

INTEGRATION OF RATIONAL FUNCTIONS WITH QUADRATIC TRINOMIAL

A perfect square of a quadratic trinomial is

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

where $ax^2 + bx + c$ is the quadratic trinomial.

EXAMPLE.

$$\int \frac{dx}{\sqrt{x^2 + 8x + 25}} = \int \frac{dx}{\sqrt{(x+4)^2 + 9}}$$

$$x^2 + 8x + 25 = x^2 + 2 \cdot 4 \cdot x + 16 + 9 = (x+4)^2 + 9$$

$$= \ln\left(x+4+\sqrt{(x+4)^2+9}\right) + C = \ln\left(x+4+\sqrt{x^2+8x+25}\right) + C$$

1. $\int \frac{dx}{x^2 + 4x}$	2. $\int \frac{dx}{x^2 - 2x}$	3. $\int \frac{dx}{x^2 + 3x}$
4. $\int \frac{dx}{x^2 + 6x + 8}$	5. $\int \frac{dx}{x^2 - x - 6}$	6. $\int \frac{dx}{x^2 - 8x + 15}$
7. $\int \frac{dx}{x^2 - 8x + 7}$	8. $\int \frac{dx}{x^2 - 9x + 14}$	9. $\int \frac{dx}{x^2 - 8x + 25}$
10. $\int \frac{dx}{x^2 + 6x - 7}$	11. $\int \frac{dx}{x^2 + 4x + 3}$	12. $\int \frac{dx}{x^2 + 7x + 12}$

INTEGRATION OF RATIONAL FRACTIONS

Example. $\int \frac{15x^2 - 4x - 81}{(x-3)(x+4)(x-1)} dx.$

The integrand is a proper fraction, let us decompose it into partial fractions.

$$\frac{15x^2 - 4x - 81}{(x-3)(x+4)(x-1)} = \frac{A}{x-3} + \frac{B}{x+4} + \frac{C}{x-1}.$$

Since the denominators on the left and right are equal and the fractions are identically equal, then the numerators are equal too.

Let us reduce to the common denominator and equate the numerators:

$$15x^2 - 4x - 81 = A(x+4)(x-1) + B(x-3)(x-1) + C(x-3)(x+4).$$

Let's consider the second method of finding of undetermined coefficients.

Let us substitute the first root of the denominator $x = 3$ into this expression:

$$\begin{aligned} 15 \cdot 9 - 4 \cdot 3 - 81 &= A(3+4)(3-1) + B \cdot 0 + C \cdot 0, \\ 42 &= 14A, \text{ or } A = 3. \end{aligned}$$

Let us substitute the second root of the denominator $x = -4$ into this expression:

$$\begin{aligned} 15 \cdot 16 + 4 \cdot 4 - 81 &= A \cdot 0 + B(-4-3)(-4-1) + C \cdot 0, \\ 175 &= 35B, \text{ or } B = 5. \end{aligned}$$

Let us substitute the third root of the denominator $x = 1$ into this expression:

$$\begin{aligned} 15 \cdot 1 - 4 \cdot 1 - 81 &= A \cdot 0 + B \cdot 0 + C(1-3)(1+4), \\ -70 &= -10B, \text{ or } B = 7. \end{aligned}$$

$$\text{So, } \int \frac{15x^2 - 4x - 81}{(x-3)(x+4)(x-1)} dx = \int \left(\frac{A}{x-3} + \frac{B}{x+4} + \frac{C}{x-1} \right) dx =$$

$$= \int \left(\frac{3}{x-3} + \frac{5}{x+4} + \frac{7}{x-1} \right) dx = 3 \int \frac{1}{x-3} dx + 5 \int \frac{1}{x+4} dx + 7 \int \frac{1}{x-1} dx =$$

$$= 3 \ln|x-3| + 5 \ln|x+4| + 7 \ln|x-1| + C.$$

13. $\int \frac{(x+1)dx}{x^2 + 3x - 4}$	14. $\int \frac{(3x+2)dx}{(x^2-1)(x+2)}$	15. $\int \frac{(x^2-19x+6)dx}{(x-1)(x^2+5x+6)}$
16. $\int \frac{(43x-67)dx}{(x-1)(x^2-x-12)}$	17. $\int \frac{xdx}{(x+1)(2x+1)}$	18. $\int \frac{(x+2)dx}{x(x+1)(x-2)}$
19. $\int \frac{dx}{x^2-8x+7}$	20. $\int \frac{dx}{x^2-9x+14}$	21. $\int \frac{dx}{2x^2+4x-30}$
22. $\int \frac{(x^3+1)dx}{x^3-5x^2+6x}$	23. $\int \frac{dx}{x^2-x-2}$	24. $\int \frac{dx}{(x^2-1)(x+2)}$
25. $\int \frac{(3x^2+x+2)dx}{x(x+1)(x-1)}$	26. $\int \frac{(x^2-3x+4)dx}{x(x+1)(x+2)}$	