МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ
ХАРКІВСЬКИЙ НАЦІОНАЛЬНИЙ ЕКОНОМІЧНИЙ УНІВЕРСИТЕТ
ІМЕНІ СЕМЕНА КУЗНЕЦЯ

Методичні рекомендації
до виконання практичних завдань
з функції багатьох змінних
з навчальної дисципліни "Вища та прикладна математика"
для іноземних студентів та студентів,
що навчаються англійською мовою,

напряму підготовки 6.030601 "Менеджмент" спеціалізації "Бізнес-адміністрування" денної форми навчання

Укладач: Місюра Є.Ю.

Відповідальний за випуск:
зав. кафедрою вищої математики
й економіко-математичних методів
Малярець Л. М.

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G94 Guidelines for doing practical tasks on the function of several variables on the academic discipline "Higher and Applied Mathematics" for full-time foreign students and students taught in English of training direction 6.030601 "Management", specialization "Business Administration" / compiled by Ie. Misiura. – Kh. : S. Kuznets KhNUE, 2015. – 56 p. (English)

The sufficient theoretical material on the academic discipline and typical examples are presented to help students master the material on the theme "The Function of Several Variables" and apply the obtained knowledge to practice. Individual tasks for self-study work and a list of theoretical questions are given to promote the improvement and extension of students' knowledge on the theme.

Recommended for full-time students of training direction 6.030601 "Management".

Викладено необхідний теоретичний матеріал з навчальної дисципліни та наведено типові приклади, які сприяють найбільш повному засвоєнню матеріалу з теми "Функція багатьох змінних" та застосуванню отриманих знань на практиці. Подано за-

вдання для індивідуальної роботи та перелік теоретичних питань, що сприяють удосконаленню та поглибленню знань студентів з даної теми.

Рекомендовано для студентів напряму підготовки 6.030601 "Менеджмент" денної форми навчання.

MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

SIMON KUZNETS KHARKIV NATIONAL UNIVERSITY OF ECONOMICS

Guidelines
for doing practical tasks on the function of several variables
on the academic discipline
"HIGHER AND APPLIED MATHEMATICS"
for full-time foreign students and students taught in English
of training direction 6.030601 "Management",
specialization "Business Administration"

Compiled by Ie. Misiura

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Харків, ХНЕУ ім. С. Кузнеця, 2015

Introduction

Differential calculus plays a very important role in economics in particular in problems concerning the optimum, management and plans. Therefore the deep knowledge of this section of higher and applied mathematics is necessary for modern economists.

In the guidelines, only the most principal topics of differential calculus are stated in brief.

The present guidelines are the continuation of the part where notions of limits and continuity of functions had been regarded. By means of these notions we can introduce the notions of the derivative and the differential of the function which are some of the most fundamental in mathematics.

Guidelines for Differential Calculus of the Function of Several Variables

1. General Information. The Domain of the Definition, the Limit and the Continuity of the Function of Several Variables

First of all it should be stressed that in order to better understand the notions connected with functions of several or many variables it is necessary to have deep knowledge of the principles of the function of one variable.

The definition of the function of several variables was given in the textbook ˝Introduction to Analysis˝, therefore we don't find it necessary to repeat it.

We begin with the domain of the definition and the range of such a function.

Definition 1. The set A of points $x=\left(x_1, x_2, x_3, ..., x_n\right)$ for which the function $f\left(x_1, x_2, x_3, ..., x_n\right)$ is defined is called the domain of the definition of the function $y=f\left(x_1, x_2, x_3, ..., x_n\right)$, while the set B of values y is termed the range of the function y. The domain of the definition and the range is denoted as $D(f)$ and $E(f)$ respectively.

Now let us pay our attention to the function of two variables $z=f(x, y)$ because it has a simple geometrical meaning and it is the basis for studying the functions of three and more variables.

2. Geometrical interpretation of the function of two variables

Geometrically the equation $z = f(x, y)$ defines some surface. A pair of values of x and y defines a point $P(x, y)$ in the plane xOy, (in Cartesian coordinates), and $z = f(x, y)$, the z-coordinate of the corresponding point $M(x, y, z)$ on the surface. Therefore, we say that z is the function of the point $P(x, y)$ and we write $z = f(P)$.

It should be noted that the function $z = f(x, y)$ can also be written in the form $F(x, y, z) = 0$ and it usually specifies each of the variables x, y, z involved as an implicit function of the other two variables.

It should also be noted there is no geometrical interpretation for the function of three and more variables.

The domain of the definition of the function $z = f(x, y)$ is usually a part of the xOy-plane bounded by one or several lines. Let us consider the following examples.

Example 1. Indicate the domain of the definition of the function

$$z = \ln\left(x^2 + y^2 - r^2\right).$$

Solution. This function exists when $x^2 + y^2 - r^2 > 0$. Therefore, the domain of the definition is the set of points, whose coordinates satisfy the condition $x^2 + y^2 > r^2$ that is the exterior of the circle (Fig. 1).

Example 2. Consider the domain of the definition of the function

$$z = \frac{1}{\sqrt{r^2 - x^2 - y^2}}.$$

Solution. This function exists when $r^2 - x^2 - y^2 > 0$. Therefore, the domain of the definition is the set of points, whose coordinates satisfy the condition $x^2 + y^2 < r^2$ that is the interior of the circle (Fig. 2).

