MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

SIMON KUZNETS KHARKIV NATIONAL UNIVERSITY OF ECONOMICS

HIGHER MATHEMATICS

Guidelines to solving practical tasks on the definite integral for students of all specialties of the first (bachelor) level

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Introduction

Integral calculus plays a very important role in economics in particular in problems concerning the optimum management and plans. Therefore, the deep knowledge of this section of higher mathematics is necessary for modern economists.

In the guidelines, only the most principal topics of integral calculus are stated in brief.

The present guidelines are the continuation of the part where the notions of integral calculus were discussed. By means of these notions, the notions of the definite integral, geometrical and economic problems being the most fundamental ones in mathematics can be introduced.

Guidelines for the Definite Integral

Let the function $f(x)$ be determined on the interval $[a,b]$. Let us divide the interval $[a,b]$ $(b>a)$ into n subintervals by n points $a=x_0<x_1<x_2<...<x_{n-1}<x_n=b$ (Fig. 1).

On each subinterval $[x_{i-1},x_i]$ we choose some point ξ_i and compound the sum

$$S_n=\sum_{i=1}^{n} f(\xi_i)\Delta x_i = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + ... + f(\xi_n)\Delta x_n,$$

where $\Delta x_i = x_i - x_{i-1}$ is the length of each interval.

The definite integral of the function $f(x)$ on the interval $[a,b]$ is called the limit

$$\lim_{max\,\Delta x_i\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx,$$

where a is the upper limit, b is the lower limit of integration.

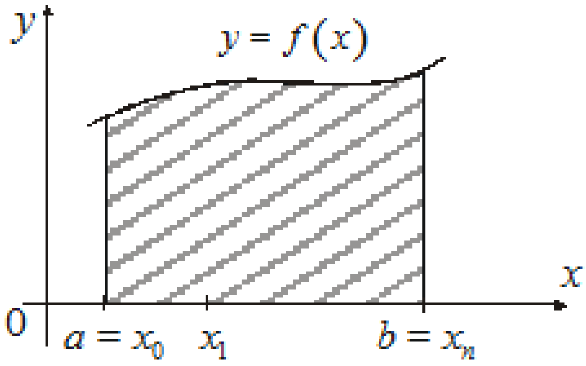


Fig. 1. The area under the function $f(x)$ on the interval $[a,b]$

If $F(x)$ is an antiderivative for a continuous function $f(x)$ on the interval $[a,b]$, then the formula of Newton-Leibnitz is true:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

Let's consider *a substitution in definite integrals*. We remember that the limits are in terms of the original variable. Then we get two approaches:

1) solve the indefinite integral first;
2) change the limits.

First solve the indefinite integral to find an antiderivative. Then use that antiderivative to solve the definite integral.

Example 1. Calculate the definite integrals:

a) $\int_1^2 x^4 dx$; b) $\int_{-1}^2 \left(4+2x-x^2\right)dx$; c) $\int_1^2 e^{3x}dx$.

Solution. Let's apply the formula of Newton-Leibnitz:

a) $\int_1^2 x^4 dx = \frac{x^5}{5}\Big|_1^2 = \frac{1}{5}\left(2^5 - 1^5\right) = \frac{1}{5}(32-1) = \frac{31}{5} = 6.2$;

b) $\int_{-1}^2 \left(4+2x-x^2\right)dx = \left(4x + 2\cdot\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{-1}^2 = \left(4x + x^2 - \frac{x^3}{3}\right)\Bigg|_{-1}^2 =$

$$=\left(4\cdot 2+2^2-\frac{2^3}{3}\right)-\left(4\cdot(-1)+(-1)^2-\frac{(-1)^3}{3}\right)=8+4-\frac{8}{3}-$$

$$-\left(-4+1+\frac{1}{3}\right)=12-\frac{8}{3}+3-\frac{1}{3}=12;$$

c) $\int_1^2 e^{3x}dx=\frac{1}{3}e^{3x}\Big|_1^2=\frac{1}{3}\left(e^6-e^3\right)=\frac{e^3}{3}\left(e^3-1\right).$

Let us note the *basic properties* of definite integrals:

1) the definite integral depends on the form of the function $f(x)$ and limits of integration, but it doesn't depend on the designation of the variable of integration, i.e.

$$\int_a^b f(x)dx=\int_a^b f(t)dt.$$

2) the definite integral changes its sign to the opposite if the limits of integration are replaced:

$$\int_a^b f(x)dx=-\int_b^a f(x)dx.$$

Task 1. Verify this property calculating the definite integral: $\int_3^0 (2-x)dx$.

3) if the lower and the upper limits of integration are equal, then the definite integral is equal to zero:

$$\int_a^a f(x)dx=0.$$

Task 2. Verify this property calculating the definite integral: $\int_1^1 \frac{dx}{1+x^2}$.

4) the additivity of the definite integral:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

where $c \in [a,b]$;

5) the definite integral of the algebraic sum of functions equals the algebraic sum of definite integrals:

$$\int_a^b \left(f_1(x) \pm f_2(x) \right) dx = \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx;$$

6) it is possible to take the constant C outside the integral:

$$\int_a^b C \cdot f(x)dx = C \cdot \int_a^b f(x)dx;$$

7) calculation of the definite integral of an even function with symmetrical limits of integration is simplified as:

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx;$$

Task 3. Verify this property calculating the definite integral: $\int_{-4}^4 3x^2dx$.

8) the definite integral of an odd function with symmetrical limits of integration is equal to zero:

$$\int_{-a}^a f(x)dx = 0;$$

Task 4. Verify this property calculating the definite integral: $\int_{-\pi}^{\pi} \sin x dx$.

9) if $f(x)$ is a continuous function with the period T, then

$$\int_a^b f(x)dx = \int_{a+nT}^{b+nT} f(x)dx,\ n = 0, \pm 1, \pm 2, \ldots;$$

Task 5. Verify this property calculating the definite integral: $\int_{13\pi/2}^{7\pi} \sin x dx$.

10) if $a < b$ in the segment $[a,b]$ and $f(x) \le \varphi(x)$, then

$$\int_a^b f(x)dx \le \int_a^b \varphi(x)dx.$$

11) *the mean-value theorem*. If the function $y = f(x)$ is continuous in the interval $[a,b]$, where $a < b$, then there exists a value $c \in [a, b]$, such that

$$\int_a^b f(x)dx = f(c)(a-b).$$

Integration with the aid of the table of basic indefinite integrals is called *the direct integration*.

Example 2. Calculate the definite integral: $\int_{-2}^{0} \frac{dx}{4+x^2}$.

Solution. Let's calculate the integral:

$$\int_{-2}^{0} \frac{dx}{4+x^2} = \int_{-2}^{0} \frac{dx}{2^2+x^2} = \frac{1}{2}\operatorname{arctg}\frac{x}{2}\Big|_{-2}^{0} = \frac{1}{2}\left(\operatorname{arctg}0 - \operatorname{arctg}\left(-\frac{2}{2}\right)\right) =$$
$$= \frac{1}{2}(0 - \operatorname{arctg}(-1)) = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}.$$

Task 6. Calculate the definite integral: $\int_0^3 \frac{dx}{\sqrt{x^2+16}}$.

Table 1

The basic table of integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$	15. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left\|\frac{x-a}{x+a}\right\| + C$, $(a \neq 0)$.
2. $\int dx = x + C$	
3. $\int 0 \cdot dx = C$	16. $\int \frac{dx}{\sqrt{x^2 \pm a}} = \ln\left\|x + \sqrt{x^2 \pm a}\right\| + C$, $(a \neq 0)$
4. $\int \frac{dx}{x} = \ln\|x\| + C$	
5. $\int \sin x dx = -\cos x + C$	17. $\int \frac{dx}{1+x^2} = \begin{cases} \mathrm{arctg} x + C \\ -\mathrm{arcctg} x + C \end{cases}$
6. $\int \cos x dx = \sin x + C$	
7. $\int \mathrm{tg} x dx = -\ln\|\cos x\| + C$	
8. $\int \mathrm{ctg} x dx = \ln\|\sin x\| + C$	18. $\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C \\ -\arccos x + C \end{cases}$
9. $\int e^x dx = e^x + C$	
10. $\int a^x dx = \frac{a^x}{\ln a} + C$, $(a > 0, a \neq 1)$	19. $\int \frac{dx}{a^2 + x^2} = \begin{cases} \frac{1}{a} \mathrm{arctg} \frac{x}{a} + C \\ -\frac{1}{a} \mathrm{arcctg} \frac{x}{a} + C \end{cases}$, $(a > 0)$
11. $\int \frac{dx}{\sin^2 x} = -\mathrm{ctg} x + C$	
12. $\int \frac{dx}{\cos^2 x} = \mathrm{tg} x + C$	
13. $\int \frac{dx}{\sin x} = \ln\left\|\mathrm{tg} \frac{x}{2}\right\| + C$	20. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \arcsin \frac{x}{a} + C \\ -\arccos \frac{x}{a} + C \end{cases}$, $(a > 0)$
14. $\int \frac{dx}{\cos x} = \ln\left\|\mathrm{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right\| + C$	

Example 3. Calculate the definite integral: $\int_0^{\pi/6} \frac{\sin^2 x}{\cos x} dx$.

Solution. Let's transform the integrand:

$$\int_0^{\pi/6}\frac{\sin^2 x}{\cos x}dx=\left|\sin^2 x=1-\cos^2 x\right|=\int_0^{\pi/6}\frac{1-\cos^2 x}{\cos x}=\int_0^{\pi/6}\left(\frac{1}{\cos x}-\cos x\right)dx=$$

$$=\left[\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|-\sin x\right]_0^{\pi/6}=\ln\operatorname{tg}\frac{\pi}{3}-\sin\frac{\pi}{6}-\left(\ln\operatorname{tg}\frac{\pi}{4}-\sin 0\right)=$$

$$=\ln\sqrt{3}-\frac{1}{2}-\ln 1+0=\frac{1}{2}(\ln 3-1).$$

The Method of Change of the Variable (Substitution) in the Definite Integral

Theorem. Let the integral $\int_a^b f(x)dx$ be given, where the function $f(x)$ is continuous in the interval $[a,b]$. Let's change the variable t by the formula $x=\varphi(t)$. If

1) $\varphi(\alpha)=a$, $\varphi(\beta)=b$;
2) $\varphi(t)$ and $\varphi'(t)$ are continuous in the interval $[\alpha,\beta]$;
3) $f(\varphi(t))$ is defined and continuous in the interval $[\alpha,\beta]$, then

$$\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

We will use Table 2 for finding the substitution.

Example 4. Calculate the definite integral: $\int_0^{\sqrt{3}}\frac{xdx}{\sqrt{25-x^4}}$.

Solution. Let's use the method of change of a variable:

$$\int_0^{\sqrt{3}}\frac{xdx}{\sqrt{25-x^4}}=\left|\begin{matrix} t=x^2, t_1=0^2=0, t_2=\left(\sqrt{3}\right)^2=3 \\ dt=2xdx,\ \frac{1}{2}dt=xdx\end{matrix}\right|=\int_0^3\frac{\frac{1}{2}dt}{\sqrt{25-t^2}}=$$

$$=\frac{1}{2}\int_0^3\frac{dt}{\sqrt{25-t^2}}=\frac{1}{2}\arcsin\left.\frac{t}{5}\right|_0^3=\frac{1}{2}\left(\arcsin\frac{3}{5}-\arcsin\frac{0}{5}\right)=\frac{1}{2}\arcsin\frac{3}{5}.$$

Table 2

The table of substitutions

No.	Kind of integral	Substitution
1	$\int f\left(x^2\right)xdx$	$x^2=t$
2	$\int f\left(x^3\right)x^2dx$	$x^3=t$
3	$\int f\left(\frac{1}{x}\right)\frac{dx}{x^2}$	$\frac{1}{x}=t$
4	$\int f\left(\sqrt{x}\right)\frac{dx}{\sqrt{x}}$	$\sqrt{x}=t$
5	$\int f(\ln x)\frac{dx}{x}$	$\ln x=t$
6	$\int f(\sin x)\cos x dx$	$\sin x=t$
7	$\int f(\cos x)\sin x dx$	$\cos x=t$
8	$\int f(\mathrm{tg} x)\frac{dx}{\cos^2 x}$	$\mathrm{tg} x=t$
9	$\int f(\mathrm{ctg} x)\frac{dx}{\sin^2 x}$	$\mathrm{ctg} x=t$
10	$\int f\left(e^x\right)e^x dx$	$e^x=t$
11	$\int f(\mathrm{arctg} x)\frac{dx}{1+x^2}$	$\mathrm{arctg} x=t$
12	$\int f(\arcsin x)\frac{dx}{\sqrt{1-x^2}}$	$\arcsin x=t$
13	$\int\frac{f'(x)dx}{f(x)}$	$f(x)=t$

Example 5. Calculate the definite integral: $\int\limits_{\pi/2}^{\pi} \frac{\sin x\,dx}{\cos^2 x+1}$.

Solution. Let's use the method of change of a variable:

$$\int\limits_{\pi/2}^{\pi} \frac{\sin x\,dx}{\cos^2 x+1} = \left| \begin{array}{l} t=\cos x, t_1=\cos\frac{\pi}{2}=0, t_2=\cos\pi=-1 \\ dt=-\sin x dx,\ -dt=\sin x dx \end{array} \right| = \int\limits_{0}^{1} \frac{-dt}{t^2+1} =$$

$$= -\int\limits_{0}^{-1} \frac{dt}{t^2+1} = \int\limits_{-1}^{0} \frac{dt}{t^2+1} = arctgt\Big|_{-1}^{0} = arctg0 - acrtg(-1) = 0+\frac{\pi}{4} = \frac{\pi}{4}.$$

Example 6. Calculate the definite integral: $\int\limits_{1}^{2} \frac{e^{\frac{1}{x}}dx}{x^2}$.

Solution. Let's use the method of change of a variable:

$$\int\limits_{1}^{2} \frac{e^{\frac{1}{x}}dx}{x^2} = \left| \begin{array}{l} t=\frac{1}{x},\ t_1=\frac{1}{1}=1,\ t_2=\frac{1}{2} \\ dt=-\frac{1}{x^2}dx,\ -dt=\frac{1}{x^2}dx \end{array} \right| = \int\limits_{1}^{1/2} \left(-e^t\right)dt = -\int\limits_{1}^{1/2} e^t\,dt =$$

$$= -e^t\Big|_{1}^{1/2} = -\left(e^{1/2}-e^1\right) = e-\sqrt{e}.$$

The Method of Integration by Parts

Let the functions $u=u(x)$ and $v=v(x)$ have continuous derivatives on the interval $[a,b]$. Then

$$\int\limits_{a}^{b} udv = (uv)\Big|_{a}^{b} - \int\limits_{a}^{b} vdu ,$$

where $(uv)\Big|_{a}^{b} = u(b)v(b)-u(a)v(a)$.

There are classes of integrals (three types) which are found by means of this method (table 3).

Table 3

Three types of integral for the method of integration by parts

Type	No.	Kind of integral	The factor u	The factor dv
The first type	1	$\int P_n(x)\cdot a^x dx$, where $P_n(x)$ is a polynomial	$P_n(x)$	$a^x dx$
	2	$\int P_n(x)\cdot e^x dx$	$P_n(x)$	$e^x dx$
	3	$\int P_n(x)\cdot \sin xdx$	$P_n(x)$	$\sin xdx$
	4	$\int P_n(x)\cdot \cos xdx$	$P_n(x)$	$\cos xdx$
The second type	5	$\int P_n(x)\cdot \arccos xdx$	$\arccos x$	$P_n(x)dx$
	6	$\int P_n(x)\cdot \arcsin xdx$	$\arcsin x$	$P_n(x)dx$
	7.	$\int P_n(x)\cdot \mathrm{arctg} xdx$	$\mathrm{arctg} x$	$P_n(x)dx$
	8	$\int P_n(x)\cdot \mathrm{arcctg} xdx$	$\mathrm{arcctg} x$	$P_n(x)dx$
	9	$\int P_n(x)\cdot \ln xdx$	$\ln x$	$P_n(x)dx$
The third type	10	$\int a^x\cdot \sin xdx$	$\sin x$ or a^x	$a^x dx$ or $\sin xdx$
	11	$\int e^x\cdot \sin xdx$	$\sin x$ or e^x	$e^x dx$ or $\sin xdx$
	12	$\int a^x\cdot \cos xdx$	$\cos x$ or a^x	$a^x dx$ or $\cos xdx$
	13	$\int e^x\cdot \cos xdx$	$\cos x$ or e^x	$e^x dx$ or $\cos xdx$

Task 7. Calculate the definite integral: $\int_0^3 (4x-1)\cos\frac{\pi x}{3}dx$.

Example 7. Calculate the definite integral: $\int_0^2 (x+3)\sin\frac{\pi x}{4}dx$.

Solution. It's the first type. Let's use the formula of integration by parts:

$$\int_0^2 (x+3)\sin\frac{\pi x}{4}dx = \left| \begin{array}{l} u = x+3, dv = \sin\frac{\pi x}{4}dx \\ du = (x+3)' dx = 1\cdot dx = dx \\ v = \int \sin\frac{\pi x}{4}dx = -\frac{4}{\pi}\cos\frac{\pi x}{4}\ (C=0) \end{array} \right| =$$

$$= \left| \int_a^b u dv = (uv)\Big|_a^b - \int_a^b v du \right| = \left((x+3)\cdot\left(-\frac{4}{\pi}\cos\frac{\pi x}{4}\right)\right)\Bigg|_0^2 -$$

$$-\int_0^2 \left(-\frac{4}{\pi}\cos\frac{\pi x}{4}\right)dx = -\frac{4}{\pi}\left(5\cos\frac{2\pi}{4} - 3\cos 0\right) + \frac{4}{\pi}\int_0^2 \cos\frac{\pi x}{4}dx =$$

$$= -\frac{4}{\pi}\left(5\cos\frac{2\pi}{4} - 3\cos 0\right) + \frac{4}{\pi}\int_0^2 \cos\frac{\pi x}{4}dx = -\frac{4}{\pi}(5\cdot 0 - 3\cdot 1) +$$

$$+\frac{4}{\pi}\cdot\frac{4}{\pi}\sin\frac{\pi x}{4}\Bigg|_0^2 = \frac{12}{\pi} + \frac{16}{\pi^2}\cdot\left(\sin\frac{2\pi}{4} - \sin 0\right) = \frac{12}{\pi} + \frac{16}{\pi^2}\cdot 1 = \frac{12\pi + 16}{\pi^2}.$$

Example 8. Calculate the definite integral: $\int_0^{\sqrt{3}} x\operatorname{arctg} x dx$.

Solution. It's the second type. Let's use the formula of integration by parts:

$$\int_0^{\sqrt{3}} x\operatorname{arctg} x dx = \left| \begin{array}{l} u = \operatorname{arctg} x,\ dv = x dx \\ du = (\operatorname{arctg} x)' dx = \frac{1}{1+x^2}\cdot dx = \frac{dx}{1+x^2} \\ v = \int x dx = \frac{x^2}{2}\ (C=0) \end{array} \right| =$$

$$= \left(\frac{x^2}{2}\operatorname{arctg} x\right)\Bigg|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x^2}{2}\cdot\frac{dx}{1+x^2} = \frac{(\sqrt{3})^2}{2}\operatorname{arctg}\sqrt{3} - \frac{0^2}{2}\operatorname{arctg} 0 -$$

$$-\frac{1}{2}\int_0^{\sqrt{3}}\frac{x^2}{1+x^2}dx=\frac{3}{2}\cdot\frac{\pi}{3}-0-\frac{1}{2}\int_0^{\sqrt{3}}\frac{1+x^2-1}{1+x^2}dx=\frac{\pi}{2}-$$

$$-\frac{1}{2}\int_0^{\sqrt{3}}\left(\frac{1+x^2}{1+x^2}-\frac{1}{1+x^2}\right)dx=\frac{\pi}{2}-\frac{1}{2}\int_0^{\sqrt{3}}\left(1-\frac{1}{1+x^2}\right)dx=\frac{\pi}{2}-$$

$$-\frac{1}{2}\cdot(x-\operatorname{arctg}x)\Big|_0^{\sqrt{3}}=\frac{\pi}{2}-\frac{1}{2}\cdot\left[\sqrt{3}-\operatorname{arctg}\sqrt{3}-(0-\operatorname{arctg}0)\right]=$$

$$=\frac{\pi}{2}-\frac{1}{2}\cdot\left(\sqrt{3}-\frac{\pi}{3}\right)=\frac{\pi}{2}-\frac{\sqrt{3}}{2}+\frac{\pi}{6}=\frac{4\pi-\sqrt{3}}{6}.$$

Example 9. Calculate the definite integral: $\int_1^e x\ln^2 x dx$.

