} \cdot \vec{R} = 0, \\ &&\text{or}&& \\ &Ax_1 + By_1 + Cz_1 + D = 0, && \vec{N} \cdot \vec{r_1} + D = 0. \end{aligned} \tag{3.38}$$

Remark. Sometimes the case in which a line is entirely contained in a plane is treated as a special case of parallel straight lines and is not distinguished as an exception.

3.3.11. The condition for a line and a plane to be perpendicular. A line given by the canonical equation (3.18) and a plane given by the general equation (3.3) are perpendicular if the line is collinear to the normal to the plane (is a normal itself), i.e., if

$$\frac{A}{l} = \frac{B}{m} = \frac{C}{n}, \quad \text{or} \quad \vec{N} = \lambda \vec{R}, \quad \text{or} \quad \vec{N} \times \vec{R} = 0. \tag{3.39}$$

3.3.12. Intersection of a line and a plane. Let's consider a plane given by the general equation (3.3) and a straight line given by the parametric equation

$$x = x_1 + lt, \quad y = y_1 + mt, \quad z = z_1 + nt, \text{ or } \vec{r} = \vec{r}_1 + t\vec{R}.$$

The coordinates of the point $M_0(x_0, y_0, z_0)$ of intersection of the line with the plane (Fig. 3.16), if the point exists at all, are determined by the formulas

$$x = x_0 + lt, \quad y = y_0 + mt, \quad z = z_0 + nt, \quad \text{or} \quad \vec{r} = \vec{r}_0 + t\vec{R}. \tag{3.40}$$

where the parameter t_0 is determined from the relation

$$t_0 = -\frac{A \cdot x_1 + B \cdot y_1 + C \cdot z_1 + D}{Al + Bm + Cn} = \frac{\overrightarrow{N} \cdot \overrightarrow{r_1} + D}{\overrightarrow{N} \cdot \overrightarrow{R}}. \tag{3.41}$$

Remark. To obtain formulas (3.40) and (3.41), one should rewrite the equation of the straight line in parametric form and replace x, y and z in equation (3.3) of the plane by their expressions via t. From the resulting expression, one finds the parameter t_0 and then the coordinates x_0, y_0 and z_0 themselves.

Example 3.19. Let's find the point of intersection of the line $\frac{x}{2} = \frac{y-1}{1} = \frac{z+1}{2}$ with the plane $x + 2y + 3z - 29 = 0$.

Solution. We use formula (3.41) to find the value of the parameter t_0:

$$t_0 = -\frac{A \cdot x_1 + B \cdot y_1 + C \cdot z_1}{Al + Bm + Cn} = -\frac{1 \cdot 0 + 2 \cdot 1 + 3 \cdot (-1) - 29}{1 \cdot 2 + 2 \cdot 1 + 3 \cdot 2} = -\frac{-30}{10} = 3.$$

Then, according to (3.40), we finally obtain the coordinates of the point of intersection in the form

$$x_0 = x_1 - lt_0 = 0 - 2 \cdot 3 = -6, \qquad y_0 = y_1 - mt_0 = 1 - 1 \cdot 3 = -2,$$

$$z_0 = z_1 - nt_0 = -1 - 2 \cdot 3 = -7.$$

3.3.13. The distance from a point to a plane. The *distance from a point to a plane* is defined as the number d equal to the length of the perpendicular drawn from this point to the plane and taken with sign + if the point and the origin lie on opposite sides of the plane and with sign – if they lie on the same side of the plane. Obviously, the distance is zero for the points lying on the plane.

To obtain the distance from a point $M_0(x_0, y_0, z_0)$ to a given plane, one should replace the current Cartesian coordinates $M(x, y, z)$ on the left-hand side in the normal equation (3.7) of this plane by the coordinates of the point M_0:

$$d = |x_0 \cdot \cos\alpha + y_0 \cdot \cos\beta + z_0 \cdot \cos\gamma - p| = \left|\vec{r_1} \cdot \overrightarrow{N^0} - p\right|, \quad (3.42)$$

where $\overrightarrow{N^0} = (\cos\alpha, \cos\beta, \cos\gamma)$ is a unit vector and $\vec{r_1}$ is the position vector of the point $M_0(x_0, y_0, z_0)$. If the plane is given by the parametric equation (3.5), then the distance from a point to a plane is equal to

$$d = \frac{(\vec{r_1} - \vec{r_0})\overrightarrow{R_1}\overrightarrow{R_2}}{\left|\overrightarrow{R_1} \times \overrightarrow{R_2}\right|} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}. \quad (3.43)$$

Example 3.20. Let's find the distance from the point $M_0(5,1,-1)$ to the plane $x - 2y - 2z + 1 = 0$.