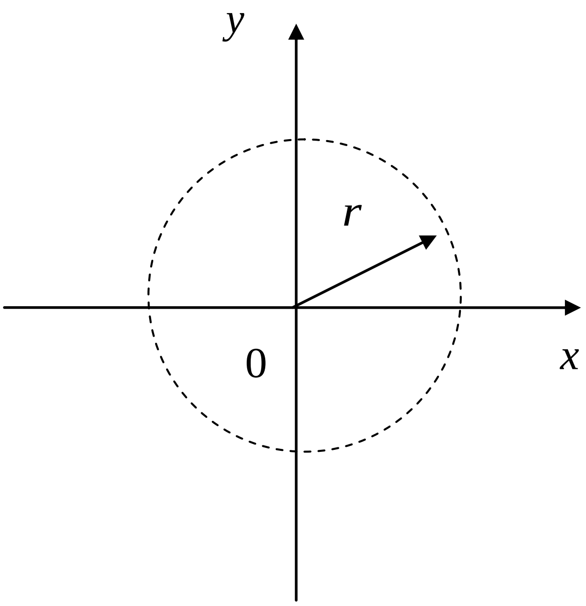

Fig. 1. **The domain of the definition (the exterior of the circle)**

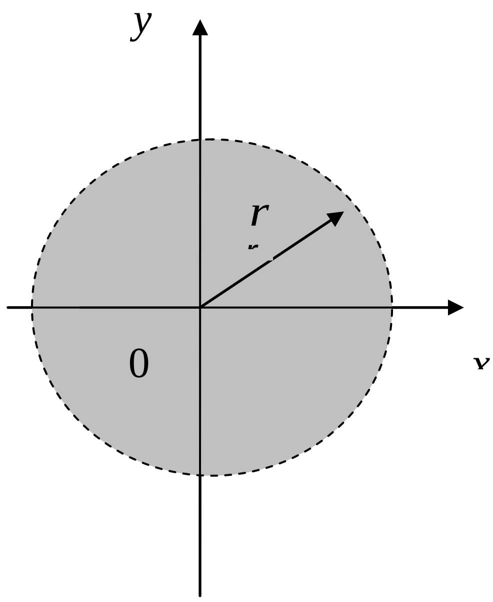

Fig. 2. **The domain of the definition (the interior of the circle)**

The notion of the limit of the function of two variables is formulated in the following way.

Definition 2. The number *A* is called the limit of the function $z = f(x, y)$ as $x \to x_0,\ y \to y_0$ if for all the values of x and y which are, respectively, sufficiently close to the numbers x_0 and y_0 the corresponding values of the function $z = f(x, y)$ are arbitrarily close to the number A. It is denoted as

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = A.$$

This definition can be restated in terms of inequalities: given an arbitrary $\varepsilon > 0$, there exists a number $\delta > 0$ so that for all the points $P(x, y)$ whose coordinates satisfy the inequality

$$0 < (x - x_0)^2 + (y - y_0)^2 < \delta^2 \quad \text{or} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta,$$

or $0 < \sqrt{\Delta x^2 + \Delta y^2} < \delta$ (that is for all points $P(x, y) \neq P_0(x_0, y_0)$ belonging to the δ-neighborhood of the point P_0) the inequality $|f(x, y) - A| < \varepsilon$ is fulfilled, the number A is the limit of the function $f(x, y)$ as $x \to x_0$, $y \to y_0$.

The notion of the limit is closely connected with the notion of continuity.

Definition 3. The function $z = f(x, y)$ is said to be continuous at the point (x_0, y_0) if

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} \left[f(x, y) - f(x_0, y_0) \right] = 0 \text{ or } \lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_0, y_0)$$

or its limit at the point coincides with a particular value of the function at this point, the converse is also true.

3. Total and Partial Increments. Partial Derivatives. A Sufficient Condition for Differentiability

The notions of the limit and the continuity make it possible to approach the notion of differentiability of the function of several variables. For this purpose let us define the total increment of the function of two variables. In the general case the increment of the function is given by the formula

$$z = f(x + \Delta x, y + \Delta y) - f(x, y),$$

where Δx and Δy are the increments of the variables x and y. And the function $z = f(x, y)$ is continuous at the point (x_0, y_0) if

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \Delta z = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \right] = 0.$$

It should be noted that this formula can serve as another definition of the continuity of the function at the given point.

The partial increments are denoted by the symbols

$$\Delta_x z = f(x + \Delta x, y) - f(x, y) \quad \text{and} \quad \Delta_y z = f(x, y + \Delta y) - f(x, y)$$

which are the increments of the function with respect to the corresponding variables x and y.

Now we can pass to the notion of partial derivatives.

Let $z = f(x, y)$ be a function of two independent variables x and y. We begin with fixing a constant value of the argument y and investigating the function of one variable x. Suppose that the function possesses the derivative with respect to x, this derivative is equal to

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = f'_x.$$

We will denote this limit as f'_x where the subscript indicates the variable x with respect to which the derivative is taken for a fixed value of y.

Definition 4. The partial derivative of the function $z = f(x, y)$ with respect to x is the function of the two variables x and y appearing when $f(x, y)$ is differentiated with respect to x on condition that y is regarded as a constant.