Solution. It's the first type. Let's use the formula of integration by parts twice:

$$\int_1^e x\ln^2 xdx=\left|\begin{array}{l}u=\ln^2 x, dv=xdx\\ du=\left(\ln^2 x\right)'dx=2\ln x\cdot\frac{1}{x}dx=\frac{2\ln x}{x}dx\\ v=\int xdx=\frac{x^2}{2}\ (C=0)\end{array}\right|=\left(\frac{x^2}{2}\ln^2 x\right)\Bigg|_1^e-$$

$$-\int_1^e\frac{x^2}{2}\cdot 2\ln x\cdot\frac{1}{x}dx=\frac{e^2}{2}\cdot\ln^2 e-\frac{1^2}{2}\cdot\ln^2 1-\int_1^e x\cdot\ln xdx=$$

$$=\left|\begin{array}{l}u=\ln x, dv=xdx\\ du=(\ln x)'dx=\frac{1}{x}dx\\ v=\int xdx=\frac{x^2}{2}\ (C=0)\end{array}\right|=\frac{e^2}{2}-0-\left[\left(\frac{x^2}{2}\ln x\right)\Bigg|_1^e-\int_1^e\frac{x^2}{2}\cdot\frac{1}{x}dx\right]=\frac{e^2}{2}-$$

$$-\left(\frac{e^2}{2}\ln e-\frac{1^2}{2}\ln 1\right)+\frac{1}{2}\int_1^e xdx=\frac{e^2}{2}-\frac{e^2}{2}+0+\frac{1}{2}\cdot\frac{x^2}{2}\Bigg|_1^e=\frac{1}{4}\cdot\left(e^2-1^2\right)=\frac{e^2-1}{4}.$$

Example 10. Calculate the definite integral: $\int_0^1 e^{2x} \cos x \, dx$.

Solution. It's the third type. Let's use the formula of integration by parts twice:

$$\int_0^1 e^{2x} \cos x dx = \left| \begin{array}{l} u = e^{2x}, dv = \cos x dx \\ du = \left(e^{2x}\right)' dx = 2 \cdot e^{2x} dx \\ v = \int \cos x dx = \sin x \; (C = 0) \end{array} \right| = \left(e^{2x} \sin x\right)\Big|_0^1 -$$

$$-\int_0^1 2e^{2x} \sin x dx = \left| \begin{array}{l} u = e^{2x}, dv = \sin x dx \\ du = \left(e^{2x}\right)' dx = 2 \cdot e^{2x} dx \\ v = \int \sin x dx = -\cos x \; (C = 0) \end{array} \right| = e^2 \sin 1 - e^0 \sin 0 -$$

$$-2\left[\left(-e^{2x} \cos x\right)\Big|_0^1 - \int_0^1 \left(-2e^{2x} \cos x\right) dx\right] = e^2 \sin 1 + 2\left(e^{2x} \cos x\right)\Big|_0^1 -$$

$$-4\int_0^1 e^{2x} \cos x dx.$$

Let's denote the initial definite integral as $\int_0^1 e^{2x} \cos x \, dx = A$. Then

$$A = e^2 \sin 1 + 2\left(e^{2x} \cos x\right)\Big|_0^1 - 4A.$$

Let's express A:

$$A + 4A = e^2 \sin 1 + 2\left(e^2 \cos 1 - e^0 \cos 0\right),$$

$$5A = e^2 \sin 1 + 2\left(e^2 \cos 1 - 1\right),$$

$$A = \frac{e^2 \sin 1 + 2\left(e^2 \cos 1 - 1\right)}{5}.$$

Integration of Rational Functions with a Quadratic Trinomial

The rational functions with a quadratic trinomial are the functions of the kinds:

$$\int \frac{A}{ax^2+bx+c}dx,\ \int \frac{A}{\sqrt{ax^2+bx+c}}dx,\ \int \frac{Ax+B}{ax^2+bx+c}dx,\ \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx,$$

where ax^2+bx+c is the quadratic trinomial.

Let's consider the first and second integrals: $\int \frac{A}{ax^2+bx+c}dx$, $\int \frac{A}{\sqrt{ax^2+bx+c}}dx$. They are reduced to tabular integrals if you allocate the perfect square in the denominator with the help of the formula:

$$(y\pm z)^2 = y^2 \pm 2yz + z^2$$

(the square of the sum or the square of the difference).

Example 11. Calculate the definite integral: $\int_0^1 \frac{dx}{x^2-5x+6}$.

Solution. Let's allocate the perfect square in the denominator:

$$x^2-5x+6 = x^2-2\cdot\frac{5}{2}x+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+6=\left(x-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2.$$

Let's make the substitution: $x-\frac{5}{2}=t,\ dx=dt$.

We change the limits of integration: if $x=0,$ then $t=0-\frac{5}{2}=-\frac{5}{2}$; if $x=1$, then $t=1-\frac{5}{2}=-\frac{3}{2}$. Thus, we have

$$\int_0^1 \frac{dx}{x^2-5x+6} = \int_0^1 \frac{dx}{\left(x-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2} = \int_{-\frac{5}{2}}^{-\frac{3}{2}} \frac{dt}{t^2-\left(\frac{1}{2}\right)^2} =$$

$$= \frac{1}{2\cdot\frac{1}{2}} \ln\left|\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right|_{-\frac{5}{2}}^{-\frac{3}{2}} = \ln\left|\frac{-2}{-1}\right| - \ln\left|\frac{-3}{-2}\right| = \ln 2 - \ln\frac{3}{2} = \ln\frac{4}{3}.$$

Let's consider the third and fourth integrals: $\int \frac{Ax+B}{ax^2+bx+c}dx$, $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$. To integrate these functions we should use the following rules:

1) to allocate the perfect square in the trinomial with the help of the formulas $(y \pm z)^2 = y^2 \pm 2yz + z^2$ (the square of the sum or the square of the difference) and obtain a new denominator $(x \pm p)^2 \pm q$;

2) to apply the substitution:

$$t = x + p; \qquad t = x - p;$$
$$dt = dx; \qquad dt = dx;$$
$$x = t - p; \qquad x = t + p;$$

3) to present the initial integral as a sum of two integrals, the first one is the tabular integral and the second one may be integrated by substitution.

Let's consider four kinds of such integrals if $a \neq 0$:

a) $\int \frac{xdx}{\sqrt{a-x^2}} = -\sqrt{a-x^2} + C$; c) $\int \frac{xdx}{\sqrt{a-x^2}} = -\sqrt{a-x^2} + C$;

b) $\int \frac{xdx}{a-x^2} = -\frac{1}{2}\ln\left|x^2-a\right| + C$; d) $\int \frac{xdx}{\sqrt{x^2 \pm a}} = \sqrt{x^2 \pm a} + C$;

4) to get back to the previous variable by substitution $t = x + p$ or $t = x - p$.

Let's give an example.

Example 12. Calculate the definite integral: $\int\limits_2^5 \frac{(x+1)dx}{x^2-4x+13}$.

Solution. Let's allocate the perfect square in the denominator, use the method of change of a variable and change the limits of integration:

$$\int\limits_2^5 \frac{(x+1)dx}{x^2-4x+13} = \left| \begin{array}{l} x^2-4x+13 = x^2-2\cdot 2x+2^2-2^2+13 = \\ =(x-2)^2+9 \end{array} \right| =$$

$$= \int\limits_2^5 \frac{(x+1)dx}{(x-2)^2+9} = \left| \begin{array}{l} t = x-2,\ dt = dx \\ x = t+2 \\ t_1 = 2-2 = 0, \\ t_2 = 5-2 = 3 \end{array} \right| = \int\limits_0^3 \frac{(t+2+1)dt}{t^2+9} = \int\limits_0^3 \frac{(t+3)dt}{t^2+9} =$$

$$= \int\limits_0^3 \frac{tdt}{t^2+9} + \int\limits_0^3 \frac{3dt}{t^2+9} = \int\limits_0^3 \frac{tdt}{t^2+9} + 3\int\limits_0^3 \frac{dt}{3^2+t^2} = \ln\left|t^2+9\right|\Big|_0^3 + 3\cdot\frac{1}{3}\operatorname{arctg}\frac{t}{3}\Big|_0^3 =$$

$$= \ln\left|3^2+9\right| - \ln\left|0^2+9\right| + \operatorname{arctg}\frac{3}{3} - \operatorname{arctg}\frac{0}{3} = \ln 18 - \ln 9 + \operatorname{arctg}1 -$$

$$- \operatorname{arctg}0 = \ln\frac{18}{9} + \frac{\pi}{4} - 0 = \ln 2 + \frac{\pi}{4}.$$

Integration of Trigonometric Functions. Integrals of Type $\int R(\sin x, \cos x)dx$

In this case, it is convenient to use the substitution $t = \operatorname{tg}\frac{x}{2}$, since the functions $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$ are rational functions of the variable x. The integration of the form $\int R(\sin x, \cos x)dx$ with the aid of the substitution $t = \operatorname{tg}\frac{x}{2}$ is sure to give the desired result, but it is because of its generality that this method may not be the best from the point of view of brevity and simplicity of the transformation involved.

Example 13. Calculate the definite integral: $\int\limits_0^{\pi/2} \frac{dx}{3+5\cos x}$.

Solution. Let's use the substitution $t = tg\frac{x}{2}$, then $x = 2arctgt$ and $dx = \frac{2dt}{1+t^2}$. Therefore $\cos x = \frac{1-t^2}{1+t^2}$ and new limits of integration: if $x_1 = 0$, then $t_1 = tg\frac{0}{2} = tg0 = 0$; if $x_2 = \frac{\pi}{2}$, then $t_2 = tg\frac{\pi}{4} = 1$.

We have

$$\int\limits_0^{\pi/2} \frac{dx}{3+5\cos x} = \int\limits_0^1 \frac{\frac{2dt}{1+t^2}}{3+5\cdot\frac{1-t^2}{1+t^2}} = 2\int\limits_0^1 \frac{dt}{3\left(1+t^2\right)+5\left(1-t^2\right)} = 2\int\limits_0^1 \frac{dt}{8-2t^2} =$$

$$= \int\limits_0^1 \frac{dt}{4-t^2} = -\int\limits_0^1 \frac{dt}{t^2-4} = \int\limits_0^1 \frac{dt}{t^2-2^2} = \frac{1}{2\cdot 2}\ln\left|\frac{t-2}{t+2}\right|\Bigg|_0^1 = \frac{1}{4}\left(\ln\left|-\frac{1}{3}\right| - \ln\left|-\frac{2}{2}\right|\right) =$$

$$= \frac{1}{4}\left(\ln\frac{1}{3} - \ln 1\right) = -\frac{\ln 3}{4}.$$

For a number of special cases, simple change of variables is possible, which can be justified in the following way.

Let the expression $\int R(\sin x, \cos x)dx$ be even with respect to $\sin x$, then the given integral is reduced to an integral of a rational function since $\cos x dx = -d(\sin x) = -dt$ considering in a similar way, we arrive at a conclusion that in the case when the expression $\int R(\sin x, \cos x)dx$ is even with respect to $\cos x$, and the other is odd, the substitution $\cos x = t$ reduces the integral $\int R(\sin x, \cos x)dx$ to an integral of a rational function.

Let's assume that the function $R(\sin x, \cos x)$ possesses one of the following properties:

a) both of them, $\sin x$ and $\cos x$, remain unchanged when $\sin x$ is replaced by $(-\sin x)$ and $\cos x$ by $(-\cos x)$;

b) both of them, $\sin x$ and $\cos x$ change the sign when $\sin x$ is replaced by $(-\sin x)$ and $\cos x$ by $(-\cos x)$. It is sufficient to consider only the first case since the second case can be reduced to the first by multiplying both the numerator and denominator of the rational function $R(\sin x, \cos x)$ by $\sin x$ or $\cos x$. The integral $\int R(\sin x, \cos x)dx$ is reduced to an integral of a rational function by the substitution $t = tgx$ or $t = ctgx$ since $\cos^2 x = \frac{1}{1+t^2}$, $\sin^2 x = \frac{t^2}{1+t^2}$ and $dx = \frac{dt}{1+t^2}$ or $\cos^2 x = \frac{t^2}{1+t^2}$, $\sin^2 x = \frac{1}{1+t^2}$ and $dx = -\frac{dt}{1+t^2}$.

And now consider the integrals of the form

$$\int \sin^m x \cos^n x dx,$$

where m and n are integers. Then

a) if $m > 0$ is odd ($m = 2k - 1$), the substitution $\cos x = t$ immediately reduces the integral to a rational function and uses $\sin^2 x = 1 - \cos^2 x$;

b) if $n > 0$ is odd ($n = 2k - 1$), the substitution $\sin x = t$ yields the same result and uses $\cos^2 x = 1 - \sin^2 x$;

c) if both exponents m and n are positive and even, the result is readily obtained by means of trigonometric transformations with multiple arguments:

$$\sin x \cos x = \frac{1}{2}\sin 2x, \ \sin^2 x = \frac{1}{2}(1 - \cos 2x), \ \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

Example 14. Calculate the definite integral: $\int_0^{\pi/2} \sin^3 x \cos^4 x dx$.

Solution. Here $m=3$, i.e. it's odd. Let's use the substitution $\cos x = t$, then $dt = -\sin x dx$ or $-dt = \sin x dx$ and then let's transform: $\sin^2 x = 1 - \cos^2 x$.

$$\int_0^{\pi/2} \sin^3 x \cos^4 x dx = \int_0^{\pi/2} \sin x \sin^2 x \cos^4 x dx = \int_0^{\pi/2} \sin x \cdot \left(1-\cos^2 x\right) \cdot \cos^4 x dx =$$

$$= \left| \begin{array}{l} t = \cos x,\ dt = \left(\cos x\right)' dx \\ dt = -\sin x dx,\ -dt = \sin x dx \\ t_1 = \cos 0 = 1,\ t_2 = \cos\dfrac{\pi}{2} = 0 \end{array} \right| = \int_1^0 \left(1-t^2\right) \cdot t^4 \cdot \left(-dt\right) = -\int_1^0 \left(t^4 - t^6\right) dt =$$

$$= \int_0^1 \left(t^4 - t^6\right) dt = \left(\frac{t^5}{5} - \frac{t^7}{7}\right)\Bigg|_0^1 = \left(\frac{1^5}{5} - \frac{1^7}{7}\right) - \left(\frac{0^5}{5} - \frac{0^7}{7}\right) = \frac{1}{5} - \frac{1}{7} - 0 = \frac{2}{35}.$$

Example 15. Calculate the definite integral: $\int_0^{\pi/4} \sin^2 x dx$.

Solution. Here $m=2$, i.e. it's even. Let's use the transformation: $\sin^2 x = \frac{1}{2}\left(1-\cos 2x\right)$. Then

$$\int_0^{\pi/4} \sin^2 x dx = \int_0^{\pi/4} \frac{1}{2}\left(1-\cos 2x\right) dx = \frac{1}{2}\int_0^{\pi/4} \left(1-\cos 2x\right) dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\Bigg|_0^{\pi/4} =$$

$$= \frac{1}{2} \cdot \left[\frac{\pi}{4} - \frac{1}{2}\sin\left(2 \cdot \frac{\pi}{4}\right) - \left(0 - \frac{1}{2}\sin\left(2 \cdot 0\right)\right)\right] = \frac{1}{2} \cdot \left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right] = \frac{\pi}{8} - \frac{1}{4}.$$

In conclusion we would like to note that integrals containing products of trigonometric functions of the form $\sin 2x$ and $\cos 2x$ are reduced to integrals of the form $\int \sin ax dx$ and $\int \cos ax dx$ by means of the following formulas:

$$\sin \alpha x \sin \beta x = \frac{1}{2}\left(\cos\left(\alpha - \beta\right)x - \cos\left(\alpha + \beta\right)x\right);$$

$$\sin\alpha x\cos\beta x=\frac{1}{2}\left(\sin\left(\alpha+\beta\right)x+\sin\left(\alpha-\beta\right)x\right);$$

(1)

$$\cos\alpha x\cos\beta x=\frac{1}{2}\left(\cos\left(\alpha+\beta\right)x+\cos\left(\alpha-\beta\right)x\right).$$

Example 16. Calculate the definite integral: $\int\limits_0^{\pi/2}\sin x\cos 3x dx$.

Solution. Let's use the transformation (1) and get:

$$\int\limits_0^{\pi/2}\sin x\cos 3xdx=\int\limits_0^{\pi/2}\frac{1}{2}\left[\sin\left(x+3x\right)+\sin\left(x-3x\right)\right]dx=$$

$$=\frac{1}{2}\int\limits_0^{\pi/2}(\sin 4x-\sin 2x)dx=\frac{1}{2}\left(-\frac{1}{4}\cos 4x+\frac{1}{2}\cos 2x\right)\Bigg|_0^{\pi/2}=\frac{1}{2}\left[-\frac{1}{4}\cos 2\pi+\right.$$

$$\left.+\frac{1}{2}\cos\pi-\left(-\frac{1}{4}\cos 0+\frac{1}{2}\cos 0\right)\right]=\frac{1}{2}\left[-\frac{1}{4}+\frac{1}{2}\cdot(-1)+\frac{1}{4}-\frac{1}{2}\right]=\frac{1}{2}\cdot(-1)=-\frac{1}{2}.$$

Integrals Containing Linear and Linear Fractional Irrationalities

1. Let's consider the indefinite integral of the form

$$\int R\left(x,x^{\frac{m}{n}},...,x^{\frac{r}{s}}\right)dx,$$

where R is a rational function of its arguments.

Let's use the substitution $x=t^k$, where k is a common denominator of the fractions $\frac{m}{n},...,\frac{r}{s}$, the integrand function is reduced to a rational function of t.

Example 17. Calculate the definite integral: $\int\limits_{27}^{64}\frac{dx}{\sqrt[3]{x}-1}$.

Solution. Let's use the substitution $x = t^3$, then $dx = 3t^2 dt$ and new limits of integration are: $t_1 = \sqrt[3]{27} = 3$, $t_2 = \sqrt[3]{64} = 4$.

Thus,

$$\int_{27}^{64} \frac{dx}{\sqrt[3]{x} - 1} = \int_3^4 \frac{3t^2 dt}{t-1} = 3\int_3^4 \frac{t^2}{t-1} dt = 3\int_3^4 \frac{t^2 - 1 + 1}{t-1} dt = 3\int_3^4 \frac{t^2 - 1}{t-1} dt + 3\int_3^4 \frac{dt}{t-1} =$$

$$= 3\int_3^4 (t+1) dt + 3\int_3^4 \frac{dt}{t-1} = 3\left(t^2 + t - \ln|t-1|\right)\Big|_3^4 = 3\Big[4^2 + 4 - \ln 4 -$$

$$-\left(3^2 + 3 - \ln 2\right)\Big] = 3\left[20 - \ln 4 - 12 + \ln 2\right] = 24 - 3\ln 2.$$

2. Let's consider the following integral:

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}} \right) dx.$$

It is reduced to the integral of the rational function with the help of the substitution:

$$\frac{ax+b}{cx+d} = t^k,$$

where k is a common denominator of the fractions $\frac{m}{n}, \ldots, \frac{r}{s}$.

3. Let's consider the following integrals:

a) $\int R\left(x, \sqrt{a^2 - x^2}\right) dx$,

b) $\int R\left(x, \sqrt{a^2 + x^2}\right) dx,$

c) $\int R\left(x, \sqrt{x^2 - a^2}\right) dx.$

Such integrals are founded with the help of substitutions:

a) $x = a \sin t$ or $x = a \cos t$,

b) $x = a\text{tg}t$ or $x = a\text{ctg}t$,

c) $x = \dfrac{a}{\cos t}$ or $x = \dfrac{a}{\sin t}$.

Example 18. Calculate the definite integral: $\int\limits_{\sqrt{3}}^{2} \dfrac{\sqrt{4-x^2}dx}{x}$.

Solution. Let's use the substitution $x = 2\sin t$, then $dx = 2\cos t dt$ and $\dfrac{x}{2} = \sin t$ or $t = \arcsin\dfrac{x}{2}$. Let's find

$$\sqrt{4-x^2} = \sqrt{4-(2\sin t)^2} = \sqrt{4-4\sin^2 t} = \sqrt{4(1-\sin^2 t)} = \sqrt{4\cos^2 t} = 2\cos t.$$

Let's calculate the new limits of integration:

if $x_1 = \sqrt{3}$, then $t_1 = \arcsin\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{3}$;

if $x_2 = 2$, then $t_1 = \arcsin\dfrac{2}{2} = \arcsin 1 = \dfrac{\pi}{2}$.