Solution. Using formula (3.43), we obtain the desired distance

$$d = \frac{|1 \cdot 5 + (-2) \cdot 1 + (-2) \cdot (-1) + 1|}{\sqrt{1^2 + (-2)^2 + (-2)^2}}.$$

3.3.14. The distance between two parallel planes. We consider two parallel planes given by the general equations $Ax + By + Cz + D_1 = 0$ and $Ax + By + Cz + D_2 = 0$. The distance between them is

$$d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}. \tag{3.44}$$

3.3.15. The distance from a point to a line. The distance from a point $M_0(x_0, y_0, z_0)$ to a line given by the canonical equation (3.18) is determined by the formula

$$d = \frac{\left|\vec{R} \times (\vec{r_1} - \vec{r_0})\right|}{\left|\vec{R}\right|} =$$

$$= \frac{\sqrt{\begin{vmatrix} m & n \\ y_1 - y_0 & z_1 - z_0 \end{vmatrix}^2 + \begin{vmatrix} n & l \\ z_1 - z_0 & x_1 - x_0 \end{vmatrix}^2 + \begin{vmatrix} l & m \\ x_1 - x_0 & y_1 - y_0 \end{vmatrix}^2}}{\sqrt{l^2 + m^2 + n^2}}. \tag{3.45}$$

Note that the last formulas are significantly simplified if $\vec{R}$ is the unit vector ($l^2 + m^2 + n^2 = 1$).

Remark. The numerator of the fraction (3.45) is the area of the triangle spanned by the vectors $\vec{r_1} - \vec{r_0}$ and $\vec{R}$, while the denominator of this fraction is the length of the base of the triangle. Hence the fraction itself is the altitude d of this triangle.

Example 3.21. Let us find the distance from the point $M_0(3,0,4)$ to the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z}{2}$.

Solution. We use formula (3.45) to obtain the desired distance

$$d = \frac{\sqrt{\begin{vmatrix} 2 & 2 \\ 1-0 & 0-4 \end{vmatrix}^2 + \begin{vmatrix} 2 & 1 \\ 0-4 & 0-3 \end{vmatrix}^2 + \begin{vmatrix} 1 & 2 \\ 0-3 & 1-0 \end{vmatrix}^2}}{\sqrt{1^2 + 2^2 + 2^2}} =$$

$$= \frac{\sqrt{\begin{vmatrix} 2 & 2 \\ 1 & -4 \end{vmatrix}^2 + \begin{vmatrix} 2 & 1 \\ -4 & -3 \end{vmatrix}^2 + \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix}^2}}{\sqrt{1+4+4}} = \frac{\sqrt{(-8-2)^2 + (-6+4)^2 + (1+6)^2}}{\sqrt{9}} =$$

$$= \frac{\sqrt{(-10)^2 + (-2)^2 + 7^2}}{3} = \frac{\sqrt{100+4+49}}{3} = \frac{\sqrt{153}}{3} \text{ (units of length).}$$

3.3.16. The distance between lines. Let's consider two nonparallel lines (Fig. 3.17) given in the canonical form

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}, \text{ or } (\vec{r} - \vec{r}_1) \times \overrightarrow{R_1} = 0,$$

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}, \text{ or } (\vec{r} - \vec{r}_2) \times \overrightarrow{R_2} = 0, \tag{3.46}$$

The distance between them can be calculated by the formula

$$d = \frac{\left| \left(\overrightarrow{r_2} - \overrightarrow{r_1}\right) \overrightarrow{R_1} \overrightarrow{R_2} \right|}{\left| \overrightarrow{R_1} \times \overrightarrow{R_2} \right|} = \frac{\pm \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2 + \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2}}. \tag{3.47}$$

The condition that the determinant in the numerator in (3.47) is zero is the *condition for the two lines in space to meet*.

Remark 1. The numerator of the fraction in (3.47) is the volume of the parallelepiped spanned by the vectors $\overrightarrow{r_2} - \overrightarrow{r_1}$, $\overrightarrow{R_1}$ and $\overrightarrow{R_2}$, while the denominator of the fraction is the area of its base. Hence the fraction itself is the altitude d of this parallelepiped.

Remark 2. If the lines are parallel (i.e., $l_2 = l_1 = l$, $m_2 = m_1 = m$ and $n_2 = n_1 = n$ or $\overrightarrow{R_2} = \overrightarrow{R_1} = \overrightarrow{R}$), then the distance between them should be calculated by formula (3.46) with $\overrightarrow{r_0}$ replaced by $\overrightarrow{r_2}$.

4. Second-Order Curves

4.1. Circle

4.1.1 Definition and canonical equation of a circle. A curve on the plane is called a *circle* if there exists a rectangular Cartesian coordinate system OXY in which the equation of this curve has the form (Fig. 4.1).

$$x^2 + y^2 = R^2 \tag{4.1}$$

where the point $O(0,0)$ is the center of the circle and $R > 0$ is its radius.

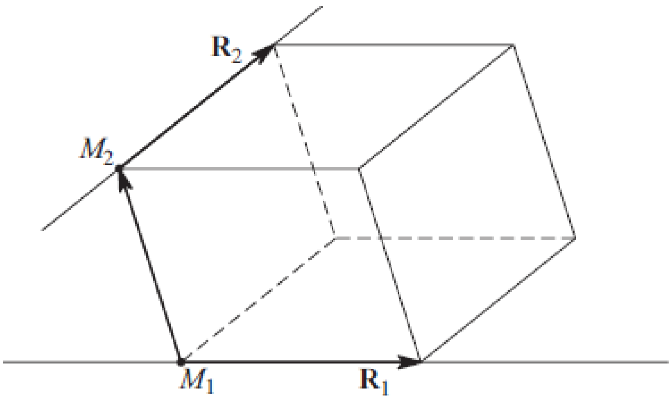


Fig. 3.17. Distance between lines

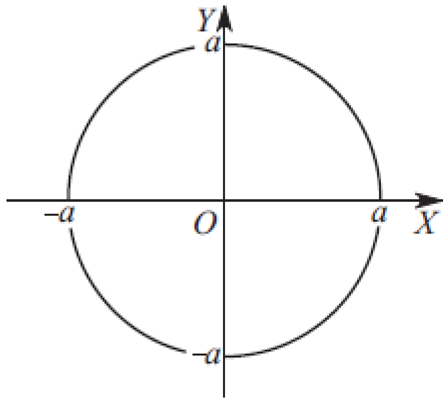


Fig. 4.1. Circle

Equation (4.1) is called the *canonical equation of a circle*.

