МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ
ХАРКІВСЬКИЙ НАЦІОНАЛЬНИЙ ЕКОНОМІЧНИЙ УНІВЕРСИТЕТ

Методичні рекомендації до проведення практичних занять з навчальної дисципліни «Вища математика» для студентів-іноземців та студентів, які навчаються англійською мовою, напряму підготовки «Менеджмент» денної форми навчання

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Методичні рекомендації до проведення практичних занять з навчальної дисципліни «Вища математика» для студентів-іноземців та студентів, які навчаються англійською мовою, напряму підготовки «Менеджмент» денної форми навчання / Укл. Є. Ю. Місюра. – Харків : Вид. ХНЕУ, 2009. – 43 с. (Англ. мов., укр. мов.)

Подано методичні рекомендації з «Лінійної алгебри», що є складовою частиною навчальної дисципліни «Вища математика».

Викладено необхідний теоретичний матеріал та наведено типові приклади, які допоможуть найбільш повному засвоєнню матеріалу.

Подано завдання для самостійної роботи та перелік теоретичних питань, що сприяють удосконаленню та поглибленню знань студентів з кожної теми.

Рекомендовано для студентів-іноземців та студентів, які навчаються англійською мовою, напряму підготовки «Менеджмент» денної форми навчання.

THE MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
KHARKIV NATIONAL UNIVERSITY OF ECONOMICS

Methodical recommendations for the conduct of the practical studies in the discipline "Higher mathematics" for foreign and English learning students of the preparatory direction "Management" of the full-time education

Composed by Ie. Misura

Chief of department L. Malyarets

Kharkiv, KNUE, 2010

Вступ

Методичні рекомендації призначені студентам-іноземцям та студентам, які навчаються англійською мовою, як додатковий матеріал для самостійного вивчення «Лінійної алгебри» з навчальної дисципліни «Вища математика».

Метою методичних рекомендацій є практичне застосування математичного апарату до розв'язання задач з «Лінійної алгебри», до складу якої входять такі теми: матриці, дії над ними, визначники, їх основні властивості, мінори та алгебраїчні доповнення, методи обчислення визначників, методи обчислення оберненої матриці за допомогою алгебраїчних доповнень та елементарних перетворень, три методи розв'язання систем лінійних алгебраїчних рівнянь (метод Крамера, матричний метод та метод Жордана–Гаусса), поняття рангу та дослідження систем рівнянь на сумісність за допомогою теореми Кронекера–Капеллі.

Для кожної з тем наведено достатню кількість розв'язаних типових прикладів, що дає можливість студентам самостійно опанувати «Лінійну алгебру» та використовувати отримані знання на практиці. Наприкінці рекомендацій надано завдання для самостійної роботи та перелік теоретичних питань, що сприяють удосконаленню і поглибленню знань студентів з кожної наведеної теми.

Introduction

Methodical recommendations are intended for foreign and English-learning students of the preparatory direction "Management" for the practical studies of “Linear algebra” in the discipline "Higher mathematics".

Its aim is the practical application of mathematic apparatus to the problem solutions in “Linear algebra”, which consists of the following themes: matrices, operations on them, determinants, their basic properties, minors and algebraic cofactors, two techniques of finding an inverse matrix, solution methods of linear algebraic equations system, finding the rank and investigating systems for compatibility using Kronecker–Capelli theorem.

The sufficient number of solved typical examples of each theme gives students the possibility to master “Linear algebra” and apply the obtained knowledge in practice on their own. At the end of methodical recommendations there are tasks for self-work and the list of theoretical questions which promote improving and extending students’ knowledge of all the themes.

1. Matrices and determinants

1.1. Matrices

Definition. A rectangular table of numbers of the form

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ or briefly } A=\left(a_{ij}\right)_{m\times n},\ i=\overline{1,m},\ j=\overline{1,n},$$

which has m rows and n columns is called *the* $m\times n$*-matrix* A or *the matrix* A *of the size* $m\times n$ [2, 4, 5, 6].

Matrices are denoted by the capital letters A, B, C and so on.

Each element of the matrix A is designated by a_{ij} and represents the entry in the matrix A on the i-th row and j-th column. For each element a_{ij} the subscript i denotes the row number and the subscript j denotes the column number.

For example,

a_{11} = element in row 1, column 1

a_{12} = element in row 1, column 2

a_{mn} = element in row m, column n.

Now let us consider the following example.

Example 1. Consider the 3×4 matrix $A=\begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$.

Here $\begin{pmatrix} 3 & 1 & 5 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$ represent the 2-nd row and the 3-rd column of the matrix A respectively, and the element 5 represents the entry in the matrix on the 2-nd row and 3-rd column.

For example, if there are 3 rows and 4 columns so there must be 4 elements in each row and 3 elements in each column.

Definition. The size of a matrix is called its *order*. *The order* is specified as:

$$\text{(number of rows)} \times \text{(number of columns)}.$$

Definition. *A matrix-column (row)* is called a matrix consisting of the only column (row):

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix}.$$

Definition. The matrix is called *zero (or null) matrix* if all its elements are equal to zero and denoted by:

$$O = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Definition. The matrix is called *square matrix*, if $m = n$. The number of rows is considered to be *the order of this matrix*.

Definition. The set of elements a_{11}, a_{22}, …, a_{nn} makes up *the main diagonal* of the matrix. The set of the elements $a_{1\,n}$, $a_{2\,n-1}$, …, $a_{n\,1}$ makes up *the secondary diagonal* of the matrix.

Definition. A square matrix is called *diagonal* if all its elements except diagonal ones are equal to zero and denoted by:

$$D = \begin{pmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}.$$

Definition. If all the diagonal elements of the diagonal matrix D are equal to 1 then the matrix is called a *unit (identity) matrix* and designated as *E*:

$$E = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{pmatrix}.$$

Definition. If all the elements of a matrix located below (above) the main diagonal are equal to zero then the matrix is called *an upper (lower) triangular matrix*

For example, $A = \begin{pmatrix} 2 & 5 & 1 & 0 \\ 0 & -4 & 2 & 3 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & 11 \end{pmatrix}$ is the upper triangular matrix.

Definition. If $a_{ij} = a_{ji}$, then the matrix *A* is called *a symmetrical matrix*.

For example, $A = \begin{pmatrix} -1 & 8 & -2 & 0 \\ 8 & 3 & 4 & 6 \\ -2 & 4 & 10 & -5 \\ 0 & 6 & -5 & -9 \end{pmatrix}$.

1.2. Operations on matrices

1. *The addition and subtraction matrices*

Matrices that have the same order can be added together, or subtracted. The addition, or subtraction, is performed on each of the corresponding elements [2, 5].

Definition. Suppose that both matrices $A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ and $B=\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$ have *m* rows and *n* columns. Then we write $A\pm B=\begin{pmatrix} a_{11}\pm b_{11} & \cdots & a_{1n}\pm b_{1n} \\ \vdots & & \vdots \\ a_{m1}\pm b_{m1} & \cdots & a_{mn}\pm b_{mn} \end{pmatrix}$ and call this *the sum (or difference) of the two matrices A and B.*

Here are the following examples.

Example 2. If $A=\begin{pmatrix} 13 & 30 \\ 8 & 15 \end{pmatrix}$ and $B=\begin{pmatrix} 7 & 35 \\ 8 & 4 \end{pmatrix}$ what are $A+B$ and $A-B$?

Solution. Matrices *A* and *B* have the same order 2×2, therefore, we can add them together, or subtract:

$$C=A+B=\begin{pmatrix} 13 & 30 \\ 8 & 15 \end{pmatrix}+\begin{pmatrix} 7 & 35 \\ 8 & 4 \end{pmatrix}=\begin{pmatrix} 13+7 & 30+35 \\ 8+8 & 15+4 \end{pmatrix}=\begin{pmatrix} 20 & 65 \\ 16 & 19 \end{pmatrix},$$

$$C=A-B=\begin{pmatrix} 13 & 30 \\ 8 & 15 \end{pmatrix}-\begin{pmatrix} 7 & 35 \\ 8 & 4 \end{pmatrix}=\begin{pmatrix} 13-7 & 30-35 \\ 8-8 & 15-4 \end{pmatrix}=\begin{pmatrix} 6 & -5 \\ 0 & 11 \end{pmatrix}.$$

Example 3. We do not have a definition for “adding” the matrices $A=\begin{pmatrix} 5 & 4 & 12 & 7 \\ 10 & 12 & 9 & 14 \end{pmatrix}$ and $B=\begin{pmatrix} 13 & 30 \\ 8 & 15 \end{pmatrix}$, because matrices have the different order.