Thus

$$\int\limits_{\sqrt{3}}^{2} \frac{\sqrt{4-x^2}dx}{x} = \int\limits_{\pi/3}^{\pi/2} \frac{2\cos t \cdot 2\cos t}{2\sin t}dt = 2\int\limits_{\pi/3}^{\pi/2} \frac{\cos^2 t}{\sin t}dt = 2\int\limits_{\pi/3}^{\pi/2} \frac{1-\sin^2 t}{\sin t}dt =$$

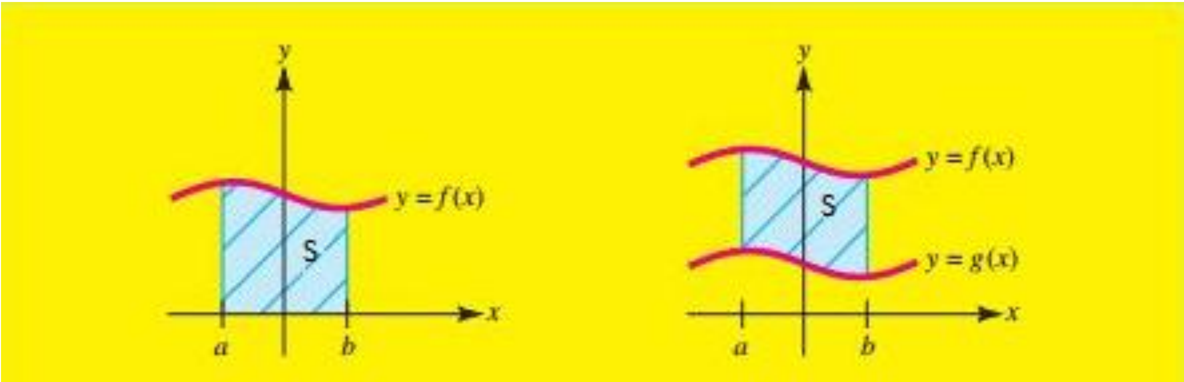
$$= 2\int\limits_{\pi/3}^{\pi/2}\left(\frac{1}{\sin t} - \sin t\right)dt = 2\left(\ln\left|tg\frac{t}{2}\right| + \cos t\right)\Bigg|_{\pi/3}^{\pi/2} = 2\left[\ln\left|tg\frac{\pi}{4}\right| + \cos\frac{\pi}{2} -\right.$$

$$\left.-\left(\ln\left|tg\frac{\pi}{6}\right| + \cos\frac{\pi}{3}\right)\right] = 2\left[\ln 1 + 0 - \left(\ln\frac{\sqrt{3}}{3} + \frac{1}{2}\right)\right] = 2\left[\frac{1}{2}\ln 3 - \frac{1}{2}\right] = \ln 3 - 1.$$

Application of the Definite Integral to Geometry Problems: Calculation of the Area of a Figure

You have already seen the procedure called definite integration in which a quantity of practical interest is first defined as the limit of a sum and then computed using antidifferentiation. Also, you know that area is just one of many quantities that can be expressed as the limit of a sum. In this section, we illustrate the definite integration procedure by using it to compute the area under and above a curve, between two curves. Moreover, you will discover that this procedure also applies to a variety of practical situations in business and economics, life and social sciences.

So, consider the area of the region under the curve $y = f(x)$ above the interval $x = a$, $x = b$, where $f(x) \geq 0$ and $f(x)$ is continuous. This region is shown in Fig. 2, a. Of course, if the region were a square, a triangle, a trapezoid, or part of a circle, we could find its area using well-known formulas, but what if the bounding of an arbitrary curve were $y = f(x)$ or two curves $y = f(x)$ − the upper curve and $y = g(x)$ − the lower curve (Fig. 2, b)?



a) **b)**

Fig. 2. The area under the curve (a) and the area between two curves (b)

In terms of the integral notation, the definition of the area under the curve can be expressed in the following compact form:

a) the region *under the curve* $y = f(x)$, where $f(x) \geq 0$ and $f(x)$ is continuous above $a \leq x \leq b$ (Fig. 2, a) has the area:

$$S = \int_a^b f(x)dx.$$

b) the region *bounded by the graphs of two curves* $y = f(x)$ and $y = g(x)$, where $f(x)$ and $g(x)$ are nonnegative functions such that $f(x) \geq g(x)$ and the vertical lines $x = a$ and $x = b$ (Fig. 2, b) has the area:

$$S = \int_a^b \left(f(x) - g(x) \right) dx.$$

Remark 1. It is easy to prove that if the function $y = f(x)$ is located under the X−axis (Fig. 3), then the area above the curve $y = f(x)$:

$$S = -\int_a^b f(x) dx.$$

Remark 2. If the figure is adjacent to the Y−axis (for example, as in Fig. 4), then the area bounded by the curve $x = \varphi(y)$ and the horizontal lines $y = c$ and $y = d$ is calculated as

$$S = \int_c^d \varphi(y) dy.$$

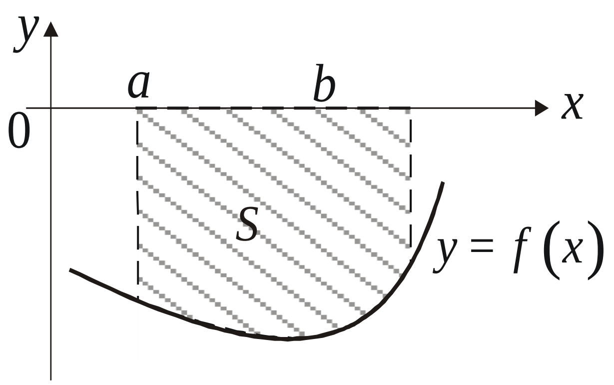


Fig. 3. The area S above the curve

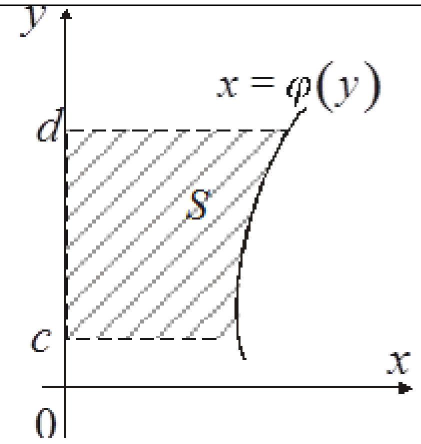


Fig. 4. The area S bounded by the curve $x = \varphi(y)$ and the lines $y = c$ and $y = d$

The following examples show how the area of the region can be computed by definite integration in different cases.

Example 19. Find the area of the region under the line $y = 2x + 1$ above the interval $1 \le x \le 3$.

Solution. First of all, let's make a drawing (see below Fig. 5).

The region under a curve $y = f(x)$ above $a \le x \le b$ has the area

$$S = \int_a^b f(x)dx.$$

As $y = 2x + 1$, $x = a = 1$ and $x = b = 3$ according to the condition of our example, the area is given by the definite integral

$$\int_1^3 (2x+1)dx = \left(x^2 + x\right)\Big|_1^3 = \left(x^2 + x\right)\Big|_{x=3} - \left(x^2 + x\right)\Big|_{x=1} = 3^2 + 3 - 1^2 - 1 = 10.$$

Thus, the area of the region under the line $y = 2x + 1$ above the interval $1 \le x \le 3$ is 10 square units.

Example 20. Find the area of the region under the curve $y = -4x^2 + 100$ above the interval $0 \le x \le 3$.

Solution. Let's make a drawing (Fig. 6).

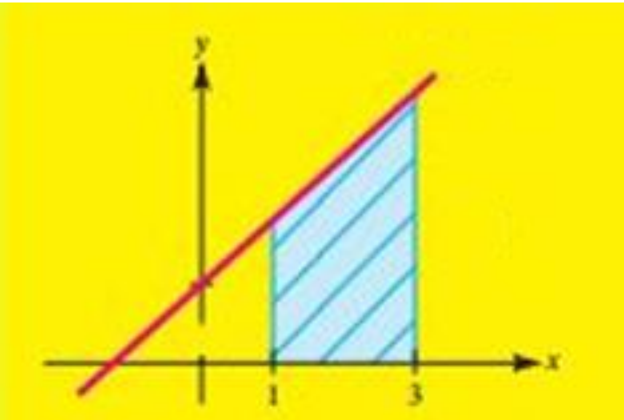


Fig. 5. The region S under the line $y = 2x + 1$ above $1 \le x \le 3$ (a trapezoid)

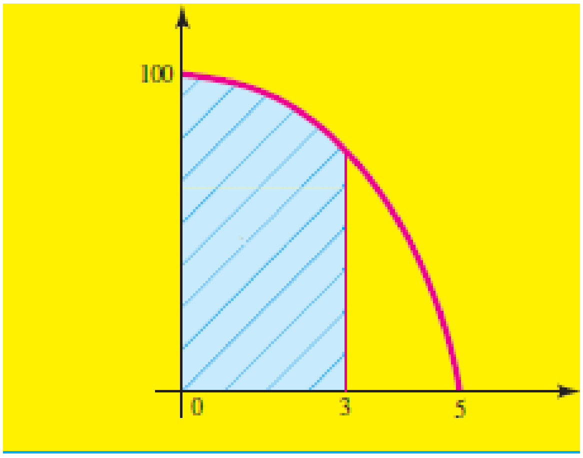


Fig. 6. The region under the curve $y = -4x^2 + 100$ above $0 \le x \le 3$

According to the formula of the area for the region under the curve $y = -4x^2 + 100$ on the given interval $[0, 3]$, we have:

$$\int_0^3 \left(-4x^2+100\right)dx = \int_0^3 4\left(-x^2+25\right)dx = 4\left(-\frac{1}{3}\cdot x^3+25x\right)\Bigg|_0^3 =$$

$$= 4(-9+75) = 4\cdot 66 = 264.$$

So, the area of the region under the curve $y=-4x^2+100$ above the interval $1\le x\le 3$ is 264 square units.

Example 21. Calculate the area of the figure bounded by the curve $y=-4x^2+100$ and X –axis.

Solution. First of all, let's find the limits of integration. In order to determine the vertical lines $x=a$ and $x=b$ we solve equation $y=0$. According to the condition of this task $y=4x-x^2-3$.

Thus, we have a quadric equation

$$4x-x^2-3=0$$

from which it is easy to obtain

$$-x^2+4x-3=0$$

hence

$$x_1=1 \text{ and } x_2=3.$$

Then the vertical lines $x=a=1$ and $x=b=3$ (note here, that $a<b$).

Now let's make a drawing (Fig. 7). Hence,

$$S=\int_a^b f(x)dx = \int_1^3 \left(4x-x^2-3\right)dx = \left(4\cdot\frac{x^2}{2}-\frac{x^3}{3}-3x\right)\Bigg|_1^3 =$$

$$=\left(2x^2-\frac{x^3}{3}-3x\right)\Bigg|_{x=3} - \left(2x^2-\frac{x^3}{3}-3x\right)\Bigg|_{x=1} = (18-9-9)-\left(2-\frac{1}{3}-3\right)=$$

$$=\frac{4}{3}\approx 1.333.$$

Thus, the area of the region of the figure bounded by the curve $y=4x-x^2-3$ and the X –axis is approximately 1.333 square units.

Example 22. Calculate the area of the figure bounded by the curves $y = x^2 + 4x$ and $y = 4 + x$.

Solution. We remember that the region S bounded by the graphs of two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ (it means: $f(x)$ is the upper curve, $g(x)$ is the lower curve) and the vertical lines $x = a$ and $x = b$ has the area:

$$S = \int_a^b \big(f(x) - g(x) \big) dx.$$

In order to find the intersection points of the given curves we solve a system of equations:

$$\begin{cases} y = x^2 + 4x \\ y = x + 4 \end{cases} \quad \text{or} \quad x^2 + 4x = x + 4$$

so, we have a quadratic equation

$$x^2 + 3x - 4 = 0$$

from which it is easy to obtain

$$x_1 = -4, \; x_2 = 1.$$

Let's make a drawing

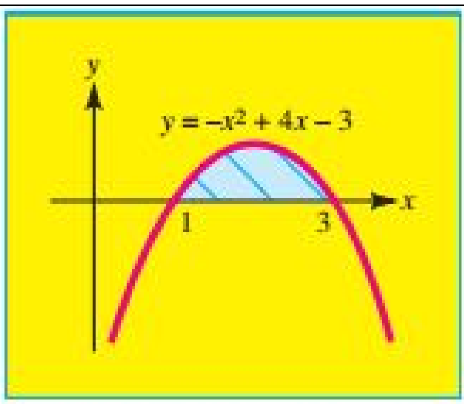
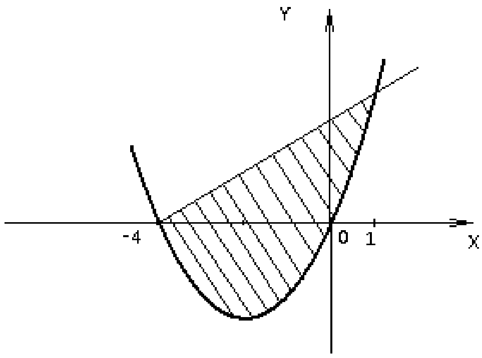
	

Fig. 7. The region bounded by the curve $y = 4x - x^2 - 3$ and the X −axis	**Fig. 8. The region bounded by two curves $y = x^2 + 4x$ and $y = 4 + x$**

Note, $y=4+x$ is the upper curve, $y=x^2+4x$ is the lower curve. According to the formula of area in our case, we obtain

$$\int_{-4}^{1}\left(x+4-x^2-4x\right)dx=\int_{-4}^{1}\left(4-3x-x^2\right)dx=$$

$$=\left(4x-\frac{3x^2}{2}-\frac{x^3}{3}\right)\Bigg|_{-4}^{1}=4-\frac{3}{2}-\frac{1}{2}+16+\frac{48}{2}-\frac{64}{3}=\frac{62}{3}\approx 20.66.$$

Thus, the area of the figure bounded by the curves $y=x^2+4x$ and $y=4+x$ is approximately 20.66 square units.

Example 23. Calculate the area of the figure bounded by the curves $y=3x^2+4$ and $y=4x^2$.

Solution. First of all, let's find the limits of integration. In order to determine a and b we find the intersection points of the given curves by solving a system of equations:

$$\begin{cases} y=3x^2+4 \\ y=4x^2 \end{cases} \quad \text{or} \quad 3x^2+4=4x^2.$$

Thus,

$$x^2-4=0$$

from which it is easy to obtain

$$x_1=2 \text{ and } x_2=-2.$$

Hence the vertical lines $x=a=-2$ and $x=b=2$ (as holds for $a<b$).

Now let's make a drawing (Fig. 9).

Here $y=3x^2+4$ is the upper curve, $y=4x^2$ is the lower curve on $[-2,2]$. According to the formula of area in this case, we have

$$S=\int_{-2}^{2}\left(\left(3x^2+4\right)-4x^2\right)dx=\left|\text{use the symmetry of the integral}\right|=$$

$$= 2\int_0^2 \left(\left(3x^2+4\right) - 4x^2 \right) dx = 2\int_0^2 \left(4 - x^2\right) dx = 2 \cdot \left(4x - \frac{x^3}{3} \right) \Bigg|_0^2 =$$

$$= 2 \cdot \left(8 - \frac{8}{3} \right) = 2 \cdot \frac{16}{3} = \frac{32}{3} \approx 10.66.$$

Thus, the area of the figure bounded by the curves $y = 3x^2 + 4$ and $y = 4x^2$ is approximately 10.66 square units.

Below is an example illustrating the computation of area above the curve located under the X −axis.

Example 24. Find the area of the figure bounded by the curves $y = 4x - x^2$, $y = 0$, $x = 4$ and $x = 5$.

Solution. Let's make a drawing (Fig. 10).

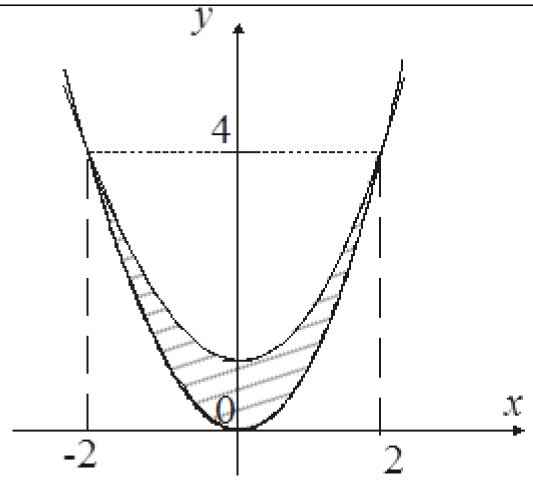


Fig. 9. The region bounded by two curves $y = 3x^2 + 4$ and $y = 4x^2$

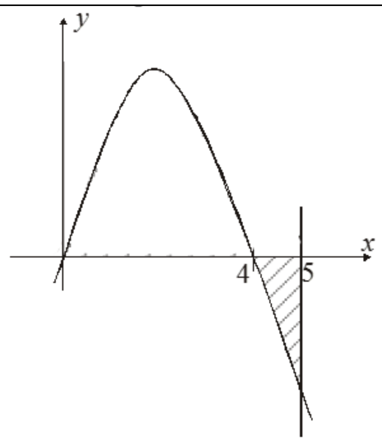


Fig. 10. The region bounded by the curves $y = 4x - x^2$, $y = 0$, $x = 4$, $x = 5$

According to the formula of area for the region above the curve $y = 4x - x^2$, located under $y = 0$ (X −axis) on the given interval $[4, 5]$, we have:

$$S = -\int_a^b f(x)dx = -\int_4^5 \left(4x - x^2\right) dx = -\left(4 \cdot \frac{x^2}{2} - \frac{x^3}{3} \right) \Bigg|_4^5 =$$

$$= -\left(2 \cdot 25 - 2 \cdot 16 - \frac{125}{3} + \frac{64}{3}\right) = \frac{7}{3} \approx 2.33.$$

Thus, the area of the figure bounded by the curves $y = 4x - x^2$, $y = 0$, $x = 4$ and $x = 5$ is approximately 2.33 square units.

The next example illustrates the computation of the area of the figure adjacent to the Y−axis.

Example 25. Calculate the area of the figure bounded by the curves $x = 0$ and $x = 4 - y^2 - 3y$.

Solution. It's clear that $x = 0$ is the Y−axis. Speaking about $x = -y^2 - 3y + 4$ it is parabola. First of all, let's find y_1 and y_2 as the roots of the equation:

$$-y^2 - 3y + 4 = 0,$$

$$y_1 = -4 \text{ and } y_2 = 1.$$

Now let's make a drawing (Fig. 11).

So, we deal with the figure adjacent to the Y−axis. Then

$$S = \int_c^d \varphi(y)dy = \int_{-4}^{1} \left(4 - 3y - y^2\right)dy = \left(4y - 3 \cdot \frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_{-4}^{1} =$$

$$= 4\left(1 - (-4)\right) - \frac{3}{2}\left(1 - (-4)^2\right) - \frac{1}{3}\left(1 - (-4)^3\right) = 4 \cdot 5 - \frac{3}{2}(-15) - \frac{1}{3} \cdot 65 =$$

$$= \frac{125}{6} \approx 20.83.$$

Thus, the area of the figure bounded by the curves $x = 0$ and $x = 4 - y^2 - 3y$ is approximately 20.83 square units.

Example 26. Calculate the area of the figure bounded by the curves $y = 4x - x^2$, $y = 0$ and $x = 5$.

Solution. Let's make a drawing (Fig. 12).

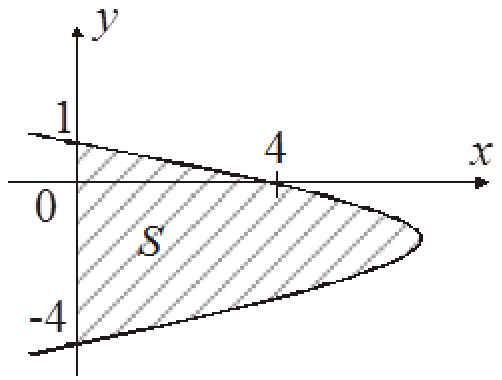
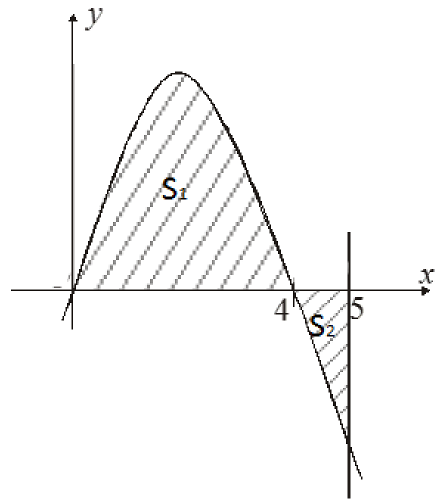
	

Fig. 11. The region bounded by the curves $x=0$ **and** $x=4-y^2-3y$	**Fig. 12. The region bounded by the curves** $y=4x-x^2$, $y=0$ **and** $x=5$

In this case the required area S is $S=S_1+S_2$.