The circle defined by the equation (4.1) is the locus of points equidistant (lying at the distance *a*) from its center. If a circle of radius R*a* is centered at a point $C(x_0, y_0)$, then its equation can be written as

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \tag{4.2}$$

Example 4.1. Find coordinates of centre C and radius R of circle

$$2x^2 + 2y^2 - 8x + 5y - 4 = 0.$$

Solution. We must allocate the full square: $(a \pm b)^2 = a^2 \pm 2ab + b^2$

$$2x^2 + 2y^2 - 8x + 5y - 4 = 0,$$

$$x^2+y^2-4x+2{,}5y-2=0,$$

$$\underbrace{x^2-4x+4}-4+\underbrace{y^2+\frac{5}{2}y+\frac{25}{16}}-\frac{25}{16}-2=0,$$

$$(x-2)^2+\left(y+\frac{5}{4}\right)^2-\frac{25}{16}-2-4=0,$$

$$(x-2)^2+\left(y+\frac{5}{4}\right)^2=\frac{121}{16}.$$

We have $C\left(2,-\frac{5}{4}\right)$ and radius $R=\frac{11}{4}$.

4.2. Ellipse

4.2.1. Definition and canonical equation of an ellipse. A curve on the plane is called an *ellipse* if there exists a rectangular Cartesian coordinate system OXY in which the equation of the curve has the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{4.3}$$

where $a \geq b > 0$ (Fig. 4.2). The coordinates in which the equation of an ellipse has the form (4.3) are called the *canonical coordinates* for this ellipse, and equation (4.3) itself is called the *canonical equation of the ellipse*.

The segments A_1A_2 and B_1B_2 joining the opposite vertices of an ellipse, as well as their lengths $2a$ and $2b$, are called the *major* and *minor axes*, respectively, *of the ellipse*. The axes of an ellipse are its axes of symmetry. In Fig. 4.2, the axes of symmetry of the ellipse coincide with the axes of the rectangular Cartesian coordinate system OXY. The numbers *a* and *b* are called the *semimajor* and *semiminor axes of the ellipse*. The number $c=\sqrt{a^2-b^2}$ is called the *linear eccentricity*, and the number $2c$ is called the *focal distance*.

The number $\varepsilon = \frac{c}{a} = \sqrt{1 - \frac{a^2}{b^2}}$, where, obviously, $0 \le \varepsilon < 10$, is called the *eccentricity* or the *numerical eccentricity*.

The point $O(0,0)$ is called the *center of the ellipse*. The points of intersection $A_1(-a,0)$, $A_2(a,0)$ and $A_1(0,-b)$, $B_2(0,b)$ of the ellipse with the axes of symmetry are called its *vertices*.

The points $F_1(-c,0)$ and $F_2(c,0)$ are the *focus of the ellipse*. This explains why the major axis of an ellipse is sometimes called its *focal axis*.

The straight lines $x = \pm\frac{a}{\varepsilon}$ $(\varepsilon \ne 0)$ are called the *directrices*. The focus $F_2(c,0)$ and the directrix $x = \frac{a}{\varepsilon}$ are said to be *right*, and the focus $F_1(-c,0)$ and the directrix $x = -\frac{a}{\varepsilon}$ are said to be *left*. A focus and a directrix are said to be *like* if both of them are right or left simultaneously.

The segments joining a point $M(x,y)$ of an ellipse with the foci $F_1(-c,0)$ and $F_2(c,0)$ are called the *left* and *right focal radii* of this point. We denote the lengths of the left and right focal radii by $r_1 = |F_1M|$ and $r_2 = |F_2M|$, respectively.

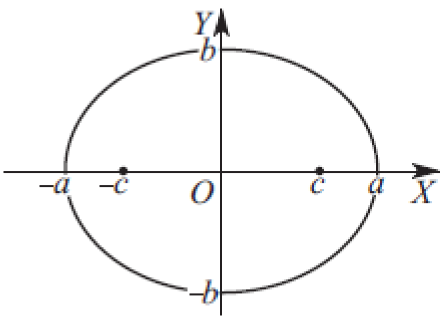


Fig. 4.2. Ellipse

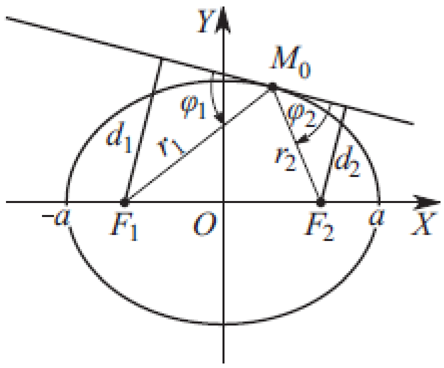


Fig. 4.3. Focal property of ellipse

Remark. For $a = b$ $(c = 0)$, equation (4.3) becomes $x^2 + y^2 = a^2$ and determines a circle; hence a circle can be considered as an ellipse for which $b = a$, $c = 0$ and $\varepsilon = 0$, i.e., the semiaxes are equal to each other, the foci coincide with the center, the eccentricity is zero (the directrices are not defined), and the focal parameter is zero.