2. *The multiplication of a matrix by a scalar value*

A matrix can be multiplied by a specific value, such as a number (scalar multiplication). Scalar multiplication simply involves the multiplication of each element in a matrix by the scalar value.

Definition. Suppose that the matrix $A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ has m rows and n columns and $\alpha \in R$. Then we write $\alpha A=\begin{pmatrix} \alpha a_{11} & \cdots & \alpha a_{1n} \\ \vdots & & \vdots \\ \alpha a_{m1} & \cdots & \alpha a_{mn} \end{pmatrix}$ and call this the product of the matrix A by the scalar α.

The operations of matrix addition (or subtraction) and matrix multiplication by some number satisfy the following laws:

1) $A+B=B+A$
2) $(A+B)+C=A+(B+C)$
3) $A+O=A$
4) $(\alpha\cdot\beta)A=\alpha(\beta\cdot A)$
5) $(\alpha\pm\beta)A=\alpha A\pm\beta A$
6) $\alpha(A\pm B)=\alpha A\pm\alpha B$

Now let us consider the following example.

Example 4. Calculate the matrix $C=3B-2A$, if $A=\begin{pmatrix} 2 & -4 & 1 \\ 3 & 0 & 7 \end{pmatrix}$ and $B=\begin{pmatrix} 5 & 1 & 2 \\ 3 & 6 & -1 \end{pmatrix}$.

Solution. Matrices *A* and *B* have the same order 2×3, therefore, we can obtain $C=3B-2A$. The entry $2A$ is multiplication the matrix A by 2:

$$2A=\begin{pmatrix} 2\cdot 2 & 2\cdot(-4) & 2\cdot 1 \\ 2\cdot 3 & 2\cdot 0 & 2\cdot 7 \end{pmatrix}=\begin{pmatrix} 4 & -8 & 2 \\ 6 & 0 & 14 \end{pmatrix}.$$

$3B$ like $2A$: $3B=\begin{pmatrix} 3\cdot 5 & 3\cdot 1 & 3\cdot 2 \\ 3\cdot 3 & 3\cdot 6 & 3\cdot(-1) \end{pmatrix}=\begin{pmatrix} 15 & 3 & 6 \\ 9 & 18 & -3 \end{pmatrix}$. Then

$$C=3B-2A=\begin{pmatrix} 15 & 3 & 6 \\ 9 & 18 & -3 \end{pmatrix}-\begin{pmatrix} 4 & -8 & 2 \\ 6 & 0 & 14 \end{pmatrix}=\begin{pmatrix} 11 & 11 & 4 \\ 3 & 18 & -17 \end{pmatrix}.$$

3. *Operation of multiplying a matrix by a matrix*

Let two matrices $A=\left(a_{ij}\right)_{m\times n}$ and $B=\left(b_{ij}\right)_{p\times q}$ be given and the number of columns at the first matrix be equal to the number of rows of the second one, i.e. $n=p$. In this case we can define the operation of multiplying the matrix A by the matrix B [3, 4, 6]. The matrix $C=A\cdot B$ of the size $m\times q$, which elements are calculated according to the following rule

$$c_{ij}=\sum_{k=1}^{n}a_{ik}\cdot b_{kj}=a_{i1}\cdot b_{1j}+\ldots+a_{in}\cdot b_{nj},\ i=\overline{1,m},\ j=\overline{1,q},$$

is called *a product of the matrix A by the matrix B*.

The rule «row by column»*:* *in order to get an element standing in the row i and the column j of the matrix C equal to the product of the matrices A and B it is necessary to multiply elements standing in the row i of the first matrix by the corresponding elements of the column j of the second matrix and then summarize the obtained products.*

In a general case the multiplication of matrices does not possess a commutative property, i. e. $A\cdot B$ and $B\cdot A$ are not equal to each other or one of the products does not exist.

The basic properties of a matrix multiplication:

1) $\alpha(AB)=(\alpha A)B=A(\alpha B)$

2) $(A+B)C=AC+BC$, $C(A+B)=CA+CB$

3) $A(BC)=A(BC)$

4) $AE=EA=A$

5) $AO=OA=O$

Now let us consider the following example.

Example 5. Multiply the following matrices: $A=\begin{pmatrix}2 & 4 & 3 & -1\\ 3 & 1 & 5 & 2\\ -1 & 0 & 7 & 6\end{pmatrix}$ and

$B=\begin{pmatrix}1 & 4\\ 2 & 3\\ 0 & -2\\ 3 & 1\end{pmatrix}$.

Solution. Note that A is a 3×4 matrix and B is a 4×2 matrix and the number of columns of the matrix A is equal to the number of rows of the matrix B, so that the product $C = A\cdot B$ is a 3×2 matrix.

Let us calculate the product

$$C = A\cdot B = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}.$$

Consider first of all c_{11}. To calculate this, we need the 1-st row of A and the 1-st column of B, so let us cover up all the unnecessary information, so that

$$C = \begin{pmatrix} 2 & 4 & 3 & -1 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \cdot \begin{pmatrix} 1 & \times \\ 2 & \times \\ 0 & \times \\ 3 & \times \end{pmatrix} = \begin{pmatrix} c_{11} & \times \\ \times & \times \\ \times & \times \end{pmatrix}.$$

From the definition of the product of the matrix A by the matrix B, we have $c_{11} = a_{11}\cdot b_{11} + a_{12}\cdot b_{21} + a_{13}\cdot b_{31} + a_{14}\cdot b_{41} = 2\cdot1 + 4\cdot2 + 3\cdot0 + (-1)\cdot3 =$ $= 2+8+0-3 = -7$ (multiply elements standing in the row 1 of A by the corresponding elements of the column 1 of the B and then summarize the obtained products).

Consider next c_{12}. To calculate this, we need the 1-st row of A and the 2-nd column of B, so let us cover up all the unnecessary information, so that

$$C = \begin{pmatrix} 2 & 4 & 3 & -1 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \cdot \begin{pmatrix} \times & 4 \\ \times & 3 \\ \times & -2 \\ \times & 1 \end{pmatrix} = \begin{pmatrix} \times & c_{12} \\ \times & \times \\ \times & \times \end{pmatrix}.$$

From the definition, we have

$$c_{12} = a_{11}\cdot b_{12} + a_{12}\cdot b_{22} + a_{13}\cdot b_{32} + a_{14}\cdot b_{42} = 2\cdot4 + 4\cdot3 + 3\cdot(-2) + (-1)\cdot1 =$$

$= -8 + 12 - 6 - 1 = 13.$

Consider next c_{21}. To calculate this, we need the 2-nd row of A and the 1-st column of B, so let us cover up all the unnecessary information, so that

$$C = \begin{pmatrix} \times & \times & \times & \times \\ 3 & 1 & 5 & 2 \\ \times & \times & \times & \times \end{pmatrix} \cdot \begin{pmatrix} 1 & \times \\ 2 & \times \\ 0 & \times \\ 3 & \times \end{pmatrix} = \begin{pmatrix} \times & \times \\ c_{21} & \times \\ \times & \times \end{pmatrix}.$$

From the definition, we have

$$c_{21} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} + a_{24} \cdot b_{41} = 3 \cdot 1 + 1 \cdot 2 + 5 \cdot 0 + 2 \cdot 3 =$$

$$= 3 + 2 + 0 + 6 = 11.$$

According to the definition, we have the rest of the elements:

$$c_{22} = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} + a_{24} \cdot b_{42} = 3 \cdot 4 + 1 \cdot 3 + 5 \cdot (-2) + 2 \cdot 1 =$$

$$= 12 + 3 - 10 + 2 = 7,$$

$$c_{31} = a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} + a_{34} \cdot b_{41} = (-1) \cdot 1 + 0 \cdot 2 + 7 \cdot 0 + 6 \cdot 3 =$$

$$= -1 + 0 + 0 + 18 = 17,$$

$$c_{32} = a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} + a_{34} \cdot b_{42} = (-1) \cdot 4 + 0 \cdot 3 + 7 \cdot (-2) + 6 \cdot 1 =$$

$$= -4 + 0 - 14 + 6 = -12.$$

Therefore we conclude that

$$C = A \cdot B = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 13 \\ 11 & 7 \\ 17 & -12 \end{pmatrix}$$

Here is the following example.

Example 6. Consider the same matrices in example 5: $A=\begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & -2 \\ 3 & 1 \end{pmatrix}$.