1) Let's calculate the area S_1. The region under the curve $y=f(x)$ above $a\le x\le b$ has the area

$$S_1=\int_a^b f(x)dx.$$

As $f(x)=4x-x^2$, $x=a=0$ and $x=b=4$ according to the condition of our example, the area is given by the definite integral

$$\int_0^4\left(4x-x^2\right)dx=4\cdot\frac{x^2}{2}\bigg|_0^4-\frac{x^3}{3}\bigg|_0^4=2\cdot16-\frac{64}{3}=\frac{32}{3}.$$

2) Let's find the area S_2. Due to the formula of area for the region above a curve $y=4x-x^2$, located under the X –axis on $[4,5]$, we have:

$$S_2=-\int_a^b f(x)dx=-\int_4^5\left(4x-x^2\right)dx=-\left(4\cdot\frac{x^2}{2}-\frac{x^3}{3}\right)\bigg|_4^5=$$

$$=-\left(2\cdot25-2\cdot16-\frac{125}{3}+\frac{64}{3}\right)=\frac{7}{3}.$$

To sum up, $S = S_1 + S_2 = \frac{32}{3} + \frac{7}{3} = 13$.

Thus, the area of the figure bounded by the curves $y = 4x - x^2$, $y = 0$ and $x = 5$ is 13 square units.

Application of the Definite Integral to Geometry Problems: Calculation of the Volumes of Solids of Revolution

The volume of a solid of revolution obtained by rotation of a curvilinear trapezoid, bounded by the curve $y = f(x)$ and the lines $x = a$, $x = b$ and $y = 0$ round the Ox-axis, can be calculated by the formula:

$$V_x = \pi \int_a^b y^2 dx$$

by rotation round the Oy-axis:

$$V_y = 2\pi \int_a^b xy\, dx \quad \text{or} \quad V_y = \pi \int_c^d x^2 dy,$$

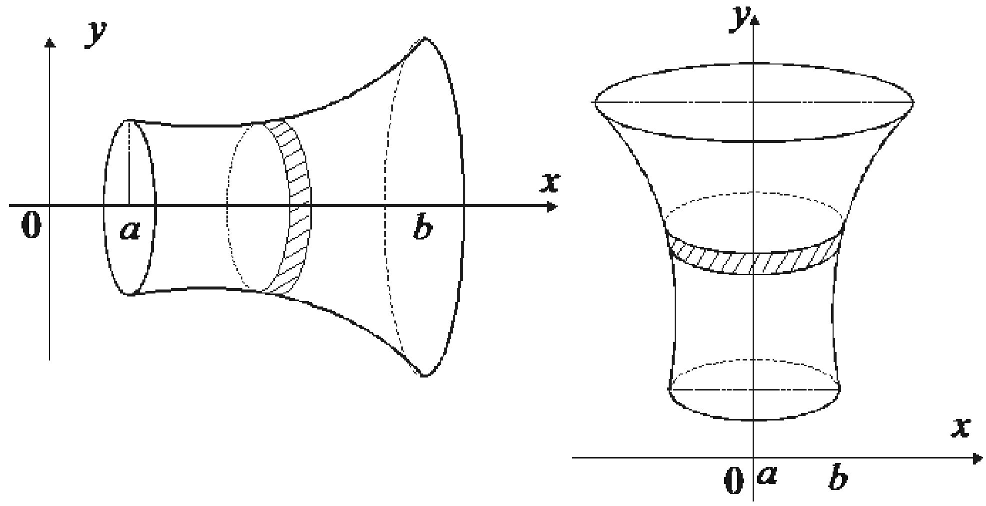
where $x = \varphi(y)$.

The volume of a solid formed by rotation of a figure, bounded by the lines $y_1 = f_1(x)$, $y_2 = f_2(x)$ $(0 \le f_1(x) \le f_2(x))$, $x = a$, $x = b$, round the Ox-axis :

$$V_x = \pi \int_a^b \left(y_2^2 - y_1^2\right) dx = \pi \int_a^b \left(f_2^2(x) - f_1^2(x)\right) dx;$$

by rotation round the Oy-axis:

$$V_y = 2\pi \int_a^b x\left(f_2(x) - f_1(x)\right) dx.$$



a) b)

Fig. 13. The volume of a solid of revolution formed by rotation of a figure, bounded by the curve $y = f(x)$ and the lines $x = a$, $x = b$ and $y = 0$ round the Ox-axis (a) and round the Oy-axis (b)

Example 27. Calculate the volume of a solid of revolution produced by rotation of a figure, bounded by the parabola $y = x^2 + 4$, straight lines $x = 1$, $x = 4$ round the Ox-axis.

Solution. Revolving the shaded area, shown in Fig. 14, 360º about the Ox-axis produces a solid of revolution given by:

$$V_x = \pi\int_1^4 y^2 dx = \pi\int_1^4 \left(x^2 + 4\right)^2 dx = \pi\int_1^4 \left(x^4 + 8x^2 + 16\right)dx = \pi\left(\frac{x^5}{5} + \frac{8x^3}{3} + 16x\right)\Bigg|_1^4 =$$

$$= \pi\left[\left(\frac{4^5}{5} + \frac{8\cdot 4^3}{3} + 16\cdot 4\right) - \left(\frac{1^5}{5} + \frac{8\cdot 1^3}{3} + 16\cdot 1\right)\right] = \pi\left[\left(204{,}6 + \frac{512}{3} + 64\right) - \right.$$

$$\left. - \left(0{,}2 + \frac{8}{3} + 16\right)\right] = \pi\left(204{,}6 + \frac{504}{3} + 48\right) = 420{,}6\pi \text{ (cubic units).}$$

Example 28. Calculate the volume of a solid of revolution produced by rotation of a figure, bounded by the curves $y = x^2$ and $y^2 = 8x$ round the Ox-axis and the Oy-axis.

Solution. Let's find the limits of integration, i.e. solve the system of equations:

$$\begin{cases} y = x^2 \\ y^2 = 8x \end{cases} \Rightarrow \begin{cases} y^2 = x^4 \\ y^2 = 8x \end{cases}.$$

We have

$$x^4 = 8x, \; x^4 - 8x = 0 \quad \text{or} \quad x\left(x^3 - 8\right) = 0.$$

Hence, the points of intersection are $x_1 = 0$, $x_2 = 2$ (Fig. 15).

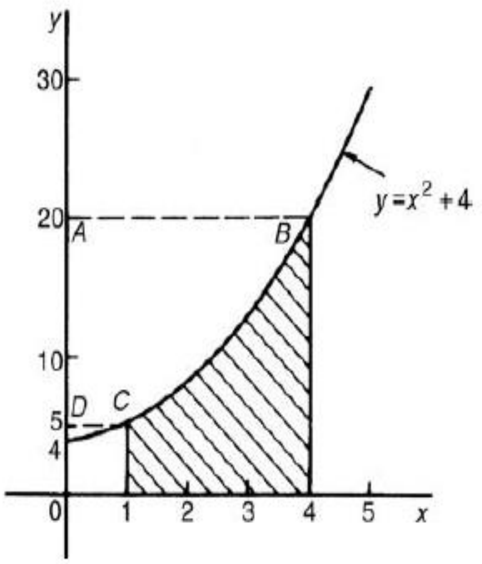


Fig. 14. The volume of a solid of revolution produced by rotation of a figure, bounded by the parabola $y = x^2 + 4$, straight lines $x = 1$, $x = 4$ round the Ox-axis

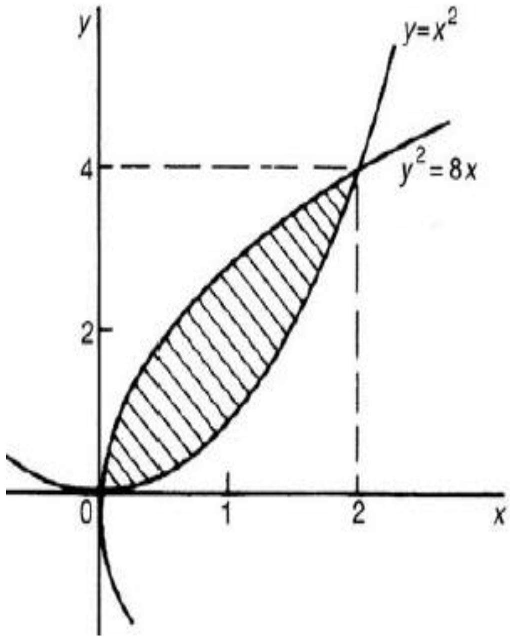


Fig. 15. The volume of a solid of revolution produced by rotation of a figure, bounded by the curves $y = x^2$ and $y^2 = 8x$ round the Ox-axis and the Oy-axis

When $x_1 = 0$, then $y_1 = 0^2 = 0$ and when $x_2 = 2$, then $y_2 = 2^2 = 4$. The points of intersection of the curves $y = x^2$ and $y^2 = 8x$ are therefore at (0; 0) and (2; 4). We have

$$V_x = \pi\int_0^2 \left(y_2^2 - y_1^2\right)dx = \pi\int_0^2 \left(8x - \left(x^2\right)^2\right)dx = \pi\int_0^2 \left(8x - x^4\right)dx = \pi\left(8\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg|_0^2 =$$

$$= \pi\left(4x^2 - \frac{x^5}{5}\right)\Bigg|_0^2 = \pi\left[4\cdot 2^2 - \frac{2^5}{5} - 0\right] = \pi\frac{16\cdot 5 - 32}{5} = \frac{48}{5}\pi \text{ (cubic units).}$$

$$V_y = \pi\int_0^2 x\left(y_2 - y_1\right)dx = \pi\int_0^2 x\left(\sqrt{8x} - x^2\right)dx = \pi\int_0^2 \left(\sqrt{8}x^2 - x^3\right)dx =$$

$$= \pi\left(\sqrt{8}\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^2 = \pi\left[\frac{2\sqrt{2}}{3}\left(2^3 - 0^3\right) - \frac{1}{4}\left(2^4 - 0^4\right)\right] = \pi\frac{64\sqrt{2} - 48}{12} =$$

$$= \pi\frac{16\sqrt{2} - 12}{3} \text{ (cubic units).}$$

Example 29. Calculate the volume of a solid of revolution produced by rotation of a figure, bounded by the straight lines $y = x$ and $y = 2 - x$ round the Oy-axis.

Solution. The initial volume consists of two volumes, produced by rotation of figures, bounded by the straight lines $y = x$ and $y = 2 - x$ round the Oy-axis, correspondingly (Fig. 16).

Let's find the limits of integration, i.e. solve the system of equations:

$$\begin{cases} y = x \\ y = 2 - x \end{cases} \Rightarrow \begin{cases} x = y \\ x = 2 - y \end{cases} \Rightarrow y = 2 - y \Rightarrow 2y = 2 \Rightarrow y = 1;$$

$$\begin{cases} x = 0 \\ y = 2 - x \end{cases} \Rightarrow \begin{cases} x = 0 \\ x = 2 - y \end{cases} \Rightarrow 2 - y = 0 \Rightarrow y = 2;$$

$$\begin{cases} x = 0 \\ y = x \end{cases} \Rightarrow y = 0.$$

Thus, we get:

$$V_y = V_1 + V_2 = \pi\int_0^1 x_1^2 dy + \pi\int_1^2 x_2^2 dy = \pi\int_0^1 y^2 dy + \pi\int_1^2 \left(y - 2\right)^2 dy = \pi\frac{y^3}{3}\Bigg|_0^1 +$$

$$+\pi \cdot \frac{1}{-1} \frac{(2-y)^3}{3}\Bigg|_1^2 = \frac{\pi}{3}\left(1^3-0^3\right)-\frac{\pi}{3}\left(0^3-1^3\right)=\frac{\pi}{3}(1+1)=\frac{2\pi}{3} \text{ (cubic units).}$$

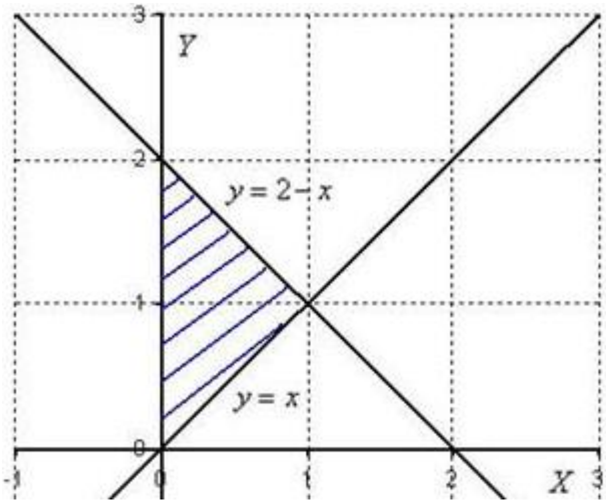


Fig. 16. The volume of a solid of revolution produced by rotation of a figure, bounded by the straight lines $y = x$ and $y = 2 - x$ round the Oy-axis

Applications of the Definite Integral in Business and Economics

The mean value of a function on the interval $[a, b]$:

$$f(c) = \frac{1}{b-a} \cdot \int_a^b f(x)dx.$$

Example 30. Let us assume that the value of the UAH expressed in the US dollars from day 1 to day 30 of a certain month is represented by the function of time: $f(t) = t^2 + \frac{900}{t^3}$. Find the mean value of the UAH in this month.

Solution. First of all, let's find the definite integral:

$$\int_1^{30}\left(x^2+\frac{900}{x^3}\right)dx = \int_1^{30}\left(x^2+900\cdot x^{-3}\right)dx = \left(\frac{1}{3}x^3-\frac{900}{2}x^{-2}\right)\Bigg|_1^{30} = \left(\frac{x^3}{3}-\frac{450}{x^2}\right)\Bigg|_{x=30} -$$

$$-\left(\frac{x^3}{3}-\frac{450}{x^2}\right)\Bigg|_{x=1}=\frac{30^3}{3}-\frac{450}{30^2}-\frac{1^3}{3}+\frac{450}{1^2}=9000-\frac{1}{2}-\frac{1}{3}+450\approx 9449.$$

According to the formula of the mean value of a function on the given interval $[1, 30]$, we have:

$$f(c)=\frac{1}{30-1}\int_1^{30}\left(x^2+\frac{900}{x^3}\right)dx\approx\frac{1}{29}\cdot 9449\approx 325.8.$$

Thus, the mean value of the UAH in this month is approximately 325.8.

Example 31. Let us assume that the production costs $f(x)=10+20x+0.03x^2$ The volume of manufacture changes from 100 UAH to 300 UAH per year. Find the mean value of the cost of production.

Solution. According to the formula of the mean value of a function on the given interval $[100, 300]$, we have:

$$f(c)=\frac{1}{300-100}\cdot\int_{100}^{300}\left(10+20x+0,03x^2\right)dx=\frac{1}{200}\cdot\left(10x+20\cdot\frac{x^2}{2}+0,03\cdot\frac{x^3}{3}\right)\Bigg|_{100}^{300}=$$

$$=\frac{1}{200}\cdot\left[\left(10x+10x^2+0,01x^3\right)\Big|_{x=300}-\left(10x+10x^2+0,01x^3\right)\Big|_{x=100}\right]=$$

$$=\frac{1}{200}\cdot\left(3000+900000+270000-1000-100000-10000\right)=$$

$$=\frac{1}{200}\cdot\left(1173000-120000\right)=5865-600=5265.$$

Thus, the mean value of the cost of production is 5 265 UAH.

Example 32. The value of productivity of equipment at certain factory from the day of start (the first day) to the next day (the second day) is represented by the function of time $f(t)=2^t$. Find the average value of productivity of this equipment.

Solution. a) In order to find the average value of productivity of equipment we will calculate the average value of the given function $f(t)=2^t$ on the interval $[0, 2]$. Due to the formula of the mean value of a function on the interval:

$$f(c) = \frac{1}{2-1}\int_1^2 2^x dx = \frac{2^x}{\ln 2}\Bigg|_1^2 = \frac{2^2}{\ln 2} - \frac{2^1}{\ln 2} = \frac{2}{\ln 2} \approx \frac{2}{0.6931472} \approx 2.88539.$$

Thus, the mean value of the cost of productivity of the given equipment is approximately 2.88539.

The income concentration index – the Gini coefficient:

$$G = 2\int_0^1 \left(x - L(x)\right)dx,$$

where $L(x)$ is the Lorenz curve defined as follows. Let's assume that the vector of incomes $x = (x_1, \ldots, x_n)$ is arranged in a non-decreasing order: $x_1 \le x_2 \le \ldots \le x_n$. The empirical Lorenz function is generated by points, whose first coordinates are numbers $i/n,$ where $i = 0, 1, \ldots, n;$ n is a fixed number, and second coordinates are determined as follows: $L(0) = 0$ and $L\left(\frac{i}{n}\right) = \frac{s_1}{s_n},$ where $s_1 = x_1 + x_2 + \ldots + x_i.$ The Lorenz curve is defined at all points $p \in (0,1)$ through linear interpolation. One can show that $L'(x) > 0$ and $L''(x) > 0,$ $L(0) = 0$ and $L(1) = 1.$

Remark. When the value of the Gini coefficient is close to 0, the underlying distribution is almost uniform, whereas the value close to 1 indicates a maximal inequality, i.e., a total wealth of a population is concentrated in the hands of one man.

Example 33. The Lorenz curve of the income distribution within a certain group is given by the formula: $L(x) = 0.8x^2 + 0.28x.$ Determine the degree of equality of the income distribution.

Solution. Let's find the Gini coefficient

$$G=2\int_0^1\left(x-(0.8x^2+0.28x)\right)dx=2\int_0^1\left(-0.8x^2+0.72x\right)dx=$$

$$=2\left(-0.8\cdot\frac{x^{2+1}}{2+1}+0.72\cdot\frac{x^{1+1}}{1+1}\right)\Bigg|_0^1=2\left(-\frac{4}{15}\cdot x+0.36\cdot x^2\right)\Bigg|_0^1=$$

$$=2\left(-\frac{4}{15}\cdot x+0.36\cdot x^2\right)\Bigg|_{x=1}-2\left(-\frac{4}{15}\cdot x+0.36\cdot x^2\right)\Bigg|_{x=0}=2\left(-\frac{4}{15}+0.36\right)\approx 0.2.$$

As the Gini coefficient is relatively small, we conclude that the given income distribution is fairly uniform.

Example 34. The Lorenz functions L_1 and L_2 of an income in the population of civil servants and teachers of a certain country are given by the formulas: $L_1(x)=x^2$ and $L_2(x)=0.3x^2+0.7x$ respectively. In which group of employees is income distributed more uniformly?

Solution. Let's find the Gini coefficients G_1 and G_2 by the formula

$$G_i=2\int_0^1\left(x-L_i(x)\right)dx,$$

where $L_i(x)$ are given.

Thus,

$$G_1=2\int_0^1\left(x-x^2\right)dx=2\cdot\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Bigg|_0^1=2\cdot\frac{1}{6}=0.3(3)$$

and

$G_2=2\int_0^1\left(x-(0.3x^2+0.7x)\right)dx=2\int_0^1\left(-0.3x^2+0.3x\right)dx=0.1$, accordingly.

Obviously, that $0.1<0.3(3)$, hence we conclude that income is more uniformly distributed in the group of teachers.

The net change

In many practical applications, we are given the rate of change $Q'(x)$ of a quantity $Q(x)$ and required to compute the net change $NC = Q(b) - Q(a)$ in $Q(x)$ as x varies from $x = a$ to $x = b$. But since $Q(x)$ is an antiderivative of $Q'(x)$, the fundamental theorem of calculus tells us that the net change is given by the definite integral

$$NC = Q(b) - Q(a) = \int_a^b Q'(x)dx.$$

Here is an example illustrating the computation of the net change by definite integration.

Example 35. A firm's monthly marginal cost of a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ of the given month if:

a) $MC(x) = x + 30$ and the level of production is raised from 4 units to 6 units;

b) $MC(x) = 3(x-5)^2$ and the level of production is raised from 6 units to 10 units.