Example 4.2. Reduce the given equation to a canonical form and draw the curve $16x^2+25y^2-32x+50y-359=0$.

Solution. Let's calculate the value $16\cdot25-0^2=400>0$, then this curve is an ellipse.

Let's group the addends:

$$\left(16x^2-32x\right)+\left(25y^2+50y\right)-359=0,$$

$$16\left(x^2-2x\right)+25\left(y^2+2y\right)-359=0.$$

Let's allocate the full squares and obtain

$$16\left(x^2-2x+1\right)-16+25\left(y^2+2y+1\right)-25-359=0,$$

$$16(x-1)^2+25\left(y+1^2\right)=400.$$

Let's divide both parts by 400 and obtain

$$\frac{(x-1)^2}{25}+\frac{(y+1)^2}{16}=1.$$

Carrying out the transformation of parallel shift of the coordinate axes with the new point of origin $O_1(1;-1)$

$$\mathrm{X}=x-1, \qquad \mathrm{Y}=y+1,$$

we obtain the following equation in the coordinate system $\mathrm{X}O_1\mathrm{Y}$:

$$\frac{\mathrm{X}^2}{25}+\frac{\mathrm{Y}^2}{16}=1.$$

This is the equation of the ellipse with semi-axes $a=5, b=4$. Let's draw the given curve in the coordinate system $\mathrm{X}O_1\mathrm{Y}$ (Fig. 4.4).

4.3. Hyperbola

4.3.1. Definition and canonical equation of hyperbola. A curve on the plane is called a *hyperbola* if there exists a rectangular Cartesian coordinate system OXY in which the equation of this curve has the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (4.4)$$

where $a > 0$ and $b > 0$ (Fig. 4.5). The coordinates in which the equation of a hyperbola has the form (4.4) are called the *canonical coordinates* for the hyperbola, and equation (4.4) itself is called the *canonical equation of the hyperbola*.

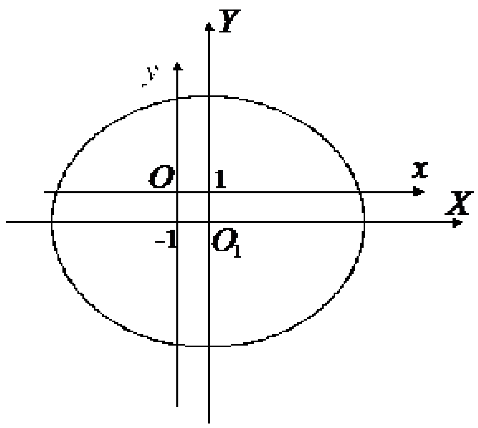


Fig. 4.4. Ellipse

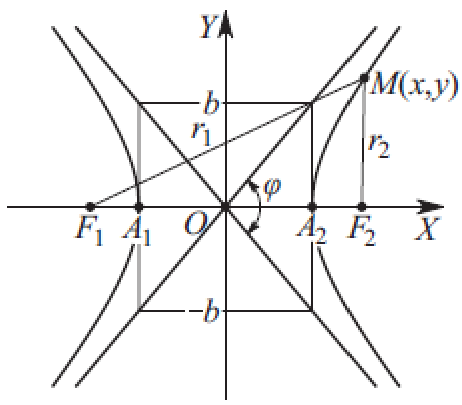


Fig. 4.5. Hyperbola

The hyperbola is a central curve of the second order. It is described by equation (4.4) and consists of two connected parts (*arms*) lying in the domains $x > a$ and $x < -a$. The hyperbola has two *asymptotes* given by the equations

$$y = \frac{b}{a}x \text{ and } y = -\frac{b}{a}x. \qquad (4.5)$$

More precisely, its arms lie in the two vertical angles formed by the asymptotes and are called the *left* and *right arms* of the hyperbola. A hyperbola is symmetric about the axes OX and OY, which are called the *principal (real, or focal, and imaginary) axes*.

The angle between the asymptotes of a hyperbola is determined by the equation

$$tg\frac{\varphi}{2}=\frac{b}{a}. \tag{4.6}$$

and if $a=b$, then $\varphi=\frac{\pi}{2}$ (an *equilateral hyperbola*).

The number a is called the *real semiaxis*, and the number b is called the *imaginary semiaxis*. The number $c=\sqrt{a^2+b^2}$ is called the *linear eccentricity*, and $2c$ is called the *focal distance*.

The number $\varepsilon=\frac{c}{a}=\frac{\sqrt{a^2+b^2}}{a}$, where, obviously, $\varepsilon>1$, is called the *eccentricity*, or the *numerical eccentricity*.