Note that B is a 4×2 matrix and A is a 3×4 matrix, so that we do not have a definition for the product $B\cdot A$, because the number of columns of the matrix A is not equal to the number of rows of the matrix B.

4. *Transposition matrix*

Definition. Consider the $m\times n$ matrix $A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$. By the transpose A^T of *A,* we mean *the transposed matrix* $A^T=\begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{pmatrix}$ obtained from *A* by transposing rows and columns.

To get the transpose of a matrix, usually written as A^T, the rows and columns are swapped around, i. e. row 1 becomes column 1 and column 1 becomes row 1, etc. If a matrix is not square then the numbers of rows and columns will alter when it is transposed.

Let us now consider the following example.

Example 7. Consider the matrix $A=\begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$. Then $A^T=\begin{pmatrix} 2 & 3 & -1 \\ 4 & 1 & 0 \\ 3 & 5 & 7 \\ -1 & 2 & 6 \end{pmatrix}$. Note that A is a 3×4 matrix and A^T is a 4×3 matrix.

Definition. For the $n \times n$ matrix and a positive integer m, *the* m*-th power of* A is $A^m = \underbrace{A \cdot A \cdot \ldots \cdot A}_{m \text{ copies of } A}$. It is also convenient to define $A^0 = E$.

1.3. Determinants

Any square matrix A can be associated with some value (number) called its *determinant* and designated as $\det A$ or $|A|$.

For example, a determinant of a matrix of the 2-nd order is calculated according to the following formula:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12},$$

i. e. the product of elements of the main diagonal minus the product of elements of the secondary diagonal.

Let us now consider the following example.

Example 8. Calculate the determinant of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

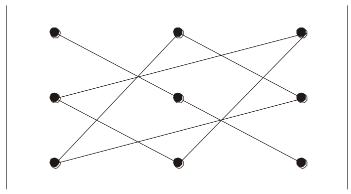
Solution. $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$.

A determinant of the 3-rd order is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} + a_{21} \cdot a_{32} \cdot a_{13} + a_{31} \cdot a_{12} \cdot a_{23} -$$

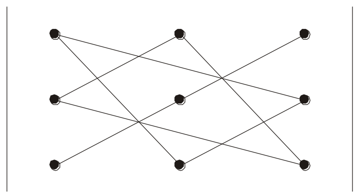
$$- a_{13} \cdot a_{22} \cdot a_{31} - a_{21} \cdot a_{12} \cdot a_{33} - a_{32} \cdot a_{23} \cdot a_{11}.$$

To memorize the last formula ***the rule of triangle (or Sarrus formula)*** is often used. It says: a product of elements from the main diagonal and 2 products of elements forming in a matrix isosceles triangles with their bases parallel to the main diagonal are taken with the sign *plus:*



+

a product of elements from the secondary diagonal and 2 products of elements forming triangles with their bases parallel to the secondary diagonal are taken with the sign *minus:*



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Let us now consider the following example.

Example 9. Calculate the determinant of the matrix $A=\begin{pmatrix}1 & 2 & 1\\ 0 & -2 & 3\\ 3 & 1 & 1\end{pmatrix}$.

Solution. Calculate the determinant, using the rule of triangle:

$$\begin{vmatrix}1 & 2 & 1\\ 0 & -2 & 3\\ 3 & 1 & 1\end{vmatrix}=1\cdot(-2)\cdot 1+2\cdot 3\cdot 3+0\cdot 1\cdot 1-1\cdot(-2)\cdot 3-0\cdot 1\cdot 2-3\cdot 1\cdot 1=$$

$$=-2+18+6-3=19.$$

1.3.1. Basic properties of determinants

1. A determinant does not change its value at the transposition of the matrix, i. e. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$ *or* $\det A = \det A^T$ *.*

Example 10. Check the property:

$$\begin{vmatrix} 3 & 2 & 5 \\ 4 & 1 & 8 \\ -1 & 2 & 4 \end{vmatrix} = 12 + 40 - 16 + 5 - 32 - 48 = -39.$$

Transpose rows and columns and obtain:

$$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 1 & 2 \\ 5 & 8 & 4 \end{vmatrix} = 12 - 16 + 40 + 5 - 32 - 48 = -39.$$

2. Transposing of two any rows (columns) the determinant changes its sign. *For example,* $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$*.*

Example 11. Check the property:

$$\begin{vmatrix} 5 & 1 & 0 \\ 2 & 5 & 6 \\ 3 & 2 & -1 \end{vmatrix} = -25 + 18 + 0 - 0 + 2 - 60 = -65.$$

Transpose the first row and the second one and obtain:

$$\begin{vmatrix} 2 & 5 & 6 \\ 5 & 1 & 0 \\ 3 & 2 & -1 \end{vmatrix} = -2 + 60 + 0 - 18 + 25 - 0 = 65.$$

3. If any row (column) of the determinant completely consists of zeros then the determinant is equal to zero*.*

Example 12*.* Check the property:

$$\begin{vmatrix} 3 & 5 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 0 \end{vmatrix} = 0+0+0-0-0-0=0.$$

4. A common factor of all elements of a row (column) can be taken out of the determinant. For example, $\begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, *where* $k \in R$*.*

Example 13. Check the property.

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ 6 & 5 & 4 \end{vmatrix} = 16+60-6-36+16-10=40.$$

The first column has a common factor 2. We take it out of the determinant and obtain

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ 6 & 5 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 2\left(8+30-3-18+8-5\right) = 2\cdot 20 = 40.$$

5. *If we add to all elements of a row (column) of the determinant the corresponding elements of other row (column) multiplied by some number then the value of the determinant will not change, i. e.*

$$\begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$, *where* $k \in R$.

Example 14. Check the property.

$$\begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 1 & 2 & 3 \end{vmatrix} = 9+20+4-5+6+24=58.$$

For example, calculate this determinant by adding to all elements of row 3 of the determinant the corresponding elements of row 1 multiplied by 2. Thus

$$\begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 1+3\cdot 2 & 2+(-1)\cdot 2 & 3+5\cdot 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 7 & 0 & 13 \end{vmatrix} =$$

$$=39+28-35+26=58.$$

6. *The determinant possessing two identical or proportional rows (columns) is equal to zero.*

Example 15. Check the property.

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 5 & 1 & 3 \end{vmatrix} = 18+8+120-120-18-8=0.$$

The determinant is equal to zero, because the first row and the second one are proportional rows: $\frac{a_{21}}{a_{11}} = \frac{a_{22}}{a_{12}} = \frac{a_{23}}{a_{13}} = 2$.

1.4. Minors and algebraic cofactors

Consider the following notions.

Definition. *The minor* M_{ij} of the element a_{ij} of the determinant of n-th order is called the determinant of the $(n-1)$-th order obtained from the given one by crossing the row and the column on which intersection the element a_{ij} is located.

Definition. *The algebraic cofactor* A_{ij} (or *the cofactor*) of the element a_{ij} of the determinant is called the following value $A_{ij} = (-1)^{i+j} \cdot M_{ij}$.

Theorem *(concerning decomposition of a determinant in its rows or columns)*. The sum of products of elements of any row (column) by their cofactors is equal to this determinant, i. e.

$$|A| = \sum_{k=1}^{n} a_{kj} A_{kj}, \quad j = \overline{1,n}.$$

Example 16. Calculate the determinant on the base of the rule of triangle and check the result using the theorem concerning decomposition of the determinant in its 1-st row and the 3-rd column.

Solution. Calculate the determinant, using the rule of triangle:

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = 1\cdot(-2)\cdot 1 + 2\cdot 3\cdot 3 + 0\cdot 1\cdot 1 - 1\cdot(-2)\cdot 3 - 0\cdot 1\cdot 2 - 3\cdot 1\cdot 1 =$$
$$= -2 + 18 + 6 - 3 = 19.$$

Checking the determinant value by decomposing in the 1-st row

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = a_{11}\cdot A_{11} + a_{12}\cdot A_{12} + a_{13}\cdot A_{13} = 1\cdot(-1)^{1+1}\cdot M_{11} + 2\cdot(-1)^{1+2}\cdot M_{12} +$$
$$+1\cdot(-1)^{1+3}\cdot M_{13} = 1\cdot 1\cdot\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} + 2\cdot(-1)\cdot\begin{vmatrix} 0 & 3 \\ 3 & 1 \end{vmatrix} + 1\cdot 1\cdot\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} = (-2-3) -$$
$$-2\cdot(0-9) + (0-(-6)) = -5 + 18 + 6 = 19$$

and the 3-rd column

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = a_{13} \cdot A_{13} + a_{23} \cdot A_{23} + a_{33} \cdot A_{33} = 1 \cdot (-1)^{1+3} \cdot M_{13} + 3 \cdot (-1)^{2+3} \cdot M_{23} +$$

$$+1 \cdot (-1)^{3+3} \cdot M_{33} = 1 \cdot 1 \cdot \begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + 3 \cdot (-1) \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 1 \cdot 1 \cdot \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = (0 - (-6)) -$$

$$-3 \cdot (1-6) + (-2-0) = 6 + 15 - 2 = 19.$$

1.5. Matrix inverse

Definition. The square matrix A is called *invertible or nonsingular*, if $\det A \neq 0$, otherwise it is called *not invertible or singular*.