Solution. It is well-known that the total cost function is the antiderivative of the marginal cost function:

$$TC(x) = \int MC(x)\,dx.$$

According to the condition of this task the level of production is raised from a units to b units. It means that the total cost function is given by the definite integral

$$TC(b) - TC(a) = \int_a^b MC(x)\,dx.$$

a) As x from $MC(x) = x + 30$ varies from $x = a = 4$ to $x = b = 6$, we obtain

$$TC(6)-TC(4)=\int_4^6 MC(x)\,dx=\int_4^6 (x+30)\,dx=\left(\frac{x^2}{2}+30x\right)\Big|_4^6=$$

$$=\frac{6^2}{2}+30\cdot 6-\left(\frac{4^2}{2}+30\cdot 4\right)=18+180-8-120=70.$$

Thus, the total cost function in the given month is 70 UAH.

b) For $MC(x)=3(x-5)^2$, according to the condition, x varies from $x=a=6$ to $x=b=10$, we obtain

$$TC(10)-TC(6)=\int_6^{10} MC(x)\,dx=\int_6^{10} 3(x-5)^2 dx=(x-5)^3\Big|_6^{10}=$$

$$=(10-5)^3-(6-5)^3=125-1=124.$$

Thus, the total cost function in the given month is 124 UAH.

The net excess profit

Suppose that t years from now, two investment plans will be generating profit $P_1(t)$ and $P_2(t)$, respectively. Assume also that the respective rates of profitability $P_1'(t)$ and $P_2'(t)$ satisfy an inequality $P_2'(t)\geq P_1'(t)$ for the first N years $(0\leq t\leq N)$. Then

$$E(t)=P_2(t)-P_1(t)$$

represents the excess profit of plan 2 over plan 1 at the time t and the net excess profit

$$NE=E(N)-E(0)$$

over the time period $0\leq t\leq N$ is given by the definite integral:

$$NE = E(N) - E(0) = \int_0^N E'(t)dt = \int_0^N \left[P_2'(t) - P_1'(t) \right] dt.$$

Remark. Note here, that the net excess profit NE which referred to above can be interpreted geometrically as the area between the curves $y = P_1'(t)$ and $y = P_2'(t)$ (Fig. 17).

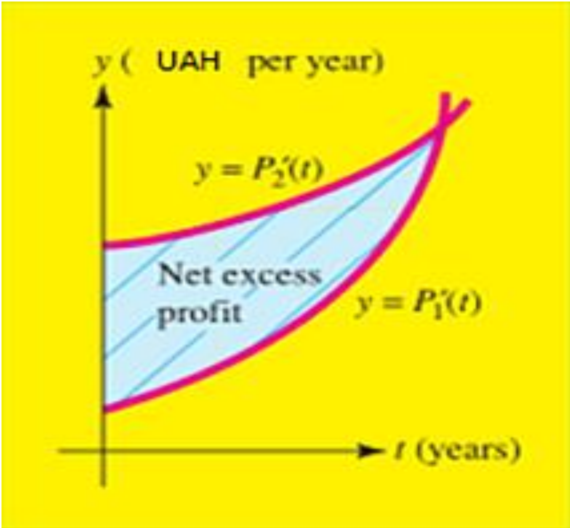


Fig. 17. The net excess profit as the area between two curves

Example 36. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t) = 5 + t^2$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t) = 10t + 80$ UAH per year.

a) For how many years does the rate of profitability of the second investment exceed that of the first?

b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a).

c) Interpret the net excess profit as an area: sketch the rate of profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b).

Solution. a) The rate of profitability of the second investment exceeds that of the first until

$$P_1'(t) = P_2'(t) \quad \text{or} \quad 5 + t^2 = 10t + 80.$$

Thus, we have a quadratic equation

$$t^2 - 10t - 75 = 0$$

from which it is easy to obtain

$$(t - 15)(t + 5) = 0,$$

hence

$$t_1 = 15 \text{ and } t_2 = -5.$$

Clearly, $t < 0$ cannot be the answer. So, we reject $t_2 = -5 < 0$ and accept the root of the quadric equation $t = 15$.

Conclusion: the rate of profitability of the second investment will exceed that of the first after 15 years.

b) The net excess profit for the time period $0 \le t \le 15$ is given by the definite integral

$$NE = \int_0^{15} \left[P_2'(t) - P_1'(t) \right] dt = \int_0^{15} \left[(80 + 10t) - (5 + t^2) \right] dt = \int_0^{15} (75 + 10t - t^2)\, dt =$$

$$= \left(75t + \frac{10}{2} \cdot t^2 - \frac{1}{3} \cdot t^3 \right) \Bigg|_0^{15} = 75 \cdot 15 + 5 \cdot 15^2 - \frac{1}{3} \cdot 15^3 = 1125 + 1125 - 1125 = 1125.$$

Thus, the net excess profit is 1125 UAH.

c) The rate of the profit curves for the two investments is shown in Fig. 18. The net excess profit is the area of the (shaded) region between the curves.

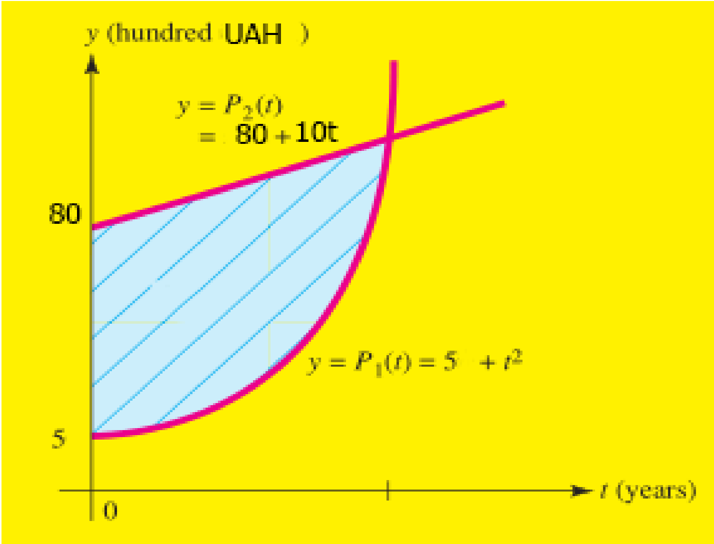


Fig. 18. The net excess profit for Example 34

The net earnings from industrial equipment

The net earnings generated by an industrial machine over a period of time is the difference between the total revenue generated by the machine and the total cost of operating and servicing the machine.

The following example shows how net earnings can be computed by definite integration.

Example 37. Suppose that when it is t years old, a particular industrial machine generates revenue at the rate $R'(t) = 500 - 2t^2$ UAH per year and that operating and servicing costs related to the machine accumulate at the rate $C'(t) = 200 + t^2$ UAH per year.

a) How many years will have pass before the profitability of the machine begins to decline?

b) Compute the net earnings generated by the machine over the time period determined in part a).

c) Interpret the net earnings as an area.

Solution. a) Let's remember the relationship between the revenue and the cost functions:

$$R(t) = P(t) + C(t),$$

where $R(t)$ is the revenue function, $P(t)$ is the profit function and $C(t)$ is the cost function. According to the condition of this task we have some information about the revenue and cost, but are asked about profit associated with the machine after t years of operation, so we will need to use the relationship

$$P(t) = R(t) - C(t).$$

Consequently, the rate of profitability is

$$P'(t) = R'(t) - C'(t) = (500 - 2t^2) - (200 + t^2) = 300 - 3t^2.$$

The profitability begins to decline when $P'(t) = 0$.

Thus, we have a quadric equation

$$300 - 3t^2 = 0,$$

from which it is easy to obtain

$$t^2 = 100,$$

hence

$$t_1 = 10 \text{ and } t_2 = -10.$$

Clearlly, $t < 0$ cannot be the answer. So, we reject $t_2 = -10$ and accept the root $t = 10$.

Conclusion: 10 years will have passed before the profitability of the machine begins to decline.

b) The net earnings NE over the time period $0 \le t \le 10$ is given by the difference

$$NE = P(10) - P(0),$$

which can be computed by the definite integral:

$$NE = P(10) - P(0) = \int_0^{10} P'(t)dt = \int_0^{10} (300 - 3t^2)dt = (300t - t^3)\Big|_0^{10} = 2000.$$

So, the net earnings generated by the machine over the time period $0 \le t \le 10$ are 2 000 UAH.

c) The rate of revenue and the rate of cost curves are sketched in Fig. 19. The net earnings are the area of the (shaded) region between the curves.

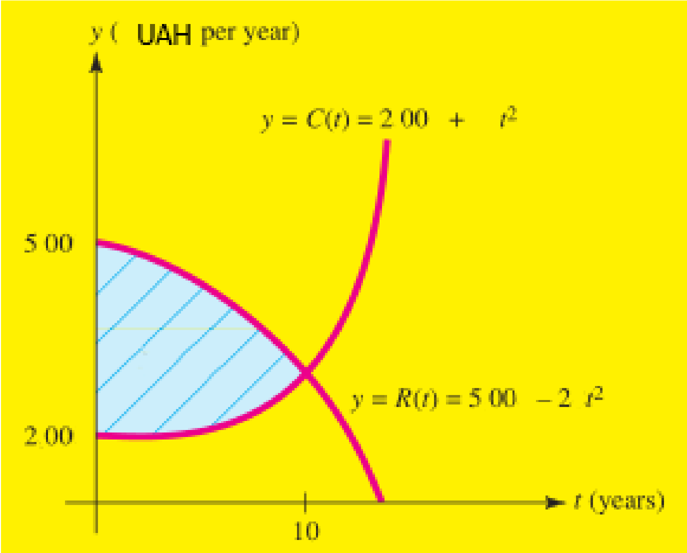


Fig. 19. The net earnings from an industrial machine

The consumers' demand curve and willingness to spend

One of the most fundamental economic models is the law of supply and demand for a certain product (bread, milk, fruit, coffee, chocolate etc.) or service (transportation, education, health care etc.) in a free-market environment. In this model the quantity of a certain item produced and sold is described by two curves called the demand and supply curves of the item.

Note here, that the consumers' demand function $p = D(q)$ can also be thought of as the rate of change with respect to q of the total amount $A(q)$ that consumers are willing to spend on q units; that is, $\frac{dA}{dq} = D(q)$. Integrating, you find that the total amount that consumers are willing to pay for q_0 units of the commodity is given by

$$A(q_0) - A(0) = \int_0^{q_0} \frac{dA}{dq} dq = \int_0^{q_0} D(q)dq.$$

Remark 1. In this context, economists call $A(q)$ the total willingness to spend and $D(q) = A'(q)$ the marginal willingness to spend.

Remark 2. In geometric terms, the total willingness to spend on q_0 units is the area under the demand curve $p = D(q)$ between $q = 0$ and $q = q_0$ (Fig. 20).

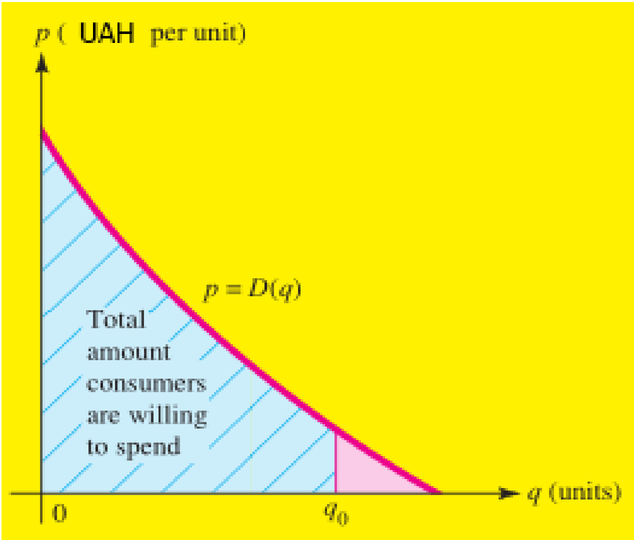


Fig. 20. The amount consumers are willing to spend is the area under the demand curve

Example 39. Suppose that the consumers' demand function for a certain commodity is $D(q) = 25 - q^2$ UAH per unit.

a) Find the total amount of money consumers are willing to spend to get 3 units of the commodity.

b) Sketch the demand curve and interpret the answer to part a) as an area.

Solution. a) Since the demand function $D(q) = 25 - q^2$, measured in UAH per unit, is the rate of change with respect to q of consumers' willingness to spend, the total amount that consumers are willing to spend to get 3 units of the commodity is given by the definite integral

$$\int_0^3 D(q)dq = \int_0^3 \left(25 - q^2\right)dq = \left(25q - \frac{1}{3}\cdot q^3\right)\Bigg|_0^3 = 75 - 9 = 66.$$

So, the total amount of money consumers is willing to spend to get 3 units of the commodity is 66 UAH.

b) The consumers' demand curve is sketched in Fig. 21. In geometric terms, the total amount, 66 UAH, that consumers are willing to spend to get 3 units of the commodity is the area under the demand curve from $q = 0$ to $q = 3$.

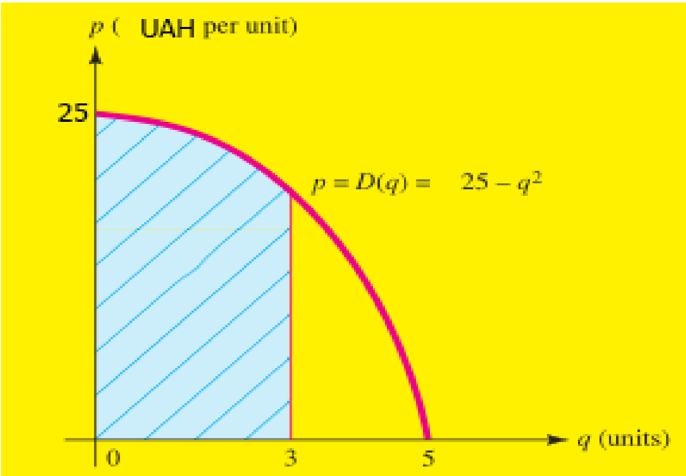


Fig. 21. Consumers' willingness to spend on 3 units when demand is given by $D(q) = 25 - q^2$

Consumers' Surplus

Clearly, that in a competitive economy, the total amount that consumers actually spend on a commodity is usually less than the total amount they would have been willing to spend. The difference between the two amounts can be thought of as savings realized by consumers and is known in economics as the consumers' surplus. That is,

$$\begin{bmatrix} Consumers' \\ surplus \end{bmatrix} = \begin{pmatrix} total\ amount\ consumers \\ would\ be\ willing\ to\ spend \end{pmatrix} - \begin{pmatrix} actual\ consumers' \\ expenditure \end{pmatrix}$$

Market conditions determine the price per unit at which a commodity is sold. Once the price, say p_0, is known, the demand equation $p = D(q)$ determines the number of units q_0 that consumers will buy. The actual consumer expenditure on q_0 units of the commodity at the price of p_0 UAH per unit is $p_0 q_0$ UAH. The consumers' surplus is calculated by subtracting this amount from the total amount consumers would have been willing to spend to get q_0 units of the commodity.

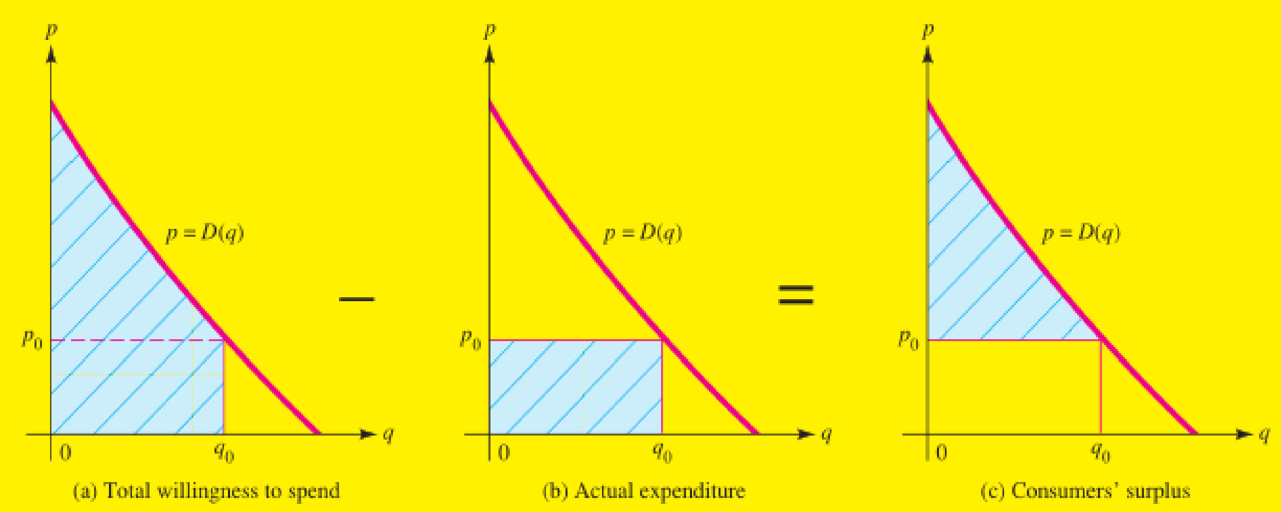


Fig. 22. Geometric interpretation of consumers' surplus

Consumers' surplus has a simple geometric interpretation, which is illustrated in Fig. 22, a, b, c. The symbols p_0 and q_0 denote the market price and corresponding demand, respectively. Fig. 22, a shows the region under the demand curve from $q = 0$ to $q = q_0$. Its area, as we have seen, represents the total amount that consumers are willing to spend to get q_0 units of the commodity.

The rectangle in Fig. 22, b has an area of $p_0 q_0$ and hence represents the actual consumer expenditure on q_0 units at p_0 UAH per unit. The difference between these two areas (Fig. 22, c) represents the consumers' surplus. That is, consumers' surplus CS is the area of the region between the demand curve $p = D(q)$ and the horizontal line $p = p_0$. Hence, if q_0 units of a commodity are sold at a price of p_0 per unit and if $p = D(q)$ is the consumers' demand function for the commodity, then

$$CS = \int_0^{q_0} [D(q) - p_0] dq = \int_0^{q_0} D(q) dq - \int_0^{q_0} p_0 dq = \int_0^{q_0} D(q) dq - p_0 q_0.$$

Thus,

$$CS = \int_0^{q_0} D(q) dq - p_0 q_0 .$$

Producers' Surplus

Producers' surplus is the other side of the coin of consumers' surplus. In particular, the supply function $p = S(q)$ gives the price per unit that producers are willing to accept in order to supply q_0 units to the marketplace. However, any producer who is willing to accept less than $p_0 = S(q_0)$ dollars for q_0 units gains from the fact that the price is p_0. Then producers' surplus is the difference between what producers would be willing to accept for supplying q_0 units and the price they actually receive. Assume that q_0 units of a commodity are sold at a price of p_0 UAH per unit and $p = S(q)$ is the producers' supply function for the commodity, then

$$PS = p_0 q_0 - \int_0^{q_0} S(q)dq.$$

Consumers' surplus has a simple geometric interpretation, which is illustrated in Fig. 23.

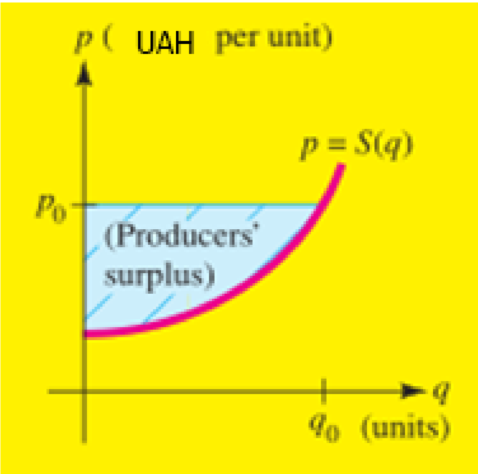


Fig. 23. Geometric interpretation of producers' surplus

Example 40. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is

$$p = D(q) = -q^2 + 116 \text{ UAH per tire}$$

and the same number of tires will be supplied when the price is

$$p = S(q) = 20 + \frac{5}{2} \cdot q \text{ UAH per tire.}$$

a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price.

b) Determine the consumers' surplus at the equilibrium price.

c) Determine the producers' surplus at the equilibrium price.

Solution. a) First of all we will find the equilibrium price. Supply equals demand when

$$-q^2 + 116 = 20 + \frac{5}{3} \cdot q$$

or

$$3q^2 + 5q - 288 = 0.$$

Let's find roots of the quadric equation obtained above: $q_1 = 9$ and $q_2 = -\frac{32}{3}$. Underline here, that $q_2 < 0$ cannot be the answer. Hence we reject $q_2 = -\frac{32}{3}$ and accept the root $q_1 = 9$.