The symbols $\frac{\partial z}{\partial x}$, z'_x, $\frac{\partial f(x, y)}{\partial x}$, $\frac{\partial}{\partial x}[f(x, y)]$ are also used for the notation of the partial derivative.

The partial derivative of the function $z = f(x, y)$ with respect to y is defined completely analogically

$$\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = f'_y$$

and it is also denoted $\frac{\partial z}{\partial y}$, z'_y, $\frac{\partial f(x, y)}{\partial y}$, $\frac{\partial}{\partial y}[f(x, y)]$.

At last the partial derivatives for the function of three and more variables are defined in a similar way.

Example 3. Find the partial derivatives for the function $z = e^x \cos y$.

Solution. For the partial derivative z'_x we suppose that y is a constant then $\cos y$ is a constant too. Hence we write

$$\frac{\partial z}{\partial x} = \left(e^x \cos y\right)'_x = \cos y\left(e^x\right)'_x = e^x \cos y .$$

Analogically for z'_y we have

$$\frac{\partial z}{\partial y} = \left(e^x \cos y\right)'_y = e^x(\cos y)'_y = -e^x \sin y .$$

Example 4. Find the partial derivatives for the function

$$u = \ln\left(x^2 + y^2 + z^2\right),$$

where $u(x, y, z)$ is the function of the three variables.

Solution. Successively supposing y and z, x and z and x and y are the constant values we get:

$$\frac{\partial u}{\partial x} = \left(\ln\left(x^2 + y^2 + z^2\right)\right)'_x = \frac{2x}{x^2 + y^2 + z^2};$$
$$\frac{\partial u}{\partial y} = \left(\ln\left(x^2 + y^2 + z^2\right)\right)'_y = \frac{2y}{x^2 + y^2 + z^2};$$
$$\frac{\partial u}{\partial z} = \left(\ln\left(x^2 + y^2 + z^2\right)\right)'_z = \frac{2z}{x^2 + y^2 + z^2}.$$

The geometrical meaning of the partial derivatives of the function $z = f(x, y)$ is the following: $f'(x_0, y_0)$ is equal to the slope, relative to Ox, of the tangent line to the section of the surface $z = f(x, y)$ by the plane $y = y_0$ drawn through the point $M_0(x_0, y_0, z_0)$, that is $f'_x(x, y) = tg\alpha$ (Fig. 3). It is similar for $f'_y(x, y)$. Now we can formulate the sufficient condition for differentiability of the function $f(x) = f(x_1, x_2, x_3, ..., x_n)$ of several variables.

Definition 5. If partial derivatives $\partial f(x)/\partial x_i$ are defined in some neighborhood of the point x and are continuous at the point $x=(x_1, x_2, ..., x_n)$ itself, then the function $f(x)$ is differentiable at this point. The function differentiable at each point of the domain of its definition is said to be differentiable in that domain.

The function $z=f(x, y)$ is said to be continuous at the point (x_0, y_0) if

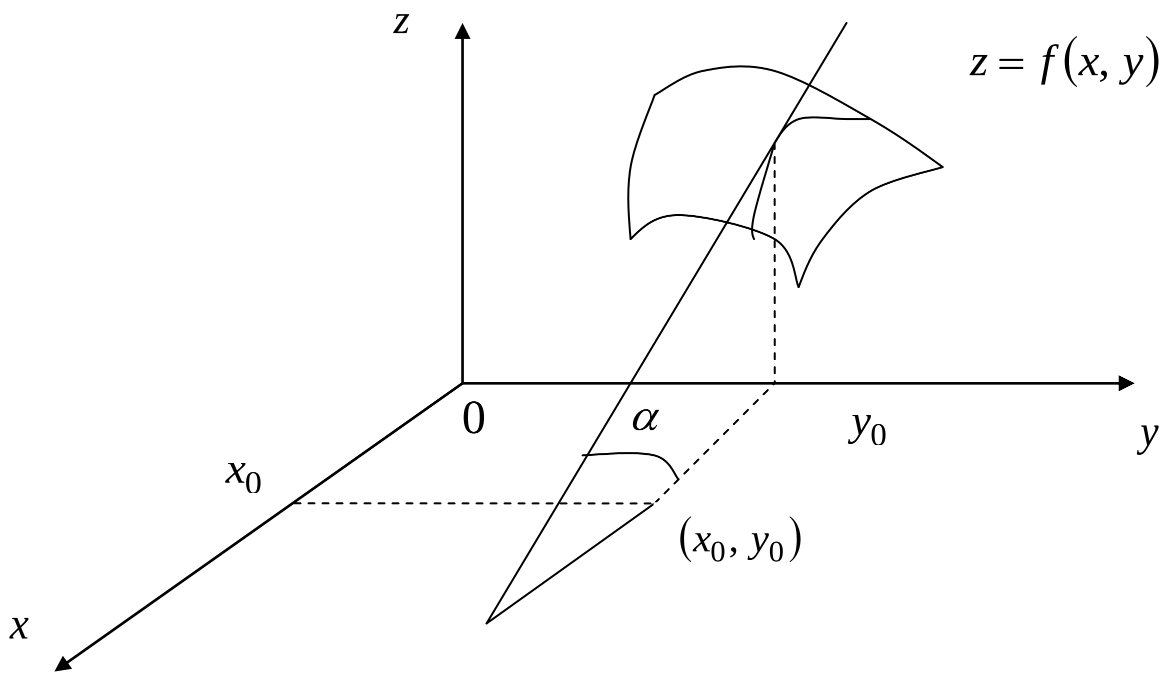

Fig. 3. **A tangent line to the section of the surface** $z=f(x, y)$ **by the plane** $y=y_0$ **drawn through the point** $M_0(x_0, y_0, z_0)$

4. Total and Partial Differentials. Applying the Total Differential to Approximate Calculation

We begin with the second definition for differentiation of the function of two variables.