Definition. The matrix A^{-1} is called *inverse* relatively to the square nonsingular matrix A if $A \cdot A^{-1} = A^{-1} \cdot A = E$, where E is the unit matrix.

The square matrix can have an inverse matrix if its determinant is non-zero, i. e. A is a nonsingular matrix.

1.5.1. Finding inverses by cofactors

An inverse matrix can be obtained according to the following formula:

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix},$$

where A_{ij} are cofactors of the elements a_{ij} of the matrix A, $i, j = \overline{1, n}$.

Now let us consider the following example.

Example 17. Find the inverse matrix for the following matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$.

Solution. Let us find the determinant of the given matrix:

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{vmatrix} = 1\cdot 0\cdot 0 + 1\cdot 3\cdot(-2) + 3\cdot 3\cdot 2 - 0\cdot(-2)\cdot 2 - 3\cdot 3\cdot 1 - 3\cdot 1\cdot 0 =$$

$$= 0 - 6 + 18 - 0 - 9 - 0 = 3 \neq 0.$$

Its determinant is non-zero. Find the cofactors for the elements of the matrix A:

$$A_{11} = (-1)^{1+1}\cdot\begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = 1\cdot(0-9) = -9,$$

$$A_{12} = (-1)^{1+2}\cdot\begin{vmatrix} 3 & 3 \\ -2 & 0 \end{vmatrix} = (-1)\cdot(0-(-6)) = -6,$$

$$A_{13} = (-1)^{1+3}\cdot\begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix} = 1\cdot(9-0) = 9,$$

$$A_{21} = (-1)^{2+1}\cdot\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} = (-1)\cdot(0-6) = 6,$$

$$A_{22} = (-1)^{2+2}\cdot\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 1\cdot(0-(-4)) = 4,$$

$$A_{23} = (-1)^{2+3}\cdot\begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} = (-1)\cdot(3-(-2)) = -5,$$

$$A_{31} = (-1)^{3+1}\cdot\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 1\cdot(3-0) = 3,$$

$$A_{32} = (-1)^{3+2}\cdot\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = (-1)\cdot(3-6) = 3,$$

$$A_{33} = (-1)^{3+3}\cdot\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = 1\cdot(0-3) = -3.$$

We obtain the inverse matrix:

$$A^{-1} = \frac{1}{3}\begin{pmatrix} -9 & 6 & 3 \\ -6 & 4 & 3 \\ 9 & -5 & -3 \end{pmatrix} = \begin{pmatrix} -3 & 2 & 1 \\ -2 & 4/3 & 1 \\ 3 & -5/3 & -1 \end{pmatrix}.$$

Checking the condition $A\cdot A^{-1}=E$:

$$A\cdot A^{-1}=\begin{pmatrix}1 & 1 & 2\\ 3 & 0 & 3\\ -2 & 3 & 0\end{pmatrix}\cdot\begin{pmatrix}-3 & 2 & 1\\ -2 & 4/3 & 1\\ 3 & -5/3 & -1\end{pmatrix}=$$

$$=\begin{pmatrix}-3-2+6 & 2+4/3-10/3 & 1+1-2\\ -9+0+9 & 6+0-5 & 3+0-3\\ 6-6+0 & -4+4+0 & -2+3+0\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E.$$

1.5.2. Finding inverses by elementary row operations

We shall consider a technique by which we can find the inverse A^{-1} of a square matrix A, if the inverse A^{-1} exists. Before we consider this technique, let us recall the *three elementary row operations (*or *elementary transformations)*. These are:

1) interchanging (exchanging) two different rows;

2) adding a multiple of one row to another row;

3) multiplying one row by a non-zero constant.

We consider an array with the matrix A on the left and the unit matrix E on the right:

$$A|E=\left(\begin{array}{ccc|ccc}a_{11} & a_{12} & a_{13} & 1 & 0 & 0\\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0\\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1\end{array}\right).$$

The obtained matrix is called *an augmented matrix* and denoted by $A|E$. We now perform elementary row operations on the array $A|E$ and try to reduce the left hand half to the unit matrix E.

Now let us consider the following example.

Example 18. Find the inverse matrix for $A=\begin{pmatrix}1 & 1 & 2\\ 3 & 0 & 3\\ -2 & 3 & 0\end{pmatrix}$ like in example 17.

Solution. To find A^{-1}, we consider the augmented matrix $A|E$:

$$A|E=\left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\3&0&3&0&1&0\\-2&3&0&0&0&1\end{array}\right).$$

We now perform elementary row operations on this array and try to reduce the left hand half to the unit matrix E.

Let's numerate the rows of an augmented matrix:

$$A|E=\left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\3&0&3&0&1&0\\-2&3&0&0&0&1\end{array}\right)\sim\begin{bmatrix}[1]\\ [2]\\ [3]\end{bmatrix}\sim$$

where $[i]$ is denoted by the number row i.

Adding –3 times row 1 to row 2 and writing down the result instead of row 2, we obtain

$$\sim\begin{bmatrix}\\ [1]\cdot(-3)+[2]\\ \\ \end{bmatrix}\sim\left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&-3&-3&-3&1&0\\-2&3&0&0&0&1\end{array}\right)\sim$$

Adding 2 times row 1 to row 3 and writing down the result instead of row 3, we obtain

$$\sim\begin{bmatrix}\\ \\ [1]\cdot 2+[3]\end{bmatrix}\sim\left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&-3&-3&-3&1&0\\0&5&4&2&0&1\end{array}\right)\sim$$

Dividing row 3 by 3 and writing down the result instead of row 2, we obtain

$$\sim\begin{bmatrix}\\ [2]:(-3)\\ \\ \end{bmatrix}\sim\left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&1&1&1&-1/3&0\\0&5&4&2&0&1\end{array}\right)\sim$$

Adding –5 times row 2 to row 3 and writing down the result instead of row 2, we obtain

$$\sim\begin{bmatrix}\\ \\ [2]\cdot(-5)+[3]\end{bmatrix}\sim\left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&1&1&1&-1/3&0\\0&0&-1&-3&5/3&1\end{array}\right)\sim$$

Adding –1 times row 2 to row 1 and writing down the result instead of row 1, we obtain

$$\sim\begin{bmatrix}[2]\cdot(-1)+[1]\\ \\ \\ \end{bmatrix}\sim\left(\begin{array}{ccc|ccc}1&0&1&0&1/3&0\\0&1&1&1&-1/3&0\\0&0&-1&-3&5/3&1\end{array}\right)\sim$$

Multiplying row 3 by (–1), we obtain

$$\sim\begin{bmatrix}\\ [3]\cdot(-1)\\ \\ \end{bmatrix}\sim\left(\begin{array}{ccc|ccc}1&0&1&0&1/3&0\\0&1&1&1&-1/3&0\\0&0&1&3&-5/3&-1\end{array}\right)\sim$$

Adding –1 times row 3 to row 2 and writing down the result instead of row 2, we obtain

$$\sim\begin{bmatrix}\\ [3]\cdot(-1)+[2]\\ \\ \end{bmatrix}\sim\left(\begin{array}{ccc|ccc}1&0&1&0&1/3&0\\0&1&0&-2&4/3&1\\0&0&1&3&-5/3&-1\end{array}\right)\sim$$

Adding –1 times row 3 to row 1 and writing down the result instead of row 1, we obtain

$$\sim\begin{bmatrix}[3]\cdot(-1)+[1]\\ \\ \\ \end{bmatrix}\sim\left(\begin{array}{ccc|ccc}1&0&0&-3&2&1\\0&1&0&-2&4/3&1\\0&0&1&3&-5/3&-1\end{array}\right).$$

Note now that the augmented matrix $A|E$ is in reduced row echelon form, and that the left hand half is the unit matrix E. It follows that the right hand half of the augmented matrix $A|E$ represents the inverse A^{-1}. Hence

$$A^{-1}=\begin{pmatrix}-3&2&1\\-2&4/3&1\\3&-5/3&-1\end{pmatrix}.$$

Both way 1.5.1 and 1.5.2 have the same result.