So, under $q_0 = 9$ we have $p_0 = D(9) = 116 - 9^2 = 116 - 81 = 35$. Thus, equilibrium occurs at a price of 35 UAH per tire.

b) Using $p_0 = 35$ and $q_0 = 9$, we find that the consumers' surplus is

$$CS = \int_0^{q_0} D(q)dq - p_0 q_0 = \int_0^9 \left(-q^2 + 116\right)dq - 35 \cdot 9 = \left(-\frac{q^3}{3} + 116q\right)\Bigg|_0^9 - 315 =$$

$$= -\frac{9^3}{3} + 116 \cdot 9 - 315 = -\frac{729}{3} + 1044 - 315 = -243 + 729 = 486.$$

So, the consumers' surplus at the equilibrium price ($p_0 = 35$) is 486 UAH.

c) According to the producers' surplus formula we have

$$PS = p_0 q_0 - \int_0^{q_0} S(q)dq = 35 \cdot 9 - \int_0^9 \left(20 + \frac{5}{2} \cdot q\right)dq = 315 - \left(20 \cdot q + \frac{5}{6} \cdot q^2\right)\Bigg|_0^9 =$$

$$= 315 - 20 \cdot 9 - \frac{5}{6} \cdot 9^2 = 315 - 180 - \frac{405}{6} = 315 - 180 - 67.5 = 67.5.$$

Thus, the producers' surplus at the equilibrium price $p_0 = 35$ is 67.5 UAH.

Volume of production

Let the function $z = f(t)$ describe the changing of the productivity of an enterprise under the time t. Then the volume of production V produced by the time $[t_1, t_2]$ is defined by the formula

$$V = \int_{t_1}^{t_2} f(t)dt.$$

Example 41. Find the volume of production V produced by an employee over the second working hour if the productivity is defined by the function:

$$f(t) = \frac{2}{3t+4} + 3 \text{ (kg)}.$$

Solution. Obviously the second working hour is the time from the first to the second hour. So, due to the formula of the volume of production V produced by the time $[1, 2]$, we have:

$$V = \int_1^2 \left(\frac{2}{3t+4} + 3\right)dt = \left(\frac{2}{3}\ln|3t+4| + 3t\right)\Bigg|_1^2 = \frac{2}{3}\ln\frac{10}{7} + 3 \approx 3.24.$$

Thus, the volume of production is 3.24 kg.

Individual tasks

Variant 1

Find the definite integrals:	
1. $\int_1^8 \left(4x - \frac{1}{3\sqrt[3]{x^2}}\right)dx$	2. $\int_{\sqrt{3}/5}^{3/5} \frac{dx}{9+25x^2}$
3. $\int_{e+1}^{e^2+1} \frac{1+\ln(x-1)}{x-1}dx$	4. $\int_{-3}^{-2} (2x+5)\sin\frac{2\pi x}{3}dx$
5. $\int_{-2}^{0} (x^2+5x+6)\cos 2x\,dx$	6. $\int_{2\,\mathrm{arctg}(1/2)}^{\pi/2} \frac{dx}{(1+\sin x-\cos x)^2}$
7. $\int_0^{\pi/3} \frac{\mathrm{tg}^2 x}{4+3\cos 2x}dx$	8. $\int_{\pi/2}^{\pi} 2^8 \sin^8 x\, dx$
9. $\int_0^1 \frac{x^2 dx}{\sqrt{4-x^2}}$	10. $\int_1^4 \frac{dx}{x^2-2x+10}$

11. Calculate the area of the plane figure bounded by the lines:

$$y=(x-2)^3, \quad y=(x-2)^2$$

12. Find the solid of revolution, formed by rotation round the axis Oy the curvilinear trapezoid, bounded by lines: $y=\frac{(x+2)^2}{2}, \quad y=2$

13. The cost of a product changes at the rate of $f(t)=\frac{3\cdot 6^t+2\cdot 3^t}{3^t}$. Find the average cost of this product if it is known that the level of production is raised from $a=3$ units to $b=5$ units

14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=0.62x^2+0.38x$. Determine the degree of equality of the income distribution

15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=0$ units to

$b = 2\pi$ units for given $MC(x) = 27\pi(1-\cos x)^3$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t) = t+5$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t) = t^2 + 5t$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q) = \dfrac{20}{q+1}$ UAH per tire, and the same number of tires will be supplied when the price is $S(q) = q+2$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 2

Find the definite integrals:	
1. $\int_{-1}^{1}\left(6x^2 - 2x - 5\right)dx$	2. $\int_{-\pi/2}^{\pi/2} \cos 3x \cos 5x\, dx$
3. $\int_{0}^{1} \dfrac{\left(x^2+1\right)dx}{\left(x^3+3x+1\right)^2}$	4. $\int_{-2}^{2}(3x+1)\cos\dfrac{\pi x}{2}dx$
5. $\int_{-1}^{1} x^2 e^{-\frac{x}{2}}\, dx$	6. $\int_{2\,\mathrm{arctg}(1/3)}^{2\,\mathrm{arctg}(1/2)} \dfrac{dx}{\sin x(1-\sin x)}$

7. $\displaystyle\int_{\arccos(1/\sqrt{10})}^{\arccos(1/\sqrt{26})} \frac{12dx}{(6+5\,\mathrm{tg}\,x)\sin 2x}$	8. $\displaystyle\int_0^{\pi} 2^4 \sin^6 x \cos^2 x\, dx$
9. $\displaystyle\int_0^{3/2} \frac{x^2 dx}{\sqrt{9-x^2}}$	10. $\displaystyle\int_0^1 \frac{dx}{\sqrt{3x^2+6x+1}}$
11. Calculate the area of the plane figure bounded by the lines: $y=(x-2)^3, \quad y=\sqrt{4-x}, \quad y=0$	
12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines: $y=\arcsin x, \quad y=\arccos x, \quad y=0$	
13. The cost of a product changes at the rate of $f(t)=\dfrac{\sqrt{t}+t^3e^t+t^2}{t^3}$. Find the average cost of this product if it is known that the level of production is raised from $a=1$ units to $b=2$ units	
14. The Lorenz curve of the distribution within a certain group is given by the following formula $L(x)=0.17x^2+0.83x$. Determine the degree of equality of the income distribution	
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=0$ units to $b=\dfrac{\pi}{2}$ units for given $MC(x)=\dfrac{16}{\pi}\sin^2 x\cos^2 x$	
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=-t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-6$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)	

17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=7-q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=-5+2q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 3

Find the definite integrals:	
1. $\int\limits_{6}^{6\sqrt{3}} \frac{dx}{x^2+36}$	2. $\int\limits_{-\pi/2}^{\pi/2} \sin 2x \sin 7x\, dx$
3. $\int\limits_{0}^{1} \frac{4\operatorname{arctg} x - x}{1+x^2}dx$	4. $\int\limits_{-2}^{0} (3x+2)\sin\frac{\pi x}{8}dx$
5. $\int\limits_{0}^{\pi} \left(9x^2+9x+11\right)\cos 3x\, dx$	6. $\int\limits_{2\operatorname{arctg} 2}^{2\operatorname{arctg} 3} \frac{dx}{\cos x\left(1-\cos x\right)}$
7. $\int\limits_{\pi/4}^{\arcsin\sqrt{2/3}} \frac{8\operatorname{tg} x\, dx}{3\cos^2 x+8\sin 2x-7}$.	8. $\int\limits_{0}^{2\pi} \sin^4 x \cos^4 x\, dx$
9. $\int\limits_{0}^{\sqrt{2}} \frac{x^4 dx}{\left(4-x^2\right)^{3/2}}$	10. $\int\limits_{0}^{1} \frac{dx}{\sqrt{3-x-2x^2}}$
11. Calculate the area of the plane figure bounded by the lines: $y=4-x^2, \quad y=x^2-2$	
12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y=x^2, \quad y=\sqrt[3]{x}$	
13. The cost of a product changes at the rate of $f(t)=e^{t^2}\cdot t$. Find the average cost of this product if it is known that the level of production is raised from $a=2$ units to $b=4$ units	
14. The Lorenz curve of the income distribution within a certain group is given	

by the following formula $L(x)=0.58x^2+0.42x$. Determine the degree of equality of the income distribution
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=0$ units to $b=\frac{\pi}{2}$ units for given $MC(x)=30\sqrt{3}\cos x\sin^2 x$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=-t+12$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b).
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=80-7q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=3q-30$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price.

Variant 4

Find the definite integrals:	
1. $\int\limits_1^{e^2} \frac{2\sqrt{x}+5-7x}{x}dx$	2. $\int\limits_0^{\pi/6} \sin 6x\,dx$
3. $\int\limits_{-0,5}^{0,5} \frac{3^x dx}{1+9^x}$	4. $\int\limits_0^3 (9x+11)\cos\frac{\pi x}{3}dx$

5. $\int_{-1}^{0}\left(x^2+2x+1\right)\sin 3x\,dx$	6. $\int_{0}^{\pi/2}\frac{\cos x-\sin x}{\left(1+\sin x\right)^2}dx$
7. $\int_{-\arcsin\left(2/\sqrt{5}\right)}^{\pi/4}\frac{2-\operatorname{tg}x}{\left(\sin x+3\cos x\right)^2}dx$	8. $\int_{0}^{2\pi}\sin^2\left(x/4\right)\cos^6\left(x/4\right)\,dx$
9. $\int_{0}^{2}\frac{dx}{\left(4+x^2\right)\sqrt{4+x^2}}$	10. $\int_{0}^{1}\frac{dx}{-4x^2+8x+5}$

11. Calculate the area of the plane figure bounded by the lines:

$$y=\sqrt{4-x^2},\quad y=0,\quad x=0,\quad x=1$$

12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines:

$$y=(x-1)^2,\quad x=0,\quad x=2,\quad y=0$$

13. The cost of a product changes at the rate of $f(t)=\frac{7^{\sqrt{t}}}{\sqrt{t}}$. Find the average cost of this product if it is known that the level of production is raised from $a=4$ units to $b=9$ units

14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=0.83x^2+0.17x$. Determine the degree of equality of the income distribution

15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=0$ units to $b=\pi$ units for given $MC(x)=\frac{2\pi x^3}{3}\cdot\sin x$

16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'\left(t\right)=-t+24$ UAH per year, while a second investment will be generating profit at the rate of $P_2'\left(t\right)=t^2+t$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret

the net excess profit as an area: sketch the rate of the profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q) = 100 - 1{,}5q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q) = 20 + 0{,}5q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 5

Find the definite integrals:	
1. $\int_0^{\pi/4} \left(tg^2 x - e^{-x}\right) dx$	2. $\int_{-2}^{0} \frac{dx}{\sqrt{x^2 + 1}}$
3. $\int_{\pi}^{2\pi} \frac{x + \cos x}{x^2 + 2\sin x} dx$	4. $\int_{-4}^{-2} (x - 5)\sin\frac{3\pi x}{8} dx$
5. $\int_0^{2\pi} \left(3 - 7x^2\right)\cos 2x\, dx$	6. $\int_{\pi/2}^{2\,\mathrm{arctg}\,2} \frac{dx}{\sin^2 x(1 - \cos x)}$
7. $\int_0^{\pi/4} \frac{4 - 7\,\mathrm{tg}\,x}{2 + 3\,\mathrm{tg}\,x} dx$	8. $\int_0^{\pi} 2^4 \cos^8(x/2)\, dx$
9. $\int_2^4 \frac{\sqrt{x^2 - 4}}{x^4} dx$	10. $\int_0^1 \frac{dx}{4x^2 + 12x + 9}$
11. Calculate the area of the plane figure bounded by the lines: $y = (x + 1)^2, \quad y^2 = x + 1$	
12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y = x^3, \quad y = x^2$	
13. The cost of a product changes at the rate of $f(t) = t \cdot e^{5t}$. Find the average cost of this product if it is known that the level of production is raised from	

$a=1$ units to $b=2$ units
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=0.97x^2+0.03x$. Determine the degree of equality of the income distribution
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=0$ units to $b=2$ units for given $MC(x)=\dfrac{36x^2}{(1+x^3)^2}$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-30$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=2220-3q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=3q-300$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 6

Find the definite integrals:	
1. $\int_{-4}^{-1/2} \frac{4x^3+2}{x^2}dx$	2. $\int_{3}^{4} \frac{dx}{25-x^2}$

3. $\int_{0}^{1/2} \frac{8x - \operatorname{arctg} 2x}{1+4x^2} dx$	4. $\int_{-3}^{3} (7x+12)\cos\frac{2\pi x}{3} dx$
5. $\int_{\frac{\pi}{4}}^{3} (3x - x^2)\sin 2x \, dx$	6. $\int_{0}^{\pi/2} \frac{\cos x \, dx}{2+\cos x}$
7. $\int_{\pi/4}^{\arccos(1/\sqrt{26})} \frac{dx}{(6-\operatorname{tg} x)\sin 2x}$	8. $\int_{-\pi/2}^{0} 2^8 \sin^8 x \, dx$
9. $\int_{0}^{1} \sqrt{4-x^2} dx$	10. $\int_{0}^{1} \frac{dx}{x^2 - 8x + 25}$
11. Calculate the area of the plane figure bounded by the lines: $$y = 2x - x^2 + 3, \quad y = x^2 - 4x + 3$$	
12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines: $y = x^2 - 2x + 1, \quad x = 2, \quad y = 0$	
13. The cost of a product changes at the rate of $f(t) = (t^3 + 3t^2 - 7)e^{-3t}$. Find the average cost of this product if it is known that the level of production is raised from $a = 0$ units to $b = 1$ units	
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x) = 0.03x^2 + 0.97x$. Determine the degree of equality of the income distribution	
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a = 0$ units to $b = \frac{\pi}{4}$ units for given $MC(x) = 14\cos 2x + 4\pi$	
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t) = 3t + 18$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t) = t^2$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net	

excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=80-30q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=30q-10$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 7

Find the definite integrals:	
1. $\int_1^6 \frac{dx}{\sqrt{3+x}}$	2. $\int_{2\sqrt{3}/3}^{2} \frac{dx}{\sqrt{16-3x^2}}$
3. $\int_0^{\pi/4} \frac{2\cos x+3\sin x}{(2\sin x-3\cos x)^3}dx$	4. $\int_{-1}^{0}(8x+1)\sin\frac{3\pi x}{4}dx$
5. $\int_0^{2\pi}(1-8x^2)\cos 4x\,dx$	6. $\int_{\pi/2}^{2\,\mathrm{arctg}\,2} \frac{dx}{\sin^2 x(1+\cos x)}$
7. $\int_0^{\arcsin 3\sqrt{10}} \frac{2\,\mathrm{tg}\,x-5}{(4\cos x-\sin x)^2}dx$	8. $\int_{\pi/2}^{\pi} 2^4\sin^6 x\cos^2 x\,dx$
9. $\int_3^6 \frac{\sqrt{x^2-9}}{x^4}dx$	10. $\int_0^1 \frac{(x-2)dx}{2x^2+5x+6}$

11. Calculate the area of the plane figure bounded by the lines: $$y^2=x+3,\quad x+2y=5$$
12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $x=\frac{1}{2}y^2,\quad 2x+2y-3=0$

13. The cost of a product changes at the rate of $f(t)=t\cdot 2^t$. Find the average cost of this product if it is known that the level of production is raised from $a=1$ units to $b=4$ units
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=0.69x^2+0.31x$. Determine the degree of equality of the income distribution
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=2$ units to $b=3$ units for given $MC(x)=(2x^2+3x-2)^{-1}$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=2t+8$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-t-20$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=-2q+100$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=q-60$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 8

Find the definite integrals:	
1. $\int_{\pi/12}^{\pi/6} ctg\,2x\,dx$	2. $\int_1^4 \frac{dx}{(1+2x)^2}$

3. $\int\limits_1^4 \frac{1/\left(2\sqrt{x}\right)+1}{\left(\sqrt{x}+x\right)^2}dx$	4. $\int\limits_0^2 (2x+1)\cos\frac{\pi x}{8}dx$
5. $\int\limits_0^3 \left(x^2-3x\right)\sin 2x\,dx$	6. $\int\limits_{2\,\mathrm{arctg}(1/2)}^{\pi/2} \frac{\cos x \infty dx}{(1-\cos x)^3}$
7. $\int\limits_{-\arccos\left(1/\sqrt{5}\right)}^{0} \frac{11-3\,\mathrm{tg}\,x}{\mathrm{tg}\,x+3}dx$	8. $\int\limits_0^{\pi} 2^4 \sin^4 x \cos^4 x\, dx$
9. $\int\limits_0^2 \frac{x^4 dx}{\sqrt{\left(8-x^2\right)^3}}$	10. $\int\limits_0^1 \frac{(-4x+3)\,dx}{\sqrt{4x^2+8x+9}}$

11. Calculate the area of the plane figure bounded by the lines:

$$x=(y-2)^3, \quad x=4y-8$$

12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines:

$$y^2=x-2, \quad y=0, \quad y=x^3, \quad y=1$$

13. The cost of a product changes at the rate of $f(t)=e^{\sqrt{t}}$. Find the average cost of this product if it is known that the level of production is raised from $a=4$ units to $b=16$ units

14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=0.31x^2+0.69x$. Determine the degree of equality of the income distribution

15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=3$ units to $b=8$ units for given $MC(x)=\frac{x}{\sqrt{x+1}}$

16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=5t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-t-16$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that

of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q) = 30 - q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q) = 15 + 2q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 9

Find the definite integrals:	
1. $\int_0^{\pi/4} \sin^2\left(\frac{\pi}{4} - x\right)dx$	2. $\int_4^{4\sqrt{3}} \frac{dx}{\sqrt{64 - x^2}}$
3. $\int_{\ln 2}^{\ln 3} \frac{e^x dx}{\sqrt{e^{2x} - 1}}$	4. $\int_1^3 (15x - 2)\sin\frac{\pi x}{6}\,dx$
5. $\int_0^{\pi} \left(8x^2 + 16x + 17\right)\cos 4x\,dx$	6. $\int_0^{\pi/2} \frac{\cos x\,dx}{5 + 4\cos x}$
7. $\int_0^{\arcsin\sqrt{7/8}} \frac{6\sin^2 x}{4 + 3\cos 2x}dx$	8. $\int_0^{2\pi} \sin^2 x \cos^6 x\,dx$
9. $\int_0^2 \frac{dx}{\sqrt{\left(16 - x^2\right)^3}}$	10. $\int_0^1 \frac{(7 - x)dx}{\sqrt{3 + 2x - x^2}}$
11. Calculate the area of the plane figure bounded by the lines: $x = 4 - y^2, \quad x = y^2 - 2y$	
12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y = (x-1)^2, \quad y = 1$	

13. The cost of a product changes at the rate of $f(t)=\dfrac{1}{t^2+4t+5}$. Find the average cost of this product if it is known that the level of production is raised from $a=9$ units to $b=12$ units
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=0.74x^2+0.26x$. Determine the degree of equality of the income distribution
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=4$ units to $b=9$ units for given $MC(x)=\dfrac{\sqrt{x}}{\sqrt{x}-1}$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=39+4t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-6$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=500-q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=-100+2q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 10

Find the definite integrals:	
1. $\int_1^8 \frac{dx}{\sqrt{17x+8}}$	2. $\int_0^1 \left(6x^2+3\right)dx$
3. $\int_0^{\sqrt{3}} \frac{\operatorname{arctg} x + x}{1+x^2}dx$	4. $\int_{-2}^0 (5x+6)\cos\frac{\pi x}{2}dx$
5. $\int_0^{\frac{\pi}{4}} \left(x^2+17{,}5\right)\sin 2x\,dx$	6. $\int_0^{2\pi/3} \frac{1+\sin x}{1+\cos x+\sin x}dx$
7. $\int_{\pi/4}^{\arcsin\left(2/\sqrt{5}\right)} \frac{4\operatorname{tg} x-5}{4\cos^2 x-\sin 2x+1}dx$	8. $\int_0^{2\pi} \cos^8\left(x/4\right)\,dx$
9. $\int_1^{\sqrt{3}} \frac{dx}{\sqrt{\left(1+x^2\right)^3}}$	10. $\int_0^1 \frac{(3x-5)dx}{\sqrt{x^2+6x+20}}$

11. Calculate the area of the plane figure bounded by the lines:

$$x=\sqrt{4-y^2},\quad x=0,\quad y=0,\quad y=1$$

12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines:

$$y=\sqrt{1-x},\quad y=0,\quad y=1,\quad x=0{,}5$$

13. The cost of a product changes at the rate of $f(t)=\frac{t}{\sqrt{5t+1}}$. Find the average cost of this product if it is known that the level of production is raised from $a=0$ units to $b=3$ units

14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=0.26x^2+0.74x$. Determine the degree of equality of the income distribution

15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=0$ units to

units for given $MC(x)=\sqrt{(1-x^2)^3}$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=10+7t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-2t$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=-3q+21$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=6q-15$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 11

Find the definite integrals:	
1. $\int_{-1}^{1}\frac{4x^3+2}{x^2}dx$	2. $\int_{0}^{\pi/8}\sin 4x\,dx$
3. $\int_{\sqrt{3}}^{\sqrt{8}}\frac{x+1/x}{\sqrt{x^2+1}}dx$	4. $\int_{1}^{2}(6x-5)\sin\frac{\pi x}{6}dx$
5. $\int_{0}^{2\pi}\left(2x^2-15\right)\cos 3x\,dx$	6. $\int_{\pi/3}^{\pi/2}\frac{\cos x\,dx}{1+\sin x-\cos x}$
7. $\int_{0}^{\pi/4}\frac{5\operatorname{tg}x+2}{2\sin 2x+5}dx$	8. $\int_{0}^{\pi}2^4\sin^8\left(x/2\right)\,dx$