The point $O(0,0)$ is called the *center of the hyperbola*. The points $A_1(-a,0)$ and $A_2(a,0)$ of intersection of the hyperbola with the real axis are called the *vertices of the hyperbola*.

Points $F_1(-c,0)$ and $F_2(c,0)$ are called the *foci of the hyperbola*. This is why the real axis of a hyperbola is sometimes called the focal axis. The straight lines $x=\frac{a}{e}$ ($y\neq 0$) are called the *directrices of the hyperbola* corresponding to the foci F_2 and F_1. The focus $F_2(c,0)$ and the directrix $x=\frac{a}{e}$ are said to be *right*, and the focus $F_1(-c,0)$ and the directrix $x=-\frac{a}{e}$ are said to be *left*. A focus and a directrix are said to be *like* if both of them are right or left simultaneously.

The segments joining a point $M(x,y)$ of the hyperbola with the foci $F_1(-c,0)$ and $F_2(c,0)$ are called the *left* and *right focal radii* of this point. We denote the lengths of the left and right focal radii by $r_1=|F_1M|$ and $r_2=|F_2M|$, respectively.

Remark. For $a=b$, the hyperbola is said to be *equilateral*, and its asymptotes are mutually perpendicular. The equation of an equilateral hyperbola has the form $x^2-y^2=a^2$. If we take the asymptotes to be the coordinate axes, then the equation of the hyperbola becomes $xy = a^2/2$; i.e., an equilateral hyperbola is the graph of inverse proportionality.

4.3.2. Focal properties of hyperbola. The hyperbola determined by equation (4.4) is the locus of points on the plane for which the difference of the distances to the foci F_1 and F_2 has the same absolute value $2a$ (Fig. 4.5). We write this property as

$$|r_1 - r_2| = 2a. \tag{4.7}$$

Remark. One can show that equation (4.4) implies equation (4.7) and vice versa; hence the focal property of a hyperbola is often used as the definition.

Example 4.3. Reduce the given equation to a canonical form and draw the curve $9x^2 - 4y^2 + 18x + 8y - 31 = 0$.

Solution. Let's calculate the value $9 \cdot (-4) - 0^2 = -36 < 0$, then this curve is hyperbola.

Let's group the addends:

$$\left(9x^2 + 18x\right) - \left(4y^2 - 8y\right) - 31 = 0, \quad \text{or} \quad 9\left(x^2 + 2x\right) - 4\left(y^2 - 2y\right) - 31 = 0.$$

Let's allocate the full squares and obtain

$$9\left(x^2 + 2x + 1\right) - 9 - 4\left(y^2 - 2y + 1\right) + 4 - 31 = 0,$$

$$9(x+1)^2 - 4(y-1)^2 = 36.$$

Let's divide both parts by 36 and obtain

$$\frac{(x+1)^2}{4} - \frac{(y-1)^2}{9} = 1.$$

Carrying out the transformation of parallel shift of the coordinate axes with the new point of origin $O_1(-1;1)$

$$\mathrm{X} = x + 1, \qquad \mathrm{Y} = y - 1,$$

we obtain the following equation in the coordinate system XO_1Y:

$$\frac{X^2}{4} - \frac{Y^2}{9} = 1.$$

This is the equation of the hyperbola with semi-axes $a = 2, b = 3$. Let's draw the given curve in the coordinate system XO_1Y (Fig. 4.6).

4. 4. Parabola

4.4.1. Definition and canonical equation of parabola. A curve on the plane is called a *parabola* if there exists a rectangular Cartesian coordinate system OXY, in which the equation of this curve has the form

$$y^2 = 2px, \tag{4.8}$$

where $p > 0$ (Fig. 4.7). The coordinates in which the equation of a parabola has the form (4.8) are called the *canonical coordinates* for the parabola, and equation (4.8) itself is called the *canonical equation of the parabola*.

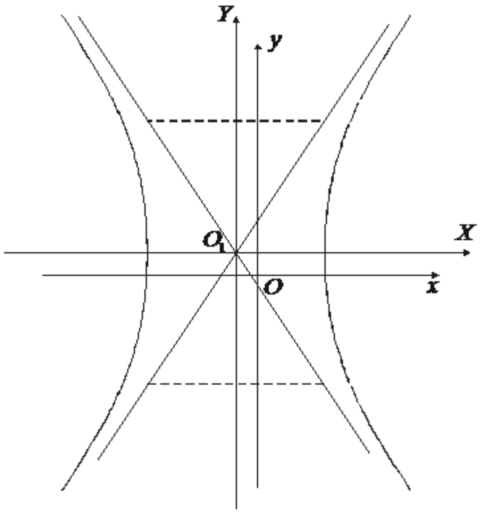


Fig. 4.6. Hyperbola

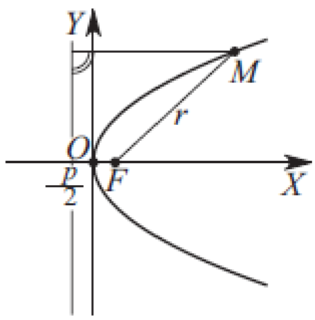


Fig. 4.7. Parabola

A parabola is a noncentral line of the second order. It consists of an infinite branch symmetric about the OX-axis. The point $O(0,0)$ is called the *vertex of the parabola*. The symmetric about the OX-axis. The point

$O(0,0)$ is called the *vertex of the parabola*. The point $F(p/2,0)$ is called the *focus of the parabola*.