1.6. Rank of a matrix

The rank of the $m\times n$-matrix A is the highest order of its non-zero minor and denoted by $r(A)$, $rg(A)$ or $rang\ A$.

For a non-zero matrix $0 \le rang\, A \le \min\{m,n\}$. If the $rang\, A = k$, then any non-zero minor of the k-th order is called *basic.*

To find a rank of a matrix we can use elementary row operations to reduce the given matrix to the triangular form:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \qquad (1.1)$$

or to the truncated-triangular form:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} & a_{1k+1} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2k} & a_{2k+1} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{kk} & a_{k\,k+1} & \dots & a_{kn} \end{pmatrix} \qquad (1.2)$$

then the number of non-zero rows of the transformed matrix defines the rank of the initial matrix.

Elementary row operations (or elementary transformations) are:

1) interchanging (exchanging) two different rows;
2) adding a multiple of one row to another row;
3) multiplying one row by a non-zero constant;
4) crossing out one of the same row;
5) crossing out of zero row.

Example 19. Calculate the rank of the matrix: $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & -3 & 2 \\ 3 & 3 & 9 \end{pmatrix}$.

Solution. Let us exchange the 1-st and the 2-nd rows:

$$A \sim \begin{pmatrix} 1 & -3 & 2 \\ 2 & -2 & 3 \\ 3 & 3 & 9 \end{pmatrix} \sim \begin{bmatrix} \\ [1]\cdot(-2)+[2] \\ [1]\cdot(-3)+[3] \end{bmatrix} \sim \begin{pmatrix} 1 & -3 & 2 \\ 0 & -4 & -1 \\ 0 & 12 & 3 \end{pmatrix} \sim \begin{bmatrix} \\ [2]:(-1) \\ [3]:3 \end{bmatrix} \sim$$

$$\sim\begin{pmatrix}1 & -3 & 2\\ 0 & 4 & 1\\ 0 & 4 & 1\end{pmatrix}\sim\begin{pmatrix}1 & -3 & 2\\ 0 & 4 & 1\end{pmatrix}.$$

The number of non-zero rows of the transformed matrix equivalent to the initial one is 2. Therefore $rang\ A = 2$.

2. Systems of linear algebraic equations

A system of m *linear algebraic equations with* n *unknown quantities* has the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}, \qquad (2.1)$$

where $a_{11}, a_{12}, \ldots, a_{mn}$ are *the coefficients of the system*; $b_1, b_2, \ldots, b_m$ are *its free terms* and $x_1, x_2, \ldots, x_n$ are *the unknown quantities*.

A solution of the system (2.1) is a set of n numbers $x_1, x_2, \ldots, x_n$ satisfying every equation of system.

Every system of the form (2.1) has either no solution, one solution or infinitely many solutions.

A system is *consistent or compatible* if there exists at least one solution, otherwise it is *inconsistent or incompatible*.

A compatible system is called *definite or determined* if it has the only solution.

A compatible system is called *indefinite or undetermined* if it has more than one various solutions.

Systems are *equivalent* if they have the same solution set.

If $b_j = 0$ for all *j* the system (2.1) is *homogeneous*.

Note that the system (2.1) can be written in a matrix form as $AX = B$, where the $m \times n$-matrix $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ is *the matrix of coefficients or the basic matrix*, the matrix-column $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is *a matrix-column of free terms* and the matrix-column $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is *the unknown matrix-column*.

2.1. Solution of system of equations using Cramer method

Let the system (2.1) consist of n linear equations with n unknown quantities and its determinant $\det A \neq 0$ then unknown quantities can be found accordingly to the formulas by Cramer:

$$x_i = \frac{\Delta_i}{\Delta},\ i = \overline{1, n},$$

where Δ is the determinant of the system; Δ_i is determinant obtained from the determinant of the system by substituting the column i by the matrix-column B:

$$\Delta_i = \begin{vmatrix} a_{11} \dots a_{1\,i-1} & b_1 & a_{1\,i+1} \dots a_{1n} \\ a_{21} \dots a_{2\,i-i} & b_2 & a_{2\,i+1} \dots a_{2n} \\ \dots & \dots & \dots \\ a_{n1} \dots a_{n\,i-1} & b_n & a_{n\,i+1} \dots a_{nn} \end{vmatrix}.$$

For example, consider the system (2.1) consists of 3 linear equations with 3 unknown quantities $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$. Then unknown quantities can be found accordingly to the formulas by Cramer:

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta},\ x_3 = \frac{\Delta_3}{\Delta},$$

where $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$,

$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$.

Let us illustrate this method by example.

Example 20. Solve the given system of equations using Cramer method:

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases}.$$

Solution. Find the determinant of the given system:

$$\Delta = \begin{vmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 5\cdot 2\cdot 2 + 4\cdot 3\cdot(-1) + 1\cdot 3\cdot(-1) - 4\cdot 2\cdot(-1) - 2\cdot 1\cdot(-1) -$$
$$- 3\cdot 3\cdot 5 = 20 - 12 - 3 + 8 + 2 - 45 = -30 \neq 0.$$

Its determinant is non-zero. Apply the formulas by Cramer:

$$\Delta_1 = \begin{vmatrix} 0 & -1 & -1 \\ 14 & 2 & 3 \\ 16 & 3 & 2 \end{vmatrix} = 0 - 48 - 42 + 32 - 0 + 28 = -30,\ x_1 = \frac{\Delta_1}{\Delta} = \frac{-30}{-30} = 1,$$

$$\Delta_2 = \begin{vmatrix} 5 & 0 & -1 \\ 1 & 14 & 3 \\ 4 & 16 & 2 \end{vmatrix} = 140 + 0 - 16 + 56 - 0 - 240 = -60,\ x_2 = \frac{\Delta_2}{\Delta} = \frac{-60}{-30} = 2,$$

$$\Delta_3 = \begin{vmatrix} 5 & -1 & 0 \\ 1 & 2 & 14 \\ 4 & 3 & 16 \end{vmatrix} = 160 - 56 + 0 - 0 - 210 + 16 = -90,\ x_3 = \frac{\Delta_3}{\Delta} = \frac{-90}{-30} = 3.$$

Checking by substitution x_1, x_2, x_3 into the initial system:

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases} \Rightarrow \begin{cases} 5 \cdot 1 - 2 - 3 = 0 \\ 1 + 2 \cdot 2 + 3 \cdot 3 = 14 \\ 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 = 16 \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ 14 = 14. \\ 16 = 16 \end{cases}$$

2.2. Solution of system of equations using an inverse matrix

Let the system (2.1) consist of n linear equations with n unknowns and its determinant $\det A \neq 0$. Write this system in a matrix form as

$$AX = B. \tag{2.2}$$

Let us multiply both parts (2.2) by the inverse matrix A^{-1} on the left. Then we obtain

$$A^{-1} \cdot AX = A^{-1} \cdot B.$$

Since $A^{-1} \cdot A = E$ and $E \cdot X = X$ we can get a solution by the formula $X = A^{-1} \cdot B$.

Let us illustrate this method by example.

Example 21. Let's find a solution of the system from example 20 by the matrix method.

Solution. Here $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 14 \\ 16 \end{pmatrix}$, $A = \begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$.