Definition 6. The function $z=f(x, y)$ is said to be differentiable at a given point (x, y) if its total increment is represented in the form $\Delta z = A\Delta x + B\Delta y + \varepsilon$, where $\varepsilon = o(\rho)$ is an infinitesimal relative to ρ, and $\rho=\sqrt{\Delta x^2+\Delta y^2}$, coefficients $A=\frac{\partial z}{\partial x}$ and $B=\frac{\partial z}{\partial y}$ are independent of Δx and Δy.

Now we can give a definition of the total differential of a function.

Definition 7. The principal part of the total increment of a differentiable function which is a linear function of the increment of the independent va-riables is called the differential of the function $dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y.$

We can see that the total differential of the function of two independent variables is equal to the sum of the products of the partial derivatives of the function by the differentials of the corresponding independent variables. These products are called partial differentials and denoted as

$$d_x z = \frac{\partial z}{\partial x}\Delta x \quad \text{and} \quad d_y z = \frac{\partial z}{\partial y}\Delta y.$$

If $\Delta x = dx$ and $\Delta y = dy$, then

$$d_x z = \frac{\partial z}{\partial x}dx, \quad d_y z = \frac{\partial z}{\partial y}dy,$$

$$dz = d_x z + d_y z = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

For the function $u = f(x) = f(x_1, x_2, x_3, ..., x_n)$ of several variables the total differential is equal to

$$du = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 + ... + \frac{\partial f}{\partial x_n}dx_n.$$

A total differential is often used for approximate calculations of a function. For instance it is required to compute the function $z = f(x, y)$ at the point $(x + \Delta x, y + \Delta y)$, i. e. $z(x + \Delta x, y + \Delta y)$. It is now

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y),$$

whence $f(x + \Delta x, y + \Delta y) = f(x, y) + \Delta z.$

If we suppose that $\Delta z \approx dz$, then

$$f(x+\Delta x, y+\Delta y) \approx f(x,y)+dz = f(x,y)+\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy.$$

So we obtain the formula for computing the function $z(x+\Delta x,\ y+\Delta y)$. This formula is valid for small $\Delta x = dx$ and $\Delta y = dy$.

Example 5. Find the total differential of the following functions:

$$1)\quad z = x^2 y;\quad 2)\quad z=\sqrt{x^2+y^2}.$$

Solution. Let us find partial derivatives for the first function, then $\frac{\partial z}{\partial x} = 2xy,\quad \frac{\partial z}{\partial y} = x^2$, and the total differential has the form

$$dz = 2xy\,dx + x^2 dy.$$

For the second function

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}},\quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}.$$

Then its total differential is written as

$$dz = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy = \frac{1}{\sqrt{x^2+y^2}}\left(x\,dx + y\,dy\right).$$

Example 6. Calculate approximately the value $(1.04)^{2.02}$.

Solution. Let

$$f(x+\Delta x,\ y+\Delta y) = (x+\Delta x)^{y+\Delta y},$$

where $x+\Delta x = 1+0.04$, $y+\Delta y = 2+0.02$ that is $\Delta x = dx = 0.04$, $x=1$, $\Delta y = dy = 0.02$, $y=2$ and $z = f(x,y) = x^y = 1^2 = 1$. Computing

$$\frac{\partial f}{\partial x} = yx^{y-1}|_{(1;2)} = 2, \quad \frac{\partial f}{\partial y} = x^y \ln x|_{(1;2)} = 0,$$

and using the relation $f(x+\Delta x, y+\Delta y) \approx f(x,y) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ we obtain

$$(1.04)^{2.02} \approx 1 + 2 \cdot 0.04 + 0 \cdot 0.02 = 1 + 0.08 = 1.08.$$

5. Differentiating Composite Functions

Let us suppose that z is a composite function of two independent variables x and y, that is, $z = f(u, v)$, where $u = \varphi(x, y)$ and $v = \psi(x, y)$ are intermediate arguments. Thus

$$z = f(\varphi(x,y), \psi(x,y)) = F(x,y).$$

We also suppose that all the functions involved possess continuous partial derivatives and therefore are differentiable.

To find $z'_x = \frac{\partial z}{\partial x}$ we must consider y constant, and then u and v become functions of only one variable x, therefore we arrive to the formula which we represent without a proof

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{and, similarly} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Hence, we can derive the following rule.

Rule. A partial derivative of a composite function is equal the sum of the products of the derivatives of the given function with respect to the intermediate arguments (i. e. u and v) by the partial derivatives of these arguments with respect to the corresponding independent variable (x or y).

This rule applies to functions of any number of independent variables and any number of intermediate arguments.