The straight line $x=-\frac{p}{2}$ is called the *directrix*. The focal parameter *p* is the distance from the focus to the directrix. The number $p/2$ is called the *focal distance*.

4.4.2. Focal properties of parabola. The parabola defined by equation (4.8) on the plane is the locus of points equidistant from the focus $F(p/2,0)$ and the directrix $x=-\frac{p}{2}$ (Fig. 4.7).

We denote the length of the focal radius by r and write this property as

$$r=x+\frac{p}{2}, \tag{4.9}$$

where r satisfies the relation

$$r=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}.$$

Remark. One can show that equation (4.8) implies equation (4.9) and vice versa; hence the focal property of a parabola is often used as the definition.

4.4.3. Parabola with vertical axis. The equation of a parabola with vertical axis has the form

$$y=ax^2+bx+c, \tag{4.10}$$

For $a>0$, the vertex of the parabola is directed downward, and for $a<0$, the vertex is directed upward. The vertex of a parabola has the coordinates

$$x_0=\frac{b}{2} \text{ and } y_0=\frac{4ac-b^2}{4a}. \tag{4.11}$$

If the center of the curve of the second order is found at the point $O_1(x_0,y_0)$, then their equations have forms:

circle: $$(x-x_0)^2+(y-y_0)^2=R^2;$$

ellipse: $$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1;$$

hyperbola: $$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1;$$

parabola: $$y-y_0=2p(x-x_0).$$

Example 4. 4. Reduce the given equation to a canonical form and draw the curve

$$y^2+8y-2x+44=0.$$

Solution. Let's calculate the value $0\cdot 1-0^2=0$, then this curve is parabola.

Let's group the addends, allocate the full squares and obtain:

$$y^2+8y+16-16-2x+44=0,$$

$$(y+4)^2-2x+28=0, \quad \text{or} \quad (y+4)^2=2(x-14).$$

Carrying out the transformation of parallel shift of the coordinate axes with the new point of origin $O_1(14;-4)$

$$\mathrm{X}=x-14, \qquad \mathrm{Y}=y+4,$$

we obtain the following equation in the coordinate system $\mathrm{X}O_1\mathrm{Y}$:

$$\mathrm{Y}^2=2\mathrm{X}.$$

This is the equation of the parabola with the parameter $p=2$. Let's draw the given curve in the coordinate system $\mathrm{X}O_1\mathrm{Y}$ (Fig. 4.8).

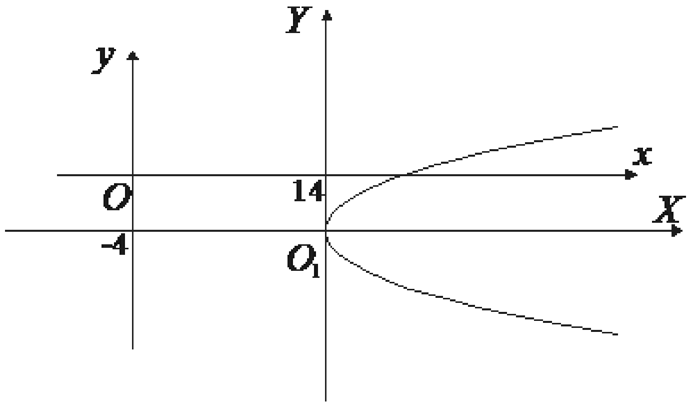


Fig. 4.8. Parabola

Control tasks. Vectors

1.1. Find projections of the vector $\vec{a} = \overline{AB} + \overline{CD}$ on the coordinate axes if $A(2,3,1)$, $B(4,1,-2)$, $C(6,3,7)$, $D(-5,-4,2)$.

1.2. Find a scalar product of the vectors $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{b} = -3\vec{i} + 4\vec{k}$ and an angle between them.

1.3. Calculate the area of the triangle ABC if $A(0,1,2)$, $B(-1,-3,5)$ and $C(1,4,-3)$. Find the length of the altitude put down from the apex B.

1.4. Calculate the area of the parallelogram based on vectors $\vec{a} + 2\vec{b}$ and $2\vec{a} + \vec{b}$ if $|\vec{a}| = 1$, $|\vec{b}| = 2$, $\varphi = \dfrac{\pi}{6}$.

1.5. Show that the vectors $\vec{a} = 2\vec{i} + 5\vec{j} + 7\vec{k}$, $\vec{b} = \vec{i} + \vec{j} - \vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} + 2\vec{k}$ are complanar.

1.6. Calculate the volume of the parallelepiped constructed on the vectors $\vec{a} = \vec{i} + \vec{j}$, $\vec{b} = \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + \vec{k}$.

1.7. At what value of *m* vectors $\vec{a} = m\vec{i} + 3\vec{j} + 4\vec{k}$ and $\vec{b} = 4\vec{i} + m\vec{j} - 7\vec{k}$ are perpendicular.

1.8. Find $(2\vec{a} + 4\vec{b}) \cdot (2\vec{a} - \vec{b})$ if $|\vec{a}| = 3$, $|\vec{b}| = 2$, $\vec{a} \perp \vec{b}$.