Calculate the determinant of the given system:

$$\Delta=\begin{vmatrix}5 & -1 & -1\\ 1 & 2 & 3\\ 4 & 3 & 2\end{vmatrix}=5\cdot2\cdot2+4\cdot3\cdot(-1)+1\cdot3\cdot(-1)-4\cdot2\cdot(-1)-2\cdot1\cdot(-1)-$$

$$-3\cdot3\cdot5=20-12-3+8+2-45=-30\neq0.$$

Its determinant is non-zero. Find the inverse matrix by cofactors:

$$A_{11}=\begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix}=-5,\qquad A_{21}=-\begin{vmatrix}-1 & -1\\ 3 & 2\end{vmatrix}=-1,\qquad A_{31}=\begin{vmatrix}-1 & -1\\ 2 & 3\end{vmatrix}=-1,$$

$$A_{12}=-\begin{vmatrix}1 & 3\\ 4 & 2\end{vmatrix}=10,\qquad A_{22}=\begin{vmatrix}5 & -1\\ 4 & 2\end{vmatrix}=14,\qquad A_{32}=-\begin{vmatrix}5 & -1\\ 1 & 3\end{vmatrix}=-16,$$

$$A_{13}=\begin{vmatrix}1 & 2\\ 4 & 3\end{vmatrix}=-5,\qquad A_{23}=-\begin{vmatrix}5 & -1\\ 4 & 3\end{vmatrix}=-19,\qquad A_{33}=\begin{vmatrix}5 & -1\\ 1 & 2\end{vmatrix}=11,$$

$$A^{-1}=\frac{1}{-30}\begin{pmatrix}-5 & -1 & -1\\ 10 & 14 & -16\\ -5 & -19 & 11\end{pmatrix}=\begin{pmatrix}\frac{5}{30} & \frac{1}{30} & \frac{1}{30}\\ -\frac{10}{30} & -\frac{14}{30} & \frac{16}{30}\\ \frac{5}{30} & \frac{19}{30} & -\frac{11}{30}\end{pmatrix}.$$

Check the condition $A\cdot A^{-1}=E$:

$$A\cdot A^{-1}=\begin{pmatrix}5 & -1 & -1\\ 1 & 2 & 3\\ 4 & 3 & 2\end{pmatrix}\begin{pmatrix}\frac{5}{30} & \frac{1}{30} & \frac{1}{30}\\ -\frac{10}{30} & -\frac{14}{30} & \frac{16}{30}\\ \frac{5}{30} & \frac{19}{30} & -\frac{11}{30}\end{pmatrix}=$$

$$=\frac{1}{30}\begin{pmatrix}25+10-5 & 5+14-19 & 5-16+11\\ 5-20+15 & 1-28+57 & 1+32-33\\ 20-30+10 & 4-42+38 & 4+48-22\end{pmatrix}=\frac{1}{30}\begin{pmatrix}30 & 0 & 0\\ 0 & 30 & 0\\ 0 & 0 & 30\end{pmatrix}=$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E.$$

The solution of given system is $X=A^{-1}\cdot B$. Then

$$X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=A^{-1}\cdot B=\begin{pmatrix}\frac{5}{30}&\frac{1}{30}&\frac{1}{30}\\-\frac{10}{30}&-\frac{14}{30}&\frac{16}{30}\\\frac{5}{30}&\frac{19}{30}&-\frac{11}{30}\end{pmatrix}\begin{pmatrix}0\\14\\16\end{pmatrix}=\begin{pmatrix}\frac{1}{6}0+\frac{14}{30}+\frac{16}{30}\\-\frac{1}{3}0-\frac{98}{15}+\frac{128}{15}\\\frac{1}{6}0+\frac{266}{30}-\frac{176}{30}\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}.$$

Thus $x_1=1,\ x_2=2,\ x_3=3.$

2.3. Solution of system of equations using Gauss method

Definition. Matrices obtained one from another by elementary row operations are called *equivalent*. The equivalence of matrices is marked by the sign $\sim$.

Gauss method is used to solve the system (2.1), which consists of *m* linear equations with *n* unknowns. This method includes sequential elimination of unknowns to following scheme.

1. Create an augmented matrix of the given system $A|B=\left(\begin{array}{cccc|c}a_{11}&a_{12}&\dots&a_{1n}&b_1\\a_{21}&a_{22}&\dots&a_{2n}&b_2\\\vdots&\vdots&\dots&\vdots&\vdots\\a_{m1}&a_{m2}&\dots&a_{mn}&b_m\end{array}\right)$. *The augmented matrix* is called an array with the matrix A on the left and the matrix-column B of free terms on the right and denoted by $A|B$. The vertical line separates the matrix-column B.

The leading row and the leading element in $A|B$ that corresponds to the choice of the leading equation and the leading unknown in the system (2.1) are chosen. The system should be transformed in order to let the leading equation be the first one.

2. The leading unknown by means of the leading equation is eliminated from the other equations. For this the certain *elementary row operations* of the matrix $A|B$ are performed it is possible:

1) to change the order of rows (that corresponds to change of the order of the equations' sequence);

2) to multiply rows by any non-zero numbers (that corresponds to multiplying the corresponding equations by these numbers);

3) to add to any row of the matrix $A|B$ its any other row multiplied by any number (that corresponds to addition to one equation of the system another equation multiplied by this number).

Due to such transformations we obtain an augmented matrix, *equivalent to the initial one* (i. e. having the same solutions).

On the second step a new leading unknown and a corresponding leading equation are chosen and then this variable is eliminated from all the other equations. The leading row in the matrix $A|B$ remains without change. After such actions the initial matrix *A* will be reduced to the triangular (1.1) or truncated-triangular form (1.2) with the elements of the main diagonal equal to 1.

Let us illustrate this method by example.

Example 22. Let's find a solution of the system from example 20 by Gauss method.

Solution. 1. Write down an augmented matrix of the given system

$$A|B=\left(\begin{array}{ccc|c} 5 & -1 & -1 & 0 \\ 1 & 2 & 3 & 14 \\ 4 & 3 & 2 & 16 \end{array}\right)\sim$$

Consider the direct way of Gauss method.

2. The leading unknown has the coefficient equal to 1, therefore find 1 in the first column. It is found at the second place (the second equation or the second row). Then exchange the first and second rows.

If there is no such an equation then we divide the first equation by *x* coefficient.

$$\sim\left(\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 5 & -1 & -1 & 0 \\ 4 & 3 & 2 & 16 \end{array}\right)\sim$$

Now 1 is found at the first place of the first column.

Let's put at the first place an equation that has the first unknown coefficient equal to 1.

3. Let's carry out elementary row operations with the matrix $A|B$. The aim is to get a triangular matrix with the elements of the main diagonal equal to 1. Let's comment on our actions.

Let’s numerate the rows of an augmented matrix:

$$\sim\left(\begin{array}{ccc|c}1 & 2 & 3 & 14\\ 5 & -1 & -1 & 0\\ 4 & 3 & 2 & 16\end{array}\right)\sim\begin{bmatrix}[1]\\ [2]\\ [3]\end{bmatrix}\sim$$

The first row is the leading row.

Let’s leave the first row unchanged and add to the second row elements the corresponding elements of the first row, multiplying by (-5) and let's write down the result instead of the second row.

Let’s add the corresponding elements to the first row multiplying by (-4) to the elements of the third row and let's write down the result replacing the third row.

$$\sim\begin{bmatrix} \\ [2]+[1]\cdot(-5)\\ [3]+[1]\cdot(-4)\end{bmatrix}\sim\left(\begin{array}{ccc|c}1 & 2 & 3 & 14\\ 0 & -5 & -10 & -40\\ 0 & -11 & -16 & -70\end{array}\right)\sim$$

Let’s simplify the received augmented matrix, dividing the second row by (-5).

$$\sim\begin{bmatrix} \\ [2]:(-5)\\ \\ \end{bmatrix}\sim\left(\begin{array}{ccc|c}1 & 2 & 3 & 14\\ 0 & 1 & 2 & 8\\ 0 & -11 & -16 & -70\end{array}\right)\sim$$

Now the second row is the leading row. Let’s work with the first and third rows.

Let's add the second row elements, multiplied by 11, to the third row. Let us write down the result instead of the third row.

$$\sim\begin{bmatrix} \\ \\ [3]+[2]\cdot 11\end{bmatrix}\sim\left(\begin{array}{ccc|c}1 & 2 & 3 & 14\\ 0 & 1 & 2 & 8\\ 0 & 0 & 6 & 18\end{array}\right)\sim$$

Let us simplify the received augmented matrix, dividing the third row by 6.

$$\sim\begin{bmatrix} \\ \\ [3]:6\end{bmatrix}\sim\left(\begin{array}{ccc|c}1 & 2 & 3 & 14\\ 0 & 1 & 2 & 8\\ 0 & 0 & 1 & 3\end{array}\right).$$

In the result we get “zeros” below the main diagonal.

Then write down the received augmented matrix as the system of questions:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 14 \\ x_2 + 2x_3 = 8 \\ x_3 = 3 \end{cases}.$$

Consider the backward way of Gauss method.

This system is solved by so-called “backward substitution”: inserting $x_3 = 3$ (obtained from the third equation) into the second equation, one finds x_2: $x_2 = 8 - 2 \cdot 3$ or $x_2 = 2$. Then inserting the values obtained for x_3 and x_2 into the first equation, one finds x_1: $x_1 = 14 - 3 \cdot 3 - 2 \cdot 2$or $x_1 = 1$.