Let z be defined as a function of arguments $u, v, \ldots, w$ which are functions of independent variables $x, y, ..., t$. Then

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+..........+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}+..........+\frac{\partial z}{\partial w}\frac{\partial w}{\partial y},$$

$$\text{..}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial t}+..........+\frac{\partial z}{\partial w}\frac{\partial w}{\partial t}.$$

In a particular case the arguments $u, v, ..., w$ may be functions of one independent variable, say, x. This means, that, ultimately, z is a function dependent solely on x. In this case its ordinary derivative (called a total derivative) is expressed by the formula:

$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\frac{du}{dx}+\frac{\partial z}{\partial v}\frac{dv}{dx}+..........+\frac{\partial z}{\partial w}\frac{dw}{dx}.$$

If x coincides with one of the arguments $u, v, ..., w$ for definiteness, $x=u$, the latter formula yields:

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial v}\frac{dv}{dx}+..........+\frac{\partial z}{\partial w}\frac{dw}{dx}.$$

Let us consider the following examples.

Example 7. Find the derivative of

$$z=u^2e^v,$$

where $u=\sin x$, $v=\cos x$.

Solution. Using the formula for the derivative of the composite function

$$dz=\frac{\partial z}{\partial u}\frac{du}{dx}+\frac{\partial z}{\partial v}\frac{dv}{dx}$$

we obtain

$$\frac{dz}{dx} = 2ue^{v}\cos x + u^{2}e^{v}(-\sin x).$$

Substituting the function u and v in terms of x into the expression of $\frac{dz}{dx}$ we have

$$\frac{dz}{dx} = e^{\cos x}(2\sin x\cos x - \sin^{3} x) = e^{\cos x}\sin x(2\cos x - \sin^{2} x).$$

Example 8. Find the partial derivatives of the function

$$z = \ln ue^{v},$$

where $u = \sin x + \cos y$, $v = \sin x - \cos y$.

Solution. Applying the formulas for partial derivatives of the composite function

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\ ;\ \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

and considering that

$$\frac{\partial u}{\partial x} = \cos x;\ \frac{\partial u}{\partial y} = -\sin y;\ \frac{\partial v}{\partial x} = \cos x;\ \frac{\partial u}{\partial y} = \sin y;$$

$$\frac{\partial z}{\partial u} = \frac{e^{v}}{u};\ \frac{\partial z}{\partial v} = \ln ue^{v},$$

we can write

$$\frac{\partial z}{\partial x} = \frac{e^{v}}{u}\cos x + \ln ue^{v}\cos x;\ \frac{\partial z}{\partial y} = \frac{e^{v}}{u}(-\sin y) + \ln ue^{v}\sin y.$$

Finally, substituting u and v in terms of x and y, we obtain

$$\frac{\partial z}{\partial x}=\frac{\cos x}{\sin x+\cos y}e^{\sin x-\cos y}+\cos x\ln(\sin x+\cos y)e^{\sin x-\cos y};$$
$$\frac{\partial z}{\partial y}=\frac{\sin y}{\sin x+\cos y}e^{\sin x-\cos y}+\sin y\ln(\sin x+\cos y)e^{\sin x-\cos y}.$$

6. Differentiating Implicit Functions

Let the function $F(x, y)$ be such that the equation $F(x, y)=0$ specifies y as the function of x: $y=\varphi(x)$. The substitution of the function $\varphi(x)$ for y into this equation leads to the identity $F(x,\varphi(x))=0$. It follows that the derivative of the function $F(x, y)$ (where $y=\varphi(x)$) with respect to x is also identically zero.

On differentiating this expression according to the differentiation rule for a composite function we find:

$$F_x' + F_y'\frac{dy}{dx}=0\text{, whence } y'=\frac{dy}{dx}=-\frac{F_x'}{F_y'}.$$

This formula expresses the derivative of the implicit function $y=\varphi(x)$ in terms of the partial derivatives of the given function $F(x, y)$. The derivative y' does not exist at the point (x, y) for which $F_y'=0$.

In the general case the equation of the form $F(x, y, z, ..., t, u)=0$ specifies u as a function of $x, y, z, ..., t$. By analogy with the foregoing case we find:

$$\frac{\partial u}{\partial x}=-\frac{F_x'}{F_u'};\quad \frac{\partial u}{\partial y}=-\frac{F_y'}{F_u'};\ldots\ldots\ldots\ldots\ \frac{\partial u}{\partial t}=-\frac{F_t'}{F_u'}.$$

Example 9. Find the derivative of $x^2+y^2=a^2$.

Solution. Let us differentiate this relation with respect to x, considering $y = \varphi(x)$. Then $2x + 2yy' = 0$, whence $y' = -x/y$.

7. Directional Derivative. Gradient

In order to study the notion of the directional derivative it is convenient to interpret this derivative as the rate of change of the function $u = f(x, y, z)$ at the given point (x, y, z) in the direction of the axis l. The direction of the axis l is given by the unit vector l_0 which forms the angles α, β and γ with corresponding axes of coordinates, i. e. Ox, Oy and Oz. The computation of the directional derivative is based on the following formula:

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma,$$

where $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines and simultaneously the coordinates of the unit vector $\vec{l}_0$.