1.9. Calculate the volume of the pyramid with the apexes $A(1,0,0)$, $B(0,1,2)$, $C(0,0,5)$, $D(-4,2,2)$.

1.10. Find the vector product of the vectors $\vec{a} = -\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$.

1.11. Apexes of a quadrangle $A(1,-2,2)$, $B(1,4,0)$, $C(-4,1,1)$, $D(-5,-5,3)$ are given. Prove that its diagonals are perpendicular.

1.12. Find such a value α for which the vectors $\vec{a} = \alpha\vec{i} - 7\vec{j} + 5\vec{k}$ and $\vec{b} = 3\vec{i} + \alpha\vec{j} + 4\vec{k}$ are mutually perpendicular.

1.13. Two vectors $\vec{a} = -\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$ are given. Determine the projections on the coordinate axes of the following vectors:
1) $\vec{a} + \vec{b}$; 2) $\vec{a} - \vec{b}$; 3) $-4\vec{a}$; 4) $-3\vec{a} + 2\vec{b}$.

1.14. Let the following apexes of a pyramid $A(1,-2,2)$, $B(4,1,-2)$, $C(-4,1,1)$, $D(-5,-4,2)$ be given. Calculate pyramid volume and the length of the altitude put down from the apex D.

1.15. The vectors $\vec{a}$ and $\vec{b}$ form the angle 45°. Find the area of the parallelogram constructed on the vectors $\vec{m} = \vec{a} - 2\vec{b}$ and $\vec{n} = 3\vec{a} + 2\vec{b}$ if $|\vec{a}| = 5$, $|\vec{b}| = 10$.

1.16. Simplify the expression: $(2\vec{a} + \vec{b}) \times (\vec{c} - \vec{a}) + (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b})$.

Control tasks. A straight line on a plane

2.1. Form a straight line through the point $M(2,-5)$ parallel to the vector $\vec{a} = (4,-3)$.

2.2. Form a straight line through the point $M(-1,4)$ perpendicular to the vector $\vec{n} = (-2,7)$.

2.3. Two straight lines are given: $y = 2x + 3$ and $y = -x + 4$. Check if they pass through the points $A(-1,1)$, $B(2,-3)$, $C(3,1)$, $D(4,0)$, $E(2,7)$, $F(0,0)$.

2.4. Write down an equation of a straight line passing through the origin of coordinates and 1) parallel to the straight line $y = 4x - 3$; 2) perpendicular to the straight line $y = \frac{1}{2}x + 1$; 3) forming 45° angle with the straight line $y = 2x + 5$.

2.5. Find the acute angle between the straight line $9x + 3y - 7 = 0$ and a straight line passing through the points $A(1,-1)$ and $B(5,7)$.

2.6. Form the straight line passing through the point $A(4,-7)$ parallel to the straight line MN, where $M(-4,3)$ and $N(2,-5)$.

2.7. The triangle ABC with the apexes $A(2,1)$, $B(-1,-1)$ and $C(3,2)$ is given. 1. Form the equations of the sides. 2. Form the equation of the altitude dropped from the apex A. 3. Form the equation of the median dropped from the apex A.

2.8. The midpoints of the triangle sides $P(1,2)$, $Q(5,-1)$ and $R(-4,3)$ are given. Form the equations of the sides.

2.9. Find the angle between the straight line $3x + y - 6 = 0$ and the straight line passing through the points $A(-3,1)$ and $B(3,3)$.

2.10. The midpoints of the triangle sides $P(-2,1)$, $Q(2,3)$ and $R(4,-1)$ are given. Find the coordinates of the triangle apexes.

2.11. Form the straight line passing through the point $A(5,-1)$ and forming 45° angle with the axis *OX*.

2.12. Form the straight line passing through the point $A(10,-6)$ and intercepting the area of 15 sq.un. from the coordinate angle.

2.13. Find the distance between two parallel straight lines: $3x + 2y - 7 = 0$ and $3x + 2y + 15 = 0$.

2.14. Reduce the equation $12x - 5y + 60 = 0$ to 1) a normal straight line equation; 2) the equation of the straight line with the slope; 3) the equation of the straight line with the intercepts on the axes.

2.15. Find the distance from the point $M(-1,-3)$ to the straight line $8x - 6y + 5 = 0$.

2.16. Find the apexes of the triangle if $7x + 3y - 25 = 0$ (*AB*), $2x - 7y - 15 = 0$ (*BC*) and $9x - 4y + 15 = 0$ (*AC*) are its sides.

2.17. Calculate the area of a square which is formed by the straight lines $4x - 3y + 15 = 0$ and $8x - 6y + 25 = 0$ as the sides.

2.18. The apex $A(2,-5)$ and the equation of a square side are given. Find the square area.

Control tasks. A straight line and a plane in a space

3.1. Form an equation of a plane passing through the point $M(3,-2,-7)$ and parallel to the plane $2x-3z+5=0$.

3.2. Form an equation of a plane passing through the point $M(3,-1,2)$ and perpendicular to two given planes $3x+y-z+2=0$ and $x+4z+1=0$.

3.3. Form an equation of a plane perpendicular to the plane $2x-2y+4z-5=0$ and cut on the coordinate axes OX and OY two segments $a=-2$ and $b=2/3$.