Thus $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

2.4. Solution of system of equations using Jordan–Gauss method

According to the method by Jordan–Gauss the leading unknown by means of the leading equation on the current step is eliminated not only from equations of the corresponding subsystem but also from the leading equations on previous steps and on any step the leading unknown has the coefficient equal to 1.

Example 23. Let’s find a solution of the system from example 20 by Jordan-Gauss method.

Solution. By elementary row operations of the augmented matrix, we obtain

$$A|B = \left(\begin{array}{ccc|c} 5 & -1 & -1 & 0 \\ 1 & 2 & 3 & 14 \\ 4 & 3 & 2 & 16 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 5 & -1 & -1 & 0 \\ 4 & 3 & 2 & 16 \end{array}\right) \sim \begin{bmatrix} \\ [2]+[1]\cdot(-5) \\ [3]+[1]\cdot(-4) \end{bmatrix} \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -5 & -10 & -40 \\ 0 & -11 & -16 & -70 \end{array}\right) \sim \begin{bmatrix} \\ [2]:(-5) \\ \end{bmatrix} \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & 2 & 8 \\ 0 & -11 & -16 & -70 \end{array}\right) \sim$$

$$\sim \begin{bmatrix} [1]+[2]\cdot(-2) \\ \\ [3]+[2]\cdot 11 \end{bmatrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 6 & 18 \end{array}\right) \sim \begin{bmatrix} \\ \\ [3]:6 \end{bmatrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right) \sim$$

$$\sim \begin{bmatrix} [1]+[3] \\ [2]+[3]\cdot(-2) \\ \end{bmatrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

A unit matrix on the left of the vertical line is obtained. The column on the right of the vertical line is values of unknown quantities.

Then write down the received augmented matrix as the system of questions:

$$\begin{cases} x_1 = 1 \\ x_2 = 2. \\ x_3 = 3 \end{cases}$$

Thus $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

2.5. Investigation of the system compatibility

Kronecker–Capelli theorem. *1.* A linear system (2.1) is consistent if its basic matrix and its augmented matrix have the same rank, i. e. $rang\ A = rang\ A|B$.

A consistent system is determined if the ranks are equal to the unknowns number, i. e. $rang\ A = rang\ A|B = n$.

2. A consistent system is undetermined if the ranks are less than the unknowns number, i. e. $rang\ A = rang\ A|B < n$.

3. A linear system is inconsistent if its basic matrix and its augmented matrix have the different rank, i. e. $rang\ A \neq rang\ A|B$.

If $rang\ A = rang\ A|B = n$, then carrying out the backward way we obtain the corresponding values of unknowns.

If $rang\ A = rang\ A|B = r < n$, then we should choose *the main (basic)* unknowns, i. e. those ones which coefficients generate the unit matrix. The basic variables are remained on the left, and other $n-r$ variables are transposed to the right parts of equations. The variables placed on the right part of the system are called *free variables.* The basic variables are expressed through free ones using the backward way. The obtained equalities are the *general solution of the system.*

Assigning to free variables any numeric values, we can find corresponding values of the basic variables. Thus we can find the *particular solutions* of the initial system of equations.

If free variables are assigned zero value, then the obtained particular solution is called *basic*.

If the values of the basic variables are not negative, then the solution is called *supporting*.

Investigation of the system compatibility is carried out using Gauss method or Jordan–Gauss method.

Example 24. Investigate the compatibility of the given system:

$$\begin{cases} x_1 + 2x_2 - 3x_3 + 4x_4 = 7 \\ 2x_1 + 4x_2 + 5x_3 - x_4 = 2 \\ 5x_1 + 10x_2 + 7x_3 + 2x_4 = 11 \end{cases}.$$

Solution. By elementary row operations of the augmented matrix, we obtain:

$$A|B = \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 2 & 4 & 5 & -1 & 2 \\ 5 & 10 & 7 & 2 & 11 \end{array}\right) \sim \begin{bmatrix} \\ [2]+[1]\cdot(-2) \\ [3]+[1]\cdot(-5) \end{bmatrix} \sim \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 11 & -9 & -12 \\ 0 & 0 & 22 & -18 & -24 \end{array}\right) \sim$$

$$\sim \begin{bmatrix} \\ [2]:11 \\ \\ \end{bmatrix} \sim \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \\ 0 & 0 & 22 & -18 & -24 \end{array}\right) \sim \begin{bmatrix} \\ \\ [3]+[2]\cdot(-22) \end{bmatrix} \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \end{array}\right) \sim \begin{bmatrix} [1]+[2]\cdot 3 \\ \\ \end{bmatrix} \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \end{array}\right) \sim \begin{bmatrix} [1]+[2]\cdot 3 \\ \\ \end{bmatrix} \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 17/11 & 41/11 \\ 0 & 0 & 1 & -9/11 & -12/11 \end{array}\right).$$

The initial system is equivalent to the following system of equations:

$$\begin{cases} x_1 + 2x_2 + \frac{17}{11}x_4 = \frac{41}{11} \\ x_3 - \frac{9}{11}x_4 = -\frac{12}{11} \end{cases}.$$

Let’s obtain the general solution:

$$\begin{cases} x_1 = \frac{41}{11} - 2x_2 - \frac{17}{11}x_4 \\ x_3 = -\frac{12}{11} + \frac{9}{11}x_4 \end{cases},$$

where x_1, x_3 are basic unknowns, x_2, x_4 are free ones.

For example, obtain the particular solution, if $x_2 = 1, x_4 = -1$:

$x_1 = \frac{41}{11} - 2 + \frac{17}{11}$ or $x_1 = \frac{36}{11}$, $\quad x_3 = -\frac{12}{11} - \frac{9}{11}$ or $x_3 = -\frac{21}{11}$.

Thus $x_1 = \frac{36}{11}$, $x_2 = 1$, $x_3 = -\frac{21}{11}$, $x_4 = -1$ are the particular solution.

For example, obtain the basic solution, if $x_2 = 0, x_4 = 0$:

$$x_1 = \frac{41}{11}, \qquad x_3 = -\frac{12}{11}.$$

Thus $x_1 = \frac{41}{11}$, $x_2 = 0$, $x_3 = -\frac{12}{11}$, $x_4 = 0$ are the basic solution.

In this example the basic solution is not the supporting one, because $x_3 = -\frac{12}{11} < 0$.

Control tasks

Task 1. For the pairs of matrices below say whether it is possible to add (subtract) them together and then, where it is possible, derive the matrices $C = A + B$, $D = A - B$, $F = 3A - 4B$, $G = (-3)A + \frac{1}{2}B$:

1) $A = \begin{pmatrix} 0 & 3 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$; 2) $A = \begin{pmatrix} 1 & 4 & 2 & 3 \\ -3 & 2 & -5 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 0 & 2 & -5 \\ 6 & 1 & 3 & 1 \end{pmatrix}$; 3) $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & -4 & -3 \\ -1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 6 & -2 \\ -1 & 5 & 3 \end{pmatrix}$.

Task 2. In each of the following cases, determine whether the products AB and BA are both defined; if so, determine also whether AB and BA have the same number of rows and the same number of columns; if so, determine also whether $AB = BA$:

a) $A = \begin{pmatrix} 0 & 3 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$; b) $A = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 3 & 6 \\ 1 & 5 \end{pmatrix}$;

c) $A = \begin{pmatrix} 3 & 1 & -4 \\ -2 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Task 3. Evaluate A^2 and A^3, where

1) $A = \begin{pmatrix} 2 & -5 \\ 3 & 1 \end{pmatrix}$; 2) $A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 3 & 2 \\ 0 & 2 & -7 \end{pmatrix}$; 3) $A = \begin{pmatrix} 2 & 4 & 2 \\ -1 & -9 & -7 \\ -4 & 3 & -1 \end{pmatrix}$.

Task 4. Carry out the operations $2(A+B)B$, $3B(B-2A)$, $(3A-2B)A$ on the given matrices and check the following properties: a) $(A+B)C = AC + BC$, b) $C(A+B) = CA + CB$, c) $A(BC) = A(BC)$.