For the function $z = f(x, y)$ of two variables $\cos\gamma = 0$ and $\cos\beta = \sin\alpha$. Then

$$\frac{\partial z}{\partial l} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha.$$

The directional derivative can be considered as the original generalization of the partial derivatives.

Indeed, if $\alpha = 0$, $\beta = \gamma = \frac{\pi}{2}$, then the direction of the axis l coincides with Ox-axis and $\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x}$. Analogously, if $\alpha = \gamma = \frac{\pi}{2}$, $\beta = 0$, then $\frac{\partial u}{\partial l} = \frac{\partial u}{\partial y}$, and for $\alpha = \beta = \frac{\pi}{2}$ and $\gamma = 0$, then $\frac{\partial u}{\partial l} = \frac{\partial u}{\partial z}$.

The notion of the gradient of the function is closely connected with the directional derivative.

Definition 8. The gradient of the function $u = f(x, y, z)$ is the vector whose projections (coordinates) are the values of the partial derivatives of the function, that is

$$grad\ u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}.$$

It should be stressed that the projections of the gradient depend on the choice of the point (x, y, z) and may vary when the coordinates of the point change.

Using the notion of the gradient we can rewrite the formula for the directional derivative in the form

$$\frac{\partial u}{\partial l} = grad\ u \cdot \vec{l}_0.$$

Consequently, the derivative of the function in a given direction is equal to the scalar product of the gradient of the function by the unit vector in that direction.

But on the other hand it is obvious that the derivative of a function in a given direction is equal to the projection of the gradient of the function on the axis l along which the differentiation is carried out, that is

$$\frac{\partial u}{\partial l} = |grad\ u| \cos\varphi,$$

where φ is the angle between the vector $grad\, u$ and the axis l (Fig. 4).

It follows immediately that the directional derivative attains its greatest value for $\cos\varphi = 1$, i. e. for $\varphi = 0$, this greatest value being equal to $|grad\ u|$. Thus $grad\, u$ is the greatest possible value of the derivative $\frac{\partial u}{\partial l}$ at the given point $P(x, y, z)$, and the direction of the vector $grad\, u$ coincides with the direction of the axis issued from the point $P(x, y, z)$, and the direction of the vector $grad\, u$ coincides with the direction of the axis issued from the point

$P(x, y, z)$ along which the rate of change of the function is the greatest, that is, the direction of the gradient is that of the fasted increase of the function.

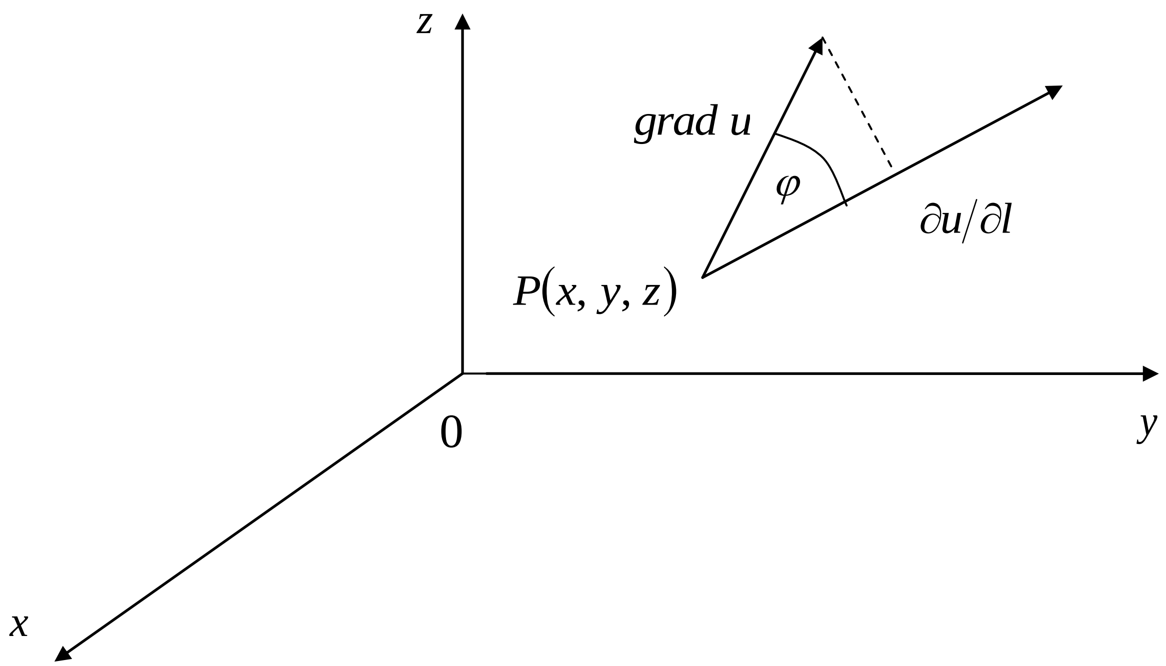

Fig. 4. **The angle** φ **between the vector** $grad\ u$ **and the axis** l

Let us consider the example.

Example 10. Find the derivative of the function $z = \dfrac{x}{y}$ at the point $M(1,1)$ in the direction of the line l: $y = x^2$ along the negative semi-axis Ox.