3.4. Find the distance between the following parallel planes $x-2y+3z+7=0$ and $x-2y+3z-1=0$.

3.5. Write down canonical equations of the straight line $\begin{cases} x-y+3z-1=0 \\ 2x+y+z-2=0 \end{cases}$.

3.6. Form an equation of a straight line passing through the point $M(3,-2,-7)$ and perpendicular to the plane $2x-3z+5=0$.

3.7. The apexes $A(2,3,4)$, $B(4,7,3)$, $C(1,2,2)$ and $D(-2,0,1)$ are given. Find: 1) the straight line and the length of its edge AB; 2) an angle between the straight lines AB and CD; 3) an equation of the plane ABC; 4) an equation of the altitude dropped from the apex D on the plane ABC; 5) an angle between the straight line AD and the plane ABC.

3.8. At which value of m the straight line $\dfrac{x+1}{3}=\dfrac{y-2}{m}=\dfrac{z+3}{-2}$ is parallel to the plane $x-3y+6z+7=0$?

3.9. Find at what values of λ and μ the couple of equations will define parallel planes $\begin{matrix} (1)\ 2x+\lambda y+3z-5=0 \\ (2)\ \mu x-6y-6z+2=0 \end{matrix}$.

3.10. Find at what value of λ the couple of equations will define perpendicular planes (1) $5x+y-3z+3=0$ (2) $2x+\lambda y-3z=0$.

Control tasks. Second-Order Curves

4.1. Reduce the given equation to a canonical form and draw the curve:

1) $4x^2+9y^2-40x+36y+100=0$; 2) $x^2+y^2-4x+8y-16=0$;

3) $9x^2+4y^2-18x-8y-23=0$; 4) $x^2-4y^2+6x+16y-11=0$;

5) $4x^2-9y^2-8x-36y-68=0$; 6) $2y^2+x-8y+3=0$.

4.2. Write down the equation of a circle if $A(3;9)$ and $B(7;3)$ are end-points of its diameter.

4.3. An ellipse passes through points $A\left(\sqrt{3};-2\right)$ and $B\left(-2\sqrt{3};1\right)$. Form an equation of its ellipse.

4.4. Form an equation of hyperbola if it passes through point $\left(10;-3\sqrt{3}\right)$ and has asymptotes $y=\pm\dfrac{3}{5}x$.

Theoretical questions

1. What do you call a scalar product?
2. Write down the property of a scalar product.
3. Write down the formula of a scalar product if two vectors are given by their coordinates.
4. Write down the formula of the angle between vectors.
5. Write down the condition of perpendicularity of vectors.
6. Write down the condition of parallelity of vectors.
7. What do you call a vector product?
8. Write down the property of a vector product.
9. Write down the formula of a vector product if two vectors are given by their coordinates.
10. Write down the formulas of the area of a parallelogram constructed on two vectors.
11. Write down the formulas of division of a segment in the given ratio.
12. What do you call a mixed product?
13. Write down the formulas of the volume of a parallelepiped constructed on three vectors.
14. Write down the condition of complanarity of three vectors.
15. What form has a general equation of a straight line?

16. Write down an equation of a straight line with a slope.
17. Write down an equation of a straight line passing through the point with a slope.
18. Write down an equation of a straight line passing through two points.
19. Write down an equation of a straight line with given intercepts on the axes.
20. Write down a canonical equation of a straight line.
21. Write down a parametric equation of a straight line.
22. Write down a normal equation of a straight line.
23. Write down the formula of the distance between the point and the straight line.
24. Write down the condition of collinearity of two straight lines.
25. Write down the condition of perpendicularity of two straight lines.
26. Write down the formula of the angle between two straight lines.
27. Describe finding of the slope from a general equation.
28. What form has a general equation of a plane?
29. Write down an equation of a plane passing through the point and perpendicular to the normal vector.
30. Write down an equation of a plane passing through three points.
31. Write down an equation of a plane with given intercepts on the axes.
32. Write down a parametric equation of a plane.
33. Write down a normal equation of a plane.
34. Write down the formula of the distance between the point and the plane.
35. Write down the condition of collinearity of two planes.
36. Write down the condition of perpendicularity of two planes.
37. Write down the formula of the angle between two planes.
38. What do you call a circle?
39. What form has a canonical equation of a circle got?
40. What is the origin of a circle?
41. What is the radius of a circle?
42. What curve is called an ellipse?
43. What form is a canonical equation of an ellipse?
44. Write down the relation between the values a, b, c for an ellipse.
45. What value is called an eccentricity of an ellipse?

46. What are focuses of an ellipse?
47. Write down the equations of the directrices of an ellipse.
48. What curve is called a hyperbola?
49. What form is a canonical equation of a hyperbola?
50. Write down the relation between the values a, b, c for a hyperbola.
51. What value is called an eccentricity of a hyperbola?
52. Write down the equations of directrixes of a hyperbola.
53. What are focuses of a hyperbola?
54. What hyperbola is called rectangular?
55. Write down the equations of asymptotes of a hyperbola.
56. What curve is called a parabola?
57. What form is a canonical equation of a parabola?
58. What is a parameter p for a parabola?
59. What point is the focus of a parabola?
60. Write down the equation of directrix of a parabola.

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