9. $\int_{-3}^{3} x^2\sqrt{9-x^2}\,dx$	10. $\int_{0}^{1} \frac{(6x-5)\,dx}{\sqrt{2x^2-12x+15}}$
11. Calculate the area of the plane figure bounded by the lines: $y=(x-1)^2, \quad y^2=x-1$	
12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y=x^3+2, \quad y=x^2+2$	
13. The cost of a product changes at the rate of $f(t)=\sqrt{9-t^2}\cdot t^2$. Find the average cost of this product if it is known that the level of production is raised from $a=-3$ units to $b=3$ units	
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=0.48x^2+0.52x$. Determine the degree of equality of the income distribution	
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=0$ units to $b=1$ units for given $MC(x)=\sqrt{x}\cdot(1+x)^{-1}$	
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=t+21$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-8t-1$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)	
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=8-0{,}7q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=0{,}3q-10$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the	

producers' surplus at the equilibrium price

Variant 12

Find the definite integrals:	
1. $\int_{-1}^{1} \frac{x^3 - x^2}{x^2} dx$	2. $\int_{-1}^{0} e^{-2x} dx$
3. $\int_{0}^{\sqrt{3}} \frac{x - (\operatorname{arctg} x)^4}{1 + x^2} dx$	4. $\int_{-2}^{2} (3x+1)\cos\frac{\pi x}{2} dx$
5. $\int_{0}^{\frac{\pi}{2}} (x^2 - 5x + 6)\sin 3x\, dx$	6. $\int_{0}^{\pi/2} \frac{(1+\cos x)dx}{1+\sin x + \cos x}$
7. $\int_{-\arccos(1/\sqrt{10})}^{0} \frac{3\operatorname{tg}^2 x - 50}{2\operatorname{tg} x + 7} dx$	8. $\int_{-\pi}^{0} 2^8 \sin^6 x \cos^2 x\, dx$
9. $\int_{0}^{2\sqrt{2}} \frac{x^4 dx}{(16 - x^2)\sqrt{16 - x^2}}$	10. $\int_{0}^{1} \frac{(8x+3)dx}{\sqrt{27 + 12x - 4x^2}}$
11. Calculate the area of the plane figure bounded by the lines: $x = 4 - (y-1)^2, \quad x = y^2 - 4y + 3$	
12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines: $y = x^2 + 1, \quad y = x, \quad x = 0, \quad x = 1$	
13. The cost of a product changes at the rate of $f(t) = t^2 \cdot e^{-t}$. Find the average cost of this product if it is known that the level of production is raised from $a = -1$ units to $b = 1$ units	
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x) = 0.52x^2 + 0.48x$. Determine the degree of equality of the income distribution	
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function	

$TC(x)$ for the month if the level of production is raised from $a=\frac{\pi}{4}$ units to $b=\frac{\pi}{3}$ units for given $MC(x)=x\sin^{-2}x$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=60$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-7t$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=-3q+200$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=q+40$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 13

Find the definite integrals:	
1. $\int_1^2 \left(\sqrt[3]{x}-\frac{1}{\sqrt[3]{x}}\right)^3 dx$	2. $\int_0^{\pi/12} \cos 2x\, dx$
3. $\int_{-\ln\sqrt{3}}^{\ln\sqrt{3}} \frac{dx}{e^x+e^{-x}}$	4. $\int_1^3 (7x+4)\sin\frac{2\pi x}{3}\, dx$
5. $\int_0^{2\pi} (3x^2+5)\cos 2x\, dx$	6. $\int_0^{\pi/2} \frac{\sin x\, dx}{1+\sin x+\cos x}$

7. $\int\limits_{\arcsin(2/\sqrt{5})}^{\arcsin(3/\sqrt{10})} \frac{2\operatorname{tg} x+5}{(5-\operatorname{tg} x)\sin 2x}dx$	8. $\int\limits_{\pi/2}^{2\pi} 2^8 \sin^4 x\cos^4 x\, dx$
9. $\int\limits_{\sqrt{2}}^{2\sqrt{2}} \frac{\sqrt{x^2-2}}{x^4}dx$	10. $\int\limits_0^1 \frac{(3-5x)dx}{4x^2+16x-9}$
11. Calculate the area of the plane figure bounded by the lines: $x=(y+1)^2, \quad x^2=y+1$	
12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y=\sin x, \quad x=0, \quad y=0, \quad x=\frac{\pi}{2}$	
13. The cost of a product changes at the rate of $f(t)=t^3\sqrt{t^2-1}$. Find the average cost of this product if it is known that the level of production is raised from $a=1$ units to $b=3$ units	
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=0.78x^2+0.22x$. Determine the degree of equality of the income distribution	
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=0$ units to $b=\pi$ units for given $MC(x)=x^3\sin x$	
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=-t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-182$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)	
17. A tire manufacturer estimates that q (thousand) radial tires will be pur-	

chased (demanded) by wholesalers when the price is $D(q)=370-0{,}5q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=-50+0{,}5q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 14

Find the definite integrals:	
1. $\int_2^5 \left(3x^2-\frac{7}{x}\right)dx$	2. $\int_0^{\pi} \sin\frac{3x}{2}\cos\frac{x}{2}\,dx$
3. $\int_2^9 \frac{x\,dx}{\sqrt[3]{x-1}}$	4. $\int_{-3}^{3} (7x+12)\cos\frac{2\pi x}{3}\,dx$
5. $\int_0^1 x^2 e^{3x}\,dx$	6. $\int_0^{2\,\mathrm{arctg}(1/2)} \frac{1+\sin x}{(1-\sin x)^2}\,dx$
7. $\int_0^{\pi/4} \frac{7+3\,\mathrm{tg}\,x}{(\sin x+2\cos x)^2}\,dx$	8. $\int_0^{\pi} 2^4\sin^2 x\cos^6 x\,dx$
9. $\int_0^{4\sqrt{3}} \frac{dx}{\sqrt{(64-x^2)^3}}$	10. $\int_0^1 \frac{(12x+11)\,dx}{9x^2-6x+2}$
11. Calculate the area of the plane figure bounded by the lines: $$y=x^2-9,\quad y=-x^2+4x-3$$	
12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines: $$y=\arccos\frac{x}{3},\quad y=\arccos x,\quad y=0$$	
13. The cost of a product changes at the rate of $f(t)=\ln t$. Find the average cost of this product if it is known that the level of production is raised from $a=1$ units to $b=e$ units	
14. The Lorenz curve of the income distribution within a certain group is given	

by the following formula $L(x)=0.38x^2+0.62x$. Determine the degree of equality of the income distribution
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=4$ units to $b=9$ units for given $MC(x)=(\sqrt{x}+1)^{-1}\cdot(x-1)$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=2t+7$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-5t-91$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=2400-900q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=-300+900q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 15

Find the definite integrals:	
1. $\int_1^2 \left(\sqrt[3]{x}+\frac{1}{\sqrt[3]{x}}\right)^3 dx$	2. $\int_1^2 \frac{dx}{2x-1}$
3. $\int_0^{\sqrt{3}} \frac{x-(\operatorname{arctg} x)^4}{1+x^2} dx$	4. $\int_2^3 (3x+2)\sin\frac{\pi x}{6} dx$

5. $\int\limits_0^{\pi}\left(2x^2+4x+7\right)\cos 2x\,dx$	6. $\int\limits_0^{\pi/2}\frac{\cos x\,dx}{1+\sin x+\cos x}$
7. $\int\limits_0^{\operatorname{arctg}(2/3)}\frac{6+\operatorname{tg}x}{9\sin^2 x+4\cos^2 x}dx$	8. $\int\limits_0^{2\pi}\cos^8 x\,dx$
9. $\int\limits_0^{4}\sqrt{16-x^2}\,dx$	10. $\int\limits_0^{1}\frac{(6x-1)\,dx}{x^2-4x+13}$

11. Calculate the area of the plane figure bounded by the lines:

$$y=x^2,\quad y=(x-2)^2,\quad y=0$$

12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y=x^2,\quad x=2,\quad y=0$

13. The cost of a product changes at the rate of $f(t)=\sqrt{1+\frac{(t-1)^2}{4t}}$. Find the average cost of this product if it is known that the level of production is raised from $a=0$ units to $b=4$ units

14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=0.42x^2+0.58x$. Determine the degree of equality of the income distribution

15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=1$ units to $b=2$ units for given $MC(x)=e^{\frac{1}{x}}\cdot x^2$

16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=7t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-120$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area repre-

sents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=10-0,2q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=-6+0,1q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 16

Find the definite integrals:	
1. $\int_0^{1/3} \frac{dx}{\sqrt{4-9x^2}}$	2. $\int_0^{\ln 3} e^{2x} dx$
3. $\int_{\sqrt{3}}^{\sqrt{8}} \frac{x-1/x}{\sqrt{x^2+1}} dx$	4. $\int_0^3 (9x+11)\cos\frac{\pi x}{3} dx$
5. $\int_{-3}^0 \left(x^2+6x+9\right)\sin 2x\, dx$	6. $\int_0^{2\operatorname{arctg}(1/3)} \frac{\cos x\, dx}{(1-\sin x)(1+\cos x)}$
7. $\int_0^{\operatorname{arctg}(2/3)} \frac{6+\operatorname{tg} x}{9\sin^2 x+4\cos^2 x} dx$	8. $\int_0^{2\pi} \sin^8(x/4)\, dx$
9. $\int_0^5 x^2\sqrt{25-x^2}\, dx$	10. $\int_0^1 \frac{(x-4)dx}{x^2+x-12}$
11. Calculate the area of the plane figure bounded by the lines: $y=\ln x,\quad x=e^2,\quad y=0$	
12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines: $y=1-x^2,\quad x=0,\quad x=\sqrt{y-2},\quad x=1$	
13. The cost of a product changes at the rate of $f(t)=8\sqrt{t}-2t\sqrt{t}+e^{3t}+5^2$. Find the average cost of this product if it is known that the level of production is raised from $a=2$ units to $b=4$ units	
14. The Lorenz curve of the income distribution within a certain group is given	

by the following formula $L(x)=\frac{7}{15}x^2+\frac{8}{15}x$. Determine the degree of equality of the income distribution	
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=0$ units to $b=1$ units for given $MC(x)=e^x\cdot(e^x-1)^4$	
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=-t+1$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2+3t-4$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)	
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=-q+15$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=q+7{,}5$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price	

Variant 17

Find the definite integrals:	
1. $\int_0^{\pi/4}\frac{x^2dx}{x^2+1}$	2. $\int_2^3 2^{4x+1}dx$
3. $\int_0^{\sin 1}\frac{(\arcsin x)^2+1}{\sqrt{1-x^2}}dx$	4. $\int_{-2}^0(2x-1)\sin\frac{\pi x}{8}dx$

5. $\int_{-2}^{0}(x+2)^2\cos 3x\,dx$	6. $\int_{-2\pi/3}^{0}\frac{\cos x\,dx}{1+\cos x-\sin x}$
7. $\int_{0}^{\operatorname{arctg}2}\frac{12+\operatorname{tg}x}{3\sin^2 x+12\cos^2 x}dx$	8. $\int_{0}^{\pi}2^4\sin^6(x/2)\cos^2(x/2)\;dx$
9. $\int_{0}^{5/2}\frac{x^2dx}{\sqrt{25-x^2}}$	10. $\int_{0}^{1}\frac{dx}{\sqrt{50x-25x^2-9}}$

11. Calculate the area of the plane figure bounded by the lines:

$$y=\sin x,\quad x=\pi,\quad x=\frac{3\pi}{2},\quad y=0$$

12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y=x^3,\quad y=\sqrt{x}$

13. The cost of a product changes at the rate of $f(t)=(1-\sin t)^{\frac{3}{2}}\cdot\cos t$. Find the average cost of this product if it is known that the level of production is raised from $a=-\frac{\pi}{2}$ units to $b=\frac{\pi}{2}$ units

14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=\frac{8}{15}x^2+\frac{7}{15}x$. Determine the degree of equality of the income distribution

15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=\frac{\pi}{6}$ units to $b=\frac{\pi}{4}$ units for given $MC(x)=\cos^{-2}x$

16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=5$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2+t-1$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you in-

vest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q) = 10(q+1)^{-1}$ UAH per tire, and the same number of tires will be supplied when the price is $S(q) = 1 + 0{,}5q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 18

Find the definite integrals:	
1. $\int_1^2 \left(\sqrt[4]{x} + \frac{1}{\sqrt{x}} \right)^2 dx$	2. $\int_{\pi/6}^{\pi/4} \frac{dx}{\sin^2 3x}$
3. $\int_1^3 \frac{1-\sqrt{x}}{\sqrt{x}(x+1)} dx$	4. $\int_0^2 (2x+1)\cos\frac{\pi x}{8} dx$
5. $\int_{-2}^0 \left(x^2+2\right)e^{\frac{x}{2}} dx$	6. $\int_{-\pi/2}^0 \frac{\cos x\, dx}{(1+\cos x - \sin x)^2}$
7. $\int_0^{\operatorname{arctg} 3} \frac{4+\operatorname{tg} x}{2\sin^2 x + 18\cos^2 x} dx$	8. $\int_{-\pi/2}^0 2^8 \sin^4 x \cos^4 x\, dx$
9. $\int_0^4 x^2\sqrt{16-x^2}\, dx$	10. $\int_0^1 \frac{dx}{4x^2-16x-9}$
11. Calculate the area of the plane figure bounded by the lines: $xy = 1, \quad x = 0, \quad x = e^2, \quad y = 0, \quad y = e$	
12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines: $y = 2x - x^2, \quad y = -x + 2, \quad x = 0$	

13. The cost of a product changes at the rate of $f(t)=\left(t^4+4t^3+50\right)e^{2t}$. Find the average cost of this product if it is known that the level of production is raised from $a=1$ units to $b=4$ units
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=\frac{3}{13}x^2+\frac{10}{13}x$. Determine the degree of equality of the income distribution
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=2$ units to $b=5$ units for given $MC(x)=(4x+3)(x-2)^{-3}$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=2-t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-10$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=35-5q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=10q-25$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 19

Find the definite integrals:	
1. $\int_1^2 \frac{\sqrt{x^2-x}+\sqrt{x^2+x}}{\sqrt{x}}dx$	2. $\int_1^2 3^{1-2x}dx$
3. $\int_{\sqrt{3}}^{\sqrt{8}} \frac{dx}{x\sqrt{x^2+1}}$	4. $\int_{-3}^{0}(7x+12)\sin\frac{2\pi x}{3}dx$
5. $\int_{-2}^{0}(x^2-4)\cos 3x\,dx$	6. $\int_0^{\pi/2}\frac{\cos x\,dx}{(1+\cos x+\sin x)^2}$
7. $\int_0^{\pi/4}\frac{6\sin^2 x}{3\cos 2x-4}dx$	8. $\int_{\pi/2}^{\pi} 2^8\sin^2 x\cos^6 x\,dx$
9. $\int_0^4 \frac{dx}{(16+x^2)^{3/2}}$	10. $\int_{-2,5}^{-1,5}\frac{dx}{\sqrt{-x^2-4x-3}}$
11. Calculate the area of the plane figure bounded by the lines: $y=e^{-x},\quad y=1+\sqrt{x},\quad x=2$	
12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y=x^2,\quad y=1,\quad x=2$	
13. The cost of a product changes at the rate of $f(t)=e^t(2t-1)$. Find the average cost of this product if it is known that the level of production is raised from $a=1$ units to $b=2$ units	
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=\frac{10}{13}x^2+\frac{3}{13}x$. Determine the degree of equality of the income distribution	
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=1$ units to $b=4$ units for given $MC(x)=\frac{8x-11}{\sqrt{2x-x^2+5}}$	
16. Suppose that t years from now, one investment will be generating profit	

at the rate of $P_1'(t)=4-t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2+t-20$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=(q+1)^{-1}\cdot 100$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=10+5q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 20

Find the definite integrals:	
1. $\int\limits_1^{3/2} \frac{\sqrt{2x-1}\,dx}{\sqrt[4]{2x-1}}$	2. $\int\limits_0^1 \frac{2^x+1}{5^x}dx$
3. $\int\limits_1^e \frac{1+\ln x}{x}dx$	4. $\int\limits_{-3}^{-2} (11x+9)\cos\frac{\pi x}{3}dx$
5. $\int\limits_1^e \sqrt{x}\ln^2 x\,dx$	6. $\int\limits_0^{2\,\mathrm{arctg}(1/2)} \frac{(1-\sin x)dx}{\cos x(1+\cos x)}$
7. $\int\limits_{\pi/4}^{\arccos(1/\sqrt{3})} \frac{\mathrm{tg}\,x}{\sin^2 x-5\cos^2 x+4}dx$	8. $\int\limits_0^{\pi} 2^4\cos^8 x\,dx$
9. $\int\limits_0^2 \sqrt{4-x^2}dx$	10. $\int\limits_4^6 \frac{dx}{-x^2+8x-5}$
11. Calculate the area of the plane figure bounded by the lines:	

$y=(x+1)^2, \quad y=1-x$.
12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines: $y=e^{1-x}, \quad y=0, \quad x=0, \quad x=1$
13. The cost of a product changes at the rate of $f(t)=\sqrt{49-t^2}$. Find the average cost of this product if it is known that the level of production is raised from $a=0$ units to $b=7$ units
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=\frac{2}{9}x^2+\frac{7}{9}x$. Determine the degree of equality of the income distribution
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=\frac{\pi}{3}$ units to $b=2\pi$ units for given $MC(x)=\sin^4 x$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=-t+30$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=222-0{,}3q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=0{,}3q-30$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 21

Find the definite integrals:	
1. $\int_0^1 \left(\sqrt[6]{x}+\sqrt{2x}\right)^2 dx$	2. $\int_0^{\sqrt{2}} \frac{dx}{8-x^2}$
3. $\int_0^1 \frac{5^{\operatorname{arctg} x}}{1+x^2} dx$	4. $\int_0^{1,5} (2x-3)\sin \pi x\, dx$
5. $\int_0^{3/2} \frac{\arccos x/3}{\sqrt{3-x}} dx$	6. $\int_0^{\pi/2} \frac{\sin x\, dx}{(1+\sin x)^2}$
7. $\int_{\pi/4}^{\operatorname{arctg} 3} \frac{1+\operatorname{ctg} x}{(\sin x+2\cos x)^2} dx$	8. $\int_0^{2\pi} \sin^8 x\, dx$
9. $\int_0^2 \frac{x^2 dx}{\sqrt{16-x^2}}$	10. $\int_{-2}^0 \frac{dx}{\sqrt{-x^2-2x+3}}$
11. Calculate the area of the plane figure bounded by the lines: $x=y^2, \quad x=6-5y^2$	
12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y=2x-x^2, \quad y=-x+2$	
13. The cost of a product changes at the rate of $f(t)=\ln(t+1)$. Find the average cost of this product if it is known that the level of production is raised from $a=0$ units to $b=e-1$ units	
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=\frac{7}{9}x^2+\frac{2}{9}x$. Determine the degree of equality of the income distribution	
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=1$ units to $b=64$ units for given $MC(x)=\frac{1}{\sqrt{x}+\sqrt[3]{x}}$	
16. Suppose that t years from now, one investment will be generating profit	

at the rate of $P_1'(t) = t + 18$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t) = t^2 - 2t$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q) = 74 - q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q) = -10 + q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 22

Find the definite integrals:	
1. $\int_0^1 \left(\sqrt[3]{x} - \sqrt{x}\right)^2 dx$	2. $\int_1^2 \frac{2^x + 5^x}{10^x} dx$
3. $\int_0^1 \frac{x^3 dx}{\left(x^2+1\right)^2}$	4. $\int_{-1}^{0,5} (2-4x)\cos\frac{3\pi x}{2} dx$
5. $\int_{-2}^{0} \frac{\arcsin x/2}{\sqrt{2-x}} dx$	6. $\int_0^{\pi/2} \frac{\sin x dx}{\left(1+\cos x+\sin x\right)^2}$
7. $\int_{-\mathrm{arctg}(1/3)}^{0} \frac{3\,\mathrm{tg}\, x+1}{2\sin 2x - 5\cos 2x + 1} dx$	8. $\int_0^{2\pi} \sin^6(x/4)\cos^2(x/4)\, dx$
9. $\int_0^1 \frac{x^4 dx}{\left(2-x^2\right)^{3/2}}$	10. $\int_4^{14} \frac{2\,dx}{x^2 - 4x + 8}$
11. Calculate the area of the plane figure bounded by the lines:	

$$y=\cos x, \quad x=-\frac{\pi}{4}, \quad x=\frac{\pi}{4}, \quad y=0$$
12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines: $x=\sqrt[3]{y}-2, \quad x=1, \quad y=1$
13. The cost of a product changes at the rate of $f(t)=12t^{-0,5}$. Find the average cost of this product if it is known that the level of production is raised from $a=100$ units to $b=140$ units
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=\frac{8}{17}x^2+\frac{9}{17}x$. Determine the degree of equality of the income distribution
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=0$ units to $b=\sqrt{3}$ units for given $MC(x)=x^2(x^2+1)^{-2}$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=t+20$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-2t-8$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=\frac{2}{q+1}$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=0,2+0,1q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 23