Solution. To compute the derivative $\dfrac{\partial z}{\partial l}$ we take the formula

$$\frac{\partial z}{\partial l} = grad\ z \cdot \vec{l}_0,$$

where $grad\ z = \dfrac{\partial z}{\partial x}\vec{i} + \dfrac{\partial z}{\partial y}\vec{j}$ and $\vec{l}_0$ is the unit vector of the direction l.

For the first we find the gradient of the function z at the point $M(1,1)$

$$\left.\frac{\partial z}{\partial x}\right|_{(1;1)} = \left.\frac{1}{y}\right|_{(1;1)} = 1; \quad \left.\frac{\partial z}{\partial y}\right|_{(1;1)} = -\left.\frac{x}{y^2}\right|_{(1;1)} = -1,$$

whence $grad\ z = \vec{i} - \vec{j}$.

In the second place we compute the unit vector $\vec{l}_0$ of the direction l, i. e. $\cos\alpha$ and $\sin\alpha$. Then $\tan\alpha = y'_x = 2x\big|_{(1;\,1)} = 2$, and

$$cos\,\alpha = -\frac{1}{\sqrt{1+tan^2\alpha}} = -\frac{1}{\sqrt{5}};\ \ sin\,\alpha = -\frac{tan\,\alpha}{\sqrt{1+tan^2\alpha}} = -\frac{2}{\sqrt{5}}.$$

Hence

$$\vec{l}_0 = (\cos\alpha; \sin\alpha) = \left(-\frac{1}{\sqrt{1+\tan^2\alpha}}; -\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}\right) = \left(-\frac{1}{\sqrt{5}}; -\frac{2}{\sqrt{5}}\right).$$

Now we find the scalar product of the vectors $grad\ z$ and $\vec{l}_0$.

We obtain

$$\frac{\partial z}{\partial l} = grad\ z \cdot \vec{l}_0 = (\vec{i} - \vec{j})\left(-\frac{1}{\sqrt{5}}\right)(\vec{i} + 2\vec{j}) = \frac{1}{\sqrt{5}}.$$

8. The extreme of the Function of Two Variable. Determining the Greatest and the Least Values of the Function

Here we consider only the case of the function of two variables, because for the function of any number n of independent variables the notion of then extreme is defined quite similarly.

The definition of the point of the extreme of the function of two variables is analogous to the corresponding definition for the function of one variable.

Definition 9. The point $P_0(x_0, y_0)$ is said to be the point of the extreme (the point of the maximum or the point of the minimum) of the function $z = f(x, y)$ if, respectively, $f(x_0, y_0)$ is the greatest or the least value of the function $f(x, y)$ in the neighborhood of the point $P_0(x_0, y_0)$.

Now we establish without proof the necessary condition for the function $z = f(x, y)$ to attain an extreme at the point $P_0(x_0, y_0)$.

A necessary Condition for the Extremum. If the differentiable function $z = f(x, y)$ attains an extremum at the point $P_0(x_0, y_0)$ its partial derivatives turn into zero at that point

$$\left.\frac{\partial z}{\partial x}\right|_{\begin{pmatrix} x=x_0 \\ y=y_0 \end{pmatrix}} = 0, \quad \left.\frac{\partial z}{\partial y}\right|_{\begin{pmatrix} x=x_0 \\ y=y_0 \end{pmatrix}} = 0.$$

It should be stressed that a continuous function of two variables may have an extreme at a point where it is not differentiable (for instance, such an extreme may correspond to a cuspidal point of the surface at the graph of the function).

Such point at which both partial derivatives of a continuous function $z = f(x, y)$ turn into zero or they don't exist are referred to as a stationary or critical point of the function $f(x, y)$.

But the necessary test for an extreme of a function of two variables established above is not sufficient. This means that the fact that the partial derivatives are zero or don't exist at a given point does not imply that this point is necessarily a point of extreme.

For instance, for the function $z = xy$ its partial derivatives $z'_x = y$ and $z'_y = x$ are equal to zero at the origin where the function has no extreme.

We must note that sufficient conditions for the extremum for a function of several independent variables are essentially more complicated than in the case of a function of one argument.

Here we shall state without proof sufficient conditions for a function of two independent variables.

Sufficient Conditions for the Extremum of a Function of Two Variables.

Let the function $z = f(x, y)$ be continuous together with its partial derivatives of the first and second orders and let $P_0(x_0, y_0)$ be a stationary point of the function, that is

$$\left.\frac{\partial z}{\partial x}\right|_{\begin{pmatrix} x=x_0 \\ y=y_0 \end{pmatrix}} = 0, \quad \left.\frac{\partial z}{\partial y}\right|_{\begin{pmatrix} x=x_0 \\ y=y_0 \end{pmatrix}} = 0.$$

Let us compute the values of the second derivatives of the function $f(x, y)$ at the point $P_0(x_0, y_0)$ and denote them, for briefness $A,\ B$ and C:

$$A = \left.\frac{\partial^2 z}{\partial x^2}\right|_{\left(\substack{x=x_0 \\ y=y_0}\right)}; \quad B = \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{\left(\substack{x=x_0 \\ y=y_0}\right)}; \quad C = \left.\frac{\partial^2 z}{\partial y^2}\right|_{\left(\substack{x=x_0 \\ y=y_0}\right)}.$$

If $AC - B^2 > 0$, the function $f(x, y)$ has an extreme at the point $P_0(x_0, y_0)$ which is a maximum if $A < 0$ and a minimum if $A > 0$ (the condition $AC - B^2 > 0$ implies that A and C are necessarily of one sign).