1) $A=\begin{pmatrix} 1 & 2 & 4 \\ 1 & -4 & -3 \\ -1 & 1 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 4 & -3 & 0 \\ 3 & 2 & -1 \\ -2 & -1 & 4 \end{pmatrix}$, $C=\begin{pmatrix} 1 & 3 & 4 \\ -2 & 0 & 3 \\ 5 & 6 & 4 \end{pmatrix}$;

2) $A=\begin{pmatrix} 4 & -1 & 5 \\ 6 & 4 & -3 \\ -2 & 0 & 5 \end{pmatrix}$, $B=\begin{pmatrix} 3 & -1 & 3 \\ 5 & 3 & -5 \\ -4 & -1 & 4 \end{pmatrix}$, $C=\begin{pmatrix} 3 & -1 & 3 \\ 5 & 3 & -5 \\ -4 & -1 & 4 \end{pmatrix}$.

Task 5. Consider the six matrices

$$A=\begin{pmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 2 \end{pmatrix}, \quad B=\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 3 \\ 1 & 1 & 5 \\ 3 & 2 & 1 \end{pmatrix}, \quad C=\begin{pmatrix} 1 & 0 & 7 \\ 2 & 1 & 2 \\ 1 & 3 & 0 \end{pmatrix}, \quad D=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

$$F=\begin{pmatrix} 1 & 7 & 2 & 9 \\ 9 & 2 & 7 & 1 \end{pmatrix}, \quad G=\begin{pmatrix} -4 & 7 & 2 \end{pmatrix}.$$

a) Calculate all possible products. b) Transpose these matrices.

Task 6. Find the inverse matrix for the given matrix by using two methods and check $A\cdot A^{-1}$ and $A^{-1}\cdot A$:

1) $A=\begin{pmatrix} 2 & 8 & -3 \\ -1 & -7 & 4 \\ -3 & -6 & 2 \end{pmatrix}$; 2) $A=\begin{pmatrix} 3 & 2 & -3 \\ 5 & 4 & 1 \\ -6 & 3 & 1 \end{pmatrix}$; 3) $A=\begin{pmatrix} 2 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$;

4) $A=\begin{pmatrix} -6 & 9 & 0 \\ 1 & 2 & 3 \\ 11 & 5 & 7 \end{pmatrix}$; 5) $A=\begin{pmatrix} 3 & -1 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix}$; 6) $A=\begin{pmatrix} 6 & 3 & 2 \\ 7 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}$.

Task 7. Calculate the determinant of the given matrices on the base of Sarrus formula and check the result using the theorem concerning the decomposition of the determinant in row 1 (2 or 3) and column 1 (2 or 3):

1) $A=\begin{pmatrix} 2 & 3 & 3 \\ 1 & 0 & -2 \\ -1 & -4 & 1 \end{pmatrix}$; 2) $A=\begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ -2 & -3 & 2 \end{pmatrix}$; 3) $A=\begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & -2 \\ -1 & -1 & 4 \end{pmatrix}$;

4) $B=\begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & -4 \\ -4 & 0 & 2 \end{pmatrix}$; 5) $B=\begin{pmatrix} 5 & 6 & 1 \\ 4 & 0 & -1 \\ -3 & -1 & 4 \end{pmatrix}$; 6) $B=\begin{pmatrix} 5 & 1 & 2 \\ 7 & 2 & -3 \\ -1 & 0 & 6 \end{pmatrix}$.

Task 8. Calculate the following determinants:

1) $\begin{vmatrix} -1 & -2 & 3 & 2 \\ 2 & 3 & 4 & -2 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 3 & 1 \end{vmatrix}$; 2) $\begin{vmatrix} 2 & 3 & 1 & -2 \\ 1 & -2 & -2 & 1 \\ -2 & 1 & 0 & -1 \\ 3 & 0 & 3 & 1 \end{vmatrix}$; 3) $\begin{vmatrix} 2 & 1 & 2 & 1 \\ 2 & 1 & 3 & -2 \\ 0 & -1 & 2 & 2 \\ -1 & 0 & 4 & -3 \end{vmatrix}$;

4) $\begin{vmatrix} 2 & 0 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 3 & 1 & -1 & -2 \\ 4 & -2 & 3 & 1 \end{vmatrix}$; 5) $\begin{vmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & -1 & -4 \\ 3 & 2 & 0 & 1 \\ 4 & -2 & 1 & 2 \end{vmatrix}$; 6) $\begin{vmatrix} 2 & -1 & 1 & -3 \\ 1 & -2 & -3 & 1 \\ -2 & 2 & 0 & -1 \\ 3 & 0 & 4 & 1 \end{vmatrix}$.

Task 9. Solve the systems using: a) Cramer method; b) an inverse matrix; c) Gauss method; d) Jordan–Gauss method.

1) $\begin{cases} 2x_1+3x_2-2x_3=5 \\ -x_1+4x_2-x_3=3 \\ x_1-x_2+x_3=0 \end{cases}$; 2) $\begin{cases} x_1+4x_2-3x_3=5 \\ -2x_1+x_2-x_3=-1 \\ 3x_1-x_2+2x_3=2 \end{cases}$; 3) $\begin{cases} 2x_1+2x_2-2x_3=-2 \\ -x_1+x_2-3x_3=-5 \\ x_1-x_2+x_3=3 \end{cases}$;

4) $\begin{cases} 2x_1+4x_2-2x_3=2 \\ 5x_1-3x_2+2x_3=-1 \\ 3x_1+x_2-x_3=0 \end{cases}$; 5) $\begin{cases} 4x_1-2x_2-x_3=-3 \\ -x_1+2x_2-x_3=1 \\ -2x_1-x_2-2x_3=-3 \end{cases}$; 6) $\begin{cases} x_1-2x_2-x_3=-3 \\ x_1+2x_2-x_3=1 \\ -3x_1-x_2-2x_3=-3 \end{cases}$.

Task 10. Investigate the compatibility of the given systems of equations and in the case of their compatibility solve them:

1) $\begin{cases} 5x_1-8x_2-6x_3=-14 \\ 3x_1-12x_2-5x_3=-4 \\ 2x_1-6x_2-x_3=0 \\ 2x_1+4x_2+3x_3=-2 \end{cases}$; 2) $\begin{cases} 2x_1+x_2+x_3=2 \\ 5x_1+x_2+3x_3=4 \\ 7x_1+2x_2+4x_3=1 \end{cases}$;

3) $\begin{cases} 3x_1-5x_2+2x_3+4x_4=2 \\ 7x_1-4x_2+x_3+3x_4=5 \\ 5x_1+7x_2-4x_3-6x_4=3 \end{cases}$; 4) $\begin{cases} x_1+x_2+x_3+x_4+x_5=7 \\ 3x_1+2x_2+x_3+x_4-3x_5=-2 \\ x_2+2x_3+2x_4+6x_5=23 \\ 5x_1+4x_2+3x_3+3x_4-x_5=12 \end{cases}$;

5) $\begin{cases} x_1 + x_2 - 3x_4 - 4x_5 = 0 \\ x_1 + x_2 - x_3 + 2x_4 - x_5 = 1 \\ 2x_1 + 2x_2 + x_3 - x_4 + 3x_5 = 0 \end{cases}$; 6) $\begin{cases} x_1 - 2x_2 - x_3 + x_4 = -2 \\ 2x_1 + x_4 - x_5 = 9 \\ -3x_1 + x_2 + x_5 - x_6 = -16 \\ 2x_1 + 2x_2 + x_3 - x_4 + x_6 = 23 \end{cases}$.

Theoretical questions

1. What do you call the matrix?
2. What is the size of a matrix or its order?
3. What matrices do you know? Call all the types of matrices.
4. What matrix is called
   a) transposed? b) singular? c) nonsingular?
5. What matrices can be:
   a) added? b) subtracted?
6. How can a matrix be multiplied by a scalar value?
7. What matrices can be multiplied? What is the rule of multiplying a matrix by a matrix?
8. How many operations on matrices do you know?
9. What matrix is called inverse to a given matrix? Does an inverse matrix exist for any matrix? Explain the rule “row by column”.
10. What methods are used for finding an inverse matrix?
11. What transformations are called elementary?
12. What do you call 2-nd and 3-rd order determinant?
13. What is the rule of triangle (or Sarrus formula)?
14. What are the basic properties of determinants?
15. What do you call minor and algebraic cofactor of any element?
16. How can the determinant order be defined?
17. What ways of calculating determinants do you know?
18. Write formulas by Cramer. What case are they used?
19. What is
    a) Gauss method?
    b) Jordan-Gauss method?
    c) matrix method?
20. What is
    a) a rank of a matrix?
    b) an augmented matrix?
21. What methods of finding a rank of a matrix do you know?
22. Formulate Kronecker–Capelli theorem.
23. What solution is called
    a) general? b) particular? c) basic? d) supporting?

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Content

НАВЧАЛЬНЕ ВИДАННЯ

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