Find the definite integrals:	
1. $\int_{-1}^{1}(2x-1)^7\,dx$	2. $\int_{0}^{\pi/8}\cos^2 2x\,dx$
3. $\int_{0}^{\pi/2}\frac{dx}{\sqrt{(1-x^2)\arcsin x}}$	4. $\int_{-2}^{2}(x-3)\sin\frac{\pi x}{2}\,dx$
5. $\int_{0}^{1}\frac{x^2\mathrm{arctg}x}{1+x^2}dx$	6. $\int_{-\pi/2}^{0}\frac{\sin x\,dx}{(1+\cos x-\sin x)^2}$
7. $\int_{0}^{\pi/4}\frac{2\,\mathrm{tg}^2 x-11\,\mathrm{tg}\,x-22}{4-\mathrm{tg}\,x}dx$	8. $\int_{0}^{\pi}2^4\sin^4(x/2)\cos^4(x/2)\,dx$
9. $\int_{0}^{\sqrt{3}}\frac{dx}{\sqrt{(4-x^2)^3}}$	10. $\int_{0}^{4}\frac{dx}{\sqrt{-x^2+6x+7}}$
11. Calculate the area of the plane figure bounded by the lines: $$y=\frac{x}{1+\sqrt{x}},\quad y=0,\quad x=1$$	
12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y=x^2,\quad y^2-x=0$	
13. The cost of a product changes at the rate of $f(t)=64\cos^4 t$. Find the average cost of this product if it is known that the level of production is raised from $a=-\frac{\pi}{2}$ units to $b=\frac{\pi}{2}$ units	
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=\frac{9}{17}x^2+\frac{8}{17}x$. Determine the degree of equality of the income distribution	
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=0$ units to	

$b=2$ units for given $MC(x)=x^2(x^2+4)^{-2}$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=9+6t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-7$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=150-q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=75+q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 24

Find the definite integrals:	
1. $\int_1^2 \left(\frac{\sqrt[4]{x}-3}{\sqrt{x}}\right)^2 dx$	2. $\int_0^{\pi/12} \sin^2 3x dx$
3. $\int_1^2 \frac{2^{2x}}{\sqrt{4^x+5}} dx$	4. $\int_2^3 (2x-1)\cos\frac{2\pi x}{3} dx$
5. $\int_0^1 \sqrt{x}\arcsin\sqrt{x}\, dx$	6. $\int_{-2\pi/3}^{0} \frac{\cos^2 x\, dx}{(1+\cos x-\sin x)^2}$
7. $\int_{\arcsin(1/\sqrt{37})}^{\pi/4} \frac{6\,\mathrm{tg}\, x\, dx}{3\sin 2x+5\cos^2 x}$	8. $\int_{-\pi/2}^{0} 2^8 \sin^2 x\cos^6 x\, dx$

9. $\int_{0}^{\sqrt{2}/2} \frac{x^4 dx}{\sqrt{(1-x^2)^3}}$	10. $\int_{0}^{4} \frac{8dx}{x^2-6x-7}$

11. Calculate the area of the plane figure bounded by the lines:

$$y=2^x, \quad y=4, \quad x=0$$

12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines:

$$y=\sin^2 x, \quad x=\frac{\pi}{2}, \quad x=0, \quad y=0$$

13. The cost of a product changes at the rate of $f(t)=2\cos^2 t$. Find the average cost of this product if it is known that the level of production is raised from $a=0$ units to $b=\frac{\pi}{3}$ units

14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=\frac{16}{19}x^2+\frac{3}{19}x$. Determine the degree of equality of the income distribution

15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=1$ units to $b=6$ units for given $MC(x)=\frac{3x-1}{\sqrt{x^2-4x+8}}$

16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=4t+45$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)

17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q) = -0,4q + 20$ UAH per tire, and the same number of tires will be supplied when the price is $S(q) = -12 + 0,2q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 25

Find the definite integrals:	
1. $\int_1^2 \left(\frac{4}{x} - 5x^4 + 2\sqrt{x} \right) dx$	2. $\int_{-\pi/4}^{0} \frac{dx}{\cos^2 x}$
3. $\int_1^e \frac{x^2 + \ln x^2}{x} dx$	4. $\int_{-2}^{0} (6x - 5) \sin \frac{\pi x}{2} dx$
5. $\int_0^1 (x^2 - 3) \cdot e^{-x} dx$	6. $\int_0^{\pi/2} \frac{\sin^2 x\, dx}{(1 + \cos x + \sin x)^2}$
7. $\int_0^{\arccos\sqrt{2/3}} \frac{\operatorname{tg} x + 2}{\sin^2 x + 2\cos^2 x - 3} dx$	8. $\int_{\pi/2}^{2\pi} 2^8 \cos^8 x\, dx$
9. $\int_1^2 \frac{\sqrt{x^2 - 1}}{x^4} dx$	10. $\int_0^4 \frac{(4 - 2x)dx}{-x^2 + 4x + 5}$
11. Calculate the area of the plane figure bounded by the lines: $$y = tgx, \quad x = \frac{\pi}{4}, \quad y = 0$$	
12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y = -x^2 + 5x - 6, \quad y = 0$	
13. The cost of a product changes at the rate of $f(t) = e^{-t} + t^2$. Find the average cost of this product if it is known that the level of production is raised from $a = 0$ units to $b = 3$ units	
14. The Lorenz curve of the income distribution within a certain group is given	

by the following formula $L(x)=\frac{3}{19}x^2+\frac{16}{19}x$. Determine the degree of equality of the income distribution
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=\frac{\pi}{4}$ units to $b=\frac{\pi}{3}$ units for given $MC(x)=e^x\cdot\sin 2x$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=8t-8$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-t-18$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=8-3q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=3q-1$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 26

Find the definite integrals:	
1. $\int_0^2 x\cdot\left(2-x^2\right)^2 dx$	2. $\int_0^{2\pi}\sin 3x\,dx$

3. $\int_{0}^{\pi/4} \operatorname{tg} x \ln \cos x \, dx$	4. $\int_{2}^{3} (1-8x)\cos\frac{3\pi x}{4} \, dx$
5. $\int_{0}^{1} \left(x^2+2\right)3^x \, dx$	6. $\int_{0}^{2\pi/3} \frac{\cos^2 x \, dx}{(1+\cos x+\sin x)^2}$
7. $\int_{0}^{\operatorname{arctg}(1/3)} \frac{(8+\operatorname{tg} x)}{18\sin^2 x+2\cos^2 x} dx$	8. $\int_{0}^{\pi} 2^4 \sin^8 x \, dx$
9. $\int_{0}^{\sqrt{5}/2} \frac{dx}{\sqrt{\left(5-x^2\right)^3}}$	10. $\int_{-1}^{3} \frac{dx}{\sqrt{-x^2+2x+8}}$
11. Calculate the area of the plane figure bounded by the lines: $y=\sqrt{x}, \quad y=3\sqrt{x}, \quad x=4$	
12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines: $y=5\cos x, \quad y=\cos x, \quad x=-\frac{\pi}{2}, \quad x=\frac{\pi}{2}$	
13. The cost of a product changes at the rate of $f(t)=(8-t)e^{-7t}$. Find the average cost of this product if it is known that the level of production is raised from $a=0$ units to $b=2$ units	
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=\frac{4}{15}x^2+\frac{11}{15}x$. Determine the degree of equality of the income distribution	
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=\frac{\pi}{6}$ units to $b=\frac{\pi}{3}$ units for given $MC(x)=3x^2\sin x$	
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=1+8t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-t-21$ UAH per year. a) For how	

many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=-q+740$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=q-100$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 27

Find the definite integrals:	
1. $\int_1^5 (2x+1)\,dx$	2. $\int_0^{\pi/2} \sin x \sin 2x\,dx$
3. $\int_0^{\pi/4} \frac{\sin x-\cos x}{(\cos x+\sin x)^5}dx$	4. $\int_0^{1,5} (2x-3)\sin \pi x\,dx$
5. $\int_{1/2}^{1} \sqrt{x}\arccos\sqrt{x}dx$	6. $\int_{\pi/2}^{2\operatorname{arctg}2} \frac{dx}{\sin x(1+\sin x)}$
7. $\int_{\pi/4}^{\operatorname{arctg}3} \frac{4\operatorname{tg}x-5}{1-\sin 2x+4\cos^2 x}dx$	8. $\int_0^{2\pi} \sin^6 x\cos^2 x\;dx$
9. $\int_0^3 \frac{dx}{(9+x^2)^{3/2}}$	10. $\int_0^2 \frac{4dx}{-x^2+2x+3}$
11. Calculate the area of the plane figure bounded by the lines: $x=\sqrt{y},\quad x=2-y,\quad y=0$	
12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines:	

$y = \arcsin\frac{x}{5}, \quad y = \arcsin\frac{x}{3}, \quad y = \frac{\pi}{2}$
13. The cost of a product changes at the rate of $f(t) = \sqrt{16 - t^2}$. Find the average cost of this product if it is known that the level of production is raised from $a = 2$ units to $b = 6$ units
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x) = \frac{11}{15}x^2 + \frac{4}{15}x$. Determine the degree of equality of the income distribution
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a = \frac{\pi}{6}$ units to $b = \frac{\pi}{2}$ units for given $MC(x) = 7x\cos x$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t) = 7t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t) = t^2 - 60$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q) = -0,75q + 50$ UAH per tire, and the same number of tires will be supplied when the price is $S(q) = 10 + 0,25q$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 28

Find the definite integrals:	
1. $\int_0^1 (3x-1)^5 dx$	2. $\int_0^{\pi/2} \sin x \sin 3x \, dx$
3. $\int_0^{1/\sqrt{2}} \frac{(\arccos x)^3 - 1}{\sqrt{1-x^2}} dx$	4. $\int_1^3 (2x-15)\cos\frac{\pi x}{6} dx$
5. $\int_0^2 \left(x^2-3\right) e^{\frac{x}{2}} dx$	6. $\int_0^{\pi/2} \frac{dx}{(1+\cos x+\sin x)^2}$
7. $\int_0^{\arccos\left(1/\sqrt{17}\right)} \frac{3+2\,\mathrm{tg}\, x}{2\sin^2 x+3\cos^2 x-1} dx$	8. $\int_0^{2\pi} \sin^4(x/4)\cos^4(x/4)\, dx$
9. $\int_0^5 \frac{dx}{\left(25+x^2\right)\sqrt{25+x^2}}$	10. $\int_2^4 \frac{(2x-1)dx}{x^2-x+4}$

11. Calculate the area of the plane figure bounded by the lines:

$$y = 2x - x^2, \quad x = 2y$$

12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines:

$$y = 3\sin x, \quad y = \sin x, \quad 0 \le x \le \frac{\pi}{6}$$

13. The cost of a product changes at the rate of $f(t) = \cos 5t \cos 3t$. Find the average cost of this product if it is known that the level of production is raised from $a = -\frac{\pi}{2}$ units to $b = \frac{\pi}{2}$ units

14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x) = \frac{3}{7}x^2 + \frac{4}{7}x$. Determine the degree of equality of the income distribution

15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a = 2$ units to

$b=6$ units for given $MC(x)=e^{\frac{x}{2}}\cdot(e^x-1)^{-1}$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=181$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2+t-1$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=160-14q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=6q-60$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 29

Find the definite integrals:	
1. $\int_1^9\left(2x-\frac{3}{\sqrt{x}}\right)dx$	2. $\int_0^{\pi}\sin\frac{x}{2}dx$
3. $\int_{-1}^{0}\frac{\mathrm{tg}(x+1)}{\cos^2(x+1)}dx$	4. $\int_4^6(1-x)\sin\frac{\pi x}{4}dx$
5. $\int_0^1 x^2 3^{\frac{x}{2}}dx$	6. $\int_0^{\pi/2}\frac{\sin x\,dx}{2+\sin x}$
7. $\int_{\arccos\left(4/\sqrt{17}\right)}^{\pi/4}\frac{2\,\mathrm{ctg}\,x+1}{(2\sin x+\cos x)^2}dx$	8. $\int_0^{\pi}2^4\sin^2(x/2)\cos^6(x/2)\,dx$

9. $\int_0^1 x^2\sqrt{1-x^2}\,dx$	10. $\int_{3,5}^{4,5} \frac{dx}{\sqrt{-x^2+8x-15}}$
11. Calculate the area of the plane figure bounded by the lines: $$y=\frac{e^{\frac{1}{x}}}{x^2},\quad y=0,\quad x=2,\quad x=1$$	
12. Find the solid of revolution, formed by rotation round the axis Oy of the curvilinear trapezoid, bounded by the lines: $y=\sin\frac{\pi x}{2},\quad y=x^2$	
13. The cost of a product changes at the rate of $f(t)=t\sin\frac{t}{2}$. Find the average cost of this product if it is known that the level of production is raised from $a=0$ units to $b=\pi$ units	
14. The Lorenz curve of the income distribution within a certain group is given by the following formula $L(x)=\frac{14}{17}x^2+\frac{3}{17}x$. Determine the degree of equality of the income distribution	
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=1$ units to $b=2$ units for given $MC(x)=\left(\frac{2\ln x+3}{x}\right)^3$	
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=91+5t$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-7-2t$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)	

17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=14-2q$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=4q-10$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Variant 30

Find the definite integrals:	
1. $\int_0^1 \sqrt{1+2x}\,dx$	2. $\int_0^{\pi} \cos\frac{x}{2}\,dx$
3. $\int_{\pi}^{2\pi} \frac{1-\cos x}{(x-\sin x)^2}\,dx$	4. $\int_2^3 (2x+3)\cos\frac{\pi x}{6}\,dx$
5. $\int_0^1 x^2 e^{-\frac{x}{2}}\,dx$	6. $\int_0^{\pi/4} \frac{dx}{\cos x(1+\cos x)}$
7. $\int_{\pi/4}^{\operatorname{arctg}3} \frac{dx}{(3\operatorname{tg}x+5)\sin 2x}$	8. $\int_0^{2\pi} \sin^4 3x\cos^4 3x\,dx$
9. $\int_0^{16} \sqrt{256-x^2}\,dx$	10. $\int_{-3}^{1} \frac{dx}{x^2+4x+5}$
11. Calculate the area of the plane figure bounded by the lines: $y=\frac{x}{x^2+1},\quad y=0,\quad x=1$	
12. Find the solid of revolution, formed by rotation round the axis Ox of the curvilinear trapezoid, bounded by the lines: $2x-x^2-y=0,\quad 2x^2-4x+y=0$	
13. The cost of a product changes at the rate of $f(t)=e^{-t}(3t+5)$. Find the average cost of this product if it is known that the level of production is raised from $a=0$ units to $b=3$ units	
14. The Lorenz curve of the income distribution within a certain group is giv-	

en by the following formula $L(x)=\frac{19}{21}x^2+\frac{2}{21}x$. Determine the degree of equality of the income distribution
15. A firm's monthly marginal cost for a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ for the month if the level of production is raised from $a=\frac{\pi}{6}$ units to $b=\frac{\pi}{4}$ units for given $MC(x)=\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$
16. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=2t+119$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t)=t^2-5t-1$ UAH per year. a) For how many years does the rate of profitability of the second investment exceed that of the first? b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a). c) Interpret the net excess profit as an area: sketch the rate of the profitability curves $y=P_1'(t)$ and $y=P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b)
17. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $D(q)=-0{,}2q+10$ UAH per tire, and the same number of tires will be supplied when the price is $S(q)=0{,}1q-6$ UAH per tire. a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price. b) Determine the consumers' surplus at the equilibrium price. c) Determine the producers' surplus at the equilibrium price

Theoretical Questions

1. A definite integral.
2. The lower limit and the upper limit.
3. The formula of Newton − Leibnitz.
4. Substitution in definite integrals.
5. Basic properties of definite integrals.
6. Calculation of the definite integral of an even function with symmetrical limits of integration.
7. Calculation of the definite integral of an odd function with symmetrical limits of integration.
8. Calculation of the definite integral of a continuous function with the period T.
9. The mean - value theorem.
10. The basic tabular integrals.
11. The direct integration.
12. The method of change of the variable in the definite Integral.
13. The table for finding the substitution.
14. The method of integration by parts.
15. Classes of integrals which are found by the method of integration by parts.
16. Integration of rational functions with a quadratic trinomial.
17. Integration of trigonometric functions.
18. Integrals of type $\int R(\sin x, \cos x)dx$.
19. Calculation of the area of a plane figure.
20. Calculation of the volume of a solid of revolution by rotation round the Ox-axis and the Oy-axis.
21. The income concentration index – the Gini coefficient.
22. The net change.
23. The net excess profit.
24. The net earnings from industrial equipment.
25. The consumers' demand curve and willingness to spend.
26. Consumers' surplus.
27. Producers' surplus.
28. The volume of production.

Bibliography

1. Англо-русский словарь математических терминов / под ред. П. С. Александрова. – Москва : Мир, 1994. – 416 с.

2. Вища математика : базовий підручник для вузів / під ред. В. С. Пономаренка. – Харків : Фоліо, 2014. – 669 с.

3. Контрольні завдання з курсу "Математика для економістів" для студентів усіх форм навчання. Ч. 1 / уклад. Е. Ю. Железнякова, А. В. Ігначкова, Л. Д. Широкорад. – Харків : Вид. ХНЕУ, 2005. – 52 с.

4. Малярець Л. М. Вища математика для економістів у прикладах, вправах і задачах : навч. посіб. / Л. М. Малярець, А. В. Ігначкова. – Харків : ВД "ІНЖЕК", 2006. – 544 с.

5. Малярець Л. М. Математика для економістів : практ. посіб. до розв'язання задач / Л. М. Малярець, Л. Д. Широкорад. – Харків : Вид. ХНЕУ, 2008. – 476 с.

6. Малярець Л. М. Математика для економістів : навч. посіб. / під ред. Л. М. Малярець. – Харків : Вид. ХНЕУ, 2011. – 568 с.

7. Малярець Л. М. Математика для економістів : навч. посіб. У 2-х ч. Ч. 1. / Л. М. Малярець, Л. М. Афанасьєва, А. В. Ігначкова. – Харків : Вид. ХНЕУ, 2011. – 393 с.

8. Малярець Л. М. Математика для економістів : навч. посіб. У 2-х ч. Ч. 2. / Л. М. Малярець, Л. М. Афанасьєва, А. В. Ігначкова. – Харків : Вид. ХНЕУ, 2011. – 368 с.

9. Borakovskiy A. Handbook for problem solving in higher mathematics / A. Borakovskiy, A. Ropavka. – Kharkiv : KNMA, 2008. – 195 p.

10. Guidelines to practical tasks for an indefinite integral on the academic discipline "Mathematical analysis and linear algebra" for full-time foreign students and students taught in English of training direction 6.030601 "Management", specialization "Business Administration" / compiled by Ie. Misiura. – Kharkiv : S. Kuznets KhNUE, 2016. – 48 p. (English)

11. Higher mathematics : handbook. Vol. 4 / under the editorship of L. Kurpa. – Kharkiv : NTU "KhPI", 2006. – 540 p.

12. Panova N. Integral calculus of functions of one variable: teaching aid for foreign students of all specialties "/ compiled by N. Panova, V. Titarev, L. Toreanik. – Kharkiv : Publishing House of KhNUE, 1995. – 20 p. (English)

НАВЧАЛЬНЕ ВИДАННЯ

Вища математика:
методичні рекомендації до виконання практичних завдань з визначеного інтегралу для студентів усіх спеціальностей першого (бакалаврського) рівня (англ.мовою)

Самостійне електронне текстове мережеве видання

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Викладено необхідний теоретичний матеріал з навчальної дисципліни та наведено типові приклади, які сприяють найбільш повному засвоєнню матеріалу з теми "Визначений інтеграл" та застосуванню отриманих знань на практиці. Подано завдання для індивідуальної роботи та перелік теоретичних питань, що сприяють удосконаленню та поглибленню знань студентів з даної теми.

Рекомендовано для студентів усіх спеціальностей першого (бакалаврського) рівня.

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