If $AC - B^2 < 0$, there is no extreme at the point $P_0(x_0, y_0)$.

If $AC - B^2 = 0$, the properties of the second derivatives don't provide any answer to the question of existence of an extreme, and further investigation is needed.

Example 11. Find the extreme of the function

$$z = x^3 y^2 (a - x - y).$$

Solution. Rewrite this function in the form:

$$z = ax^3 y^2 - x^4 y^2 - x^3 y^3.$$

Now let us find the first partial derivatives

$$\frac{\partial z}{\partial x} = 3ax^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = x^2 y^2 (3a - 4x - 3y),$$
$$\frac{\partial z}{\partial y} = 2ax^3 y - 2x^4 y - 3x^3 y^2 = x^3 y (2a - 2x - 3y).$$

It is evident that the derivatives turn into zero at the point $P_0(0, 0)$. The next critical point can be found from the system:

$$\begin{cases} 4x + 3y = 3a \\ 2x + 3y = 2a \end{cases} \Rightarrow x = a/2,\ y = a/3 \Rightarrow P_1(a/2; a/3).$$

To establish the point of the extreme we should find the second derivatives:

$$\frac{\partial^2 z}{\partial x^2} = 6axy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(a - 2x - y),$$

$$\frac{\partial^2 z}{\partial y^2} = 2ax^3 - 2x^4 - 6x^3y = 2x^3(a - x - 3y),$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6ax^2y - 8x^3y - 9x^2y^2 = x^2y(6a - 8x - 9y),$$

$$A = \frac{\partial^2 z}{\partial x^2}\Big|_{(P_0)} = 0; \quad C = \frac{\partial^2 z}{\partial y^2}\Big|_{(P_0)} = 0; \quad B = \frac{\partial^2 z}{\partial x \partial y}\Big|_{(P_0)} = 0.$$

Then $AC - B^2\Big|_{P_0} = 0$. But from the form of the function z we can say that there is no extreme at the point $P_0(0, 0)$ and $z(P_0) = 0$.

At the point $P_1(a/2, a/3)$

$$A = \frac{\partial^2 z}{\partial x^2}\Big|_{(P_1)} = -\frac{a^4}{9} < 0, C = \frac{\partial^2 z}{\partial y^2}\Big|_{(P_1)} = -\frac{a^4}{8} < 0, B = \frac{\partial^2 z}{\partial x \partial y}\Big|_{(P_1)} = -\frac{a^4}{12} < 0.$$

Then $AC - B^2\Big|_{P_1} = a^8/144 > 0$. There is an extreme at the point $P_1(a/2, a/3)$. Since A and C are less than zero, then at that point the function $z(P_1) = \frac{a^3}{8} \cdot \frac{a^2}{9}\left(a - \frac{a}{2} - \frac{a}{3}\right) = \frac{a^6}{432}$ has the maximum value.

And now we proceed to determining the greatest and the least values of the functions of two variables in some closed domain.

Suppose that it is required to determine the greatest and the least values of the function $z = f(x, y)$ in a closed domain. If one of these values (or both) is attained inside the domain it must of course be an extreme's value, But it may turn out that the greatest or the least value of the function (or both) is attained at a point belonging to the boundary of the domain. Therefore, to determine these values, we have to find a local extreme either at the interior points of the domain or at the boundary points and compare their magnitudes.

Example 12. Find the greatest or the least value of the function

$$z = x^2 y(4 - x - y)$$

in the closed domain, bounded by the lines: $x = 0$; $y = 0$; $x + y = 6$.

Solution. The given domain S is the triangle ΔOBC, including the boundary (see Fig. 5).

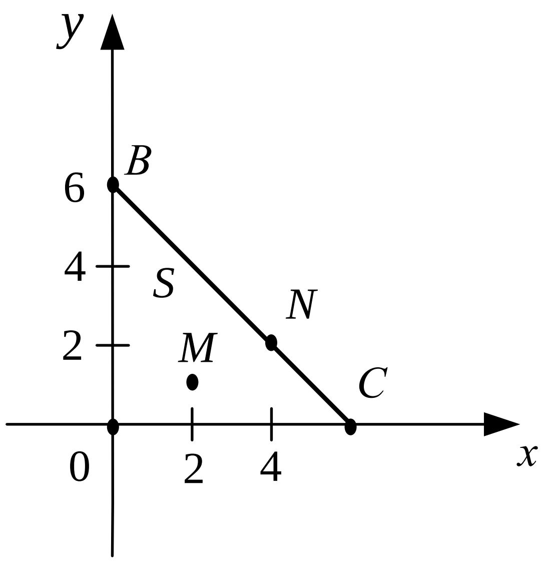

Fig. 5. **The triangle ΔOBC or the domain S**

As we know the continuous function attains the greatest and the least values either inside the domain or at the boundary points of the domain.

Let us research this function inside the domain. Using the necessary condition of the extreme of the function of two variables, we obtain:

$$\frac{\partial z}{\partial x} = 8xy - 3x^2 y - 2xy^2 = xy(8 - 3x - 2y),$$

$$\frac{\partial z}{\partial y} = 4x^2 - x^3 - 2x^2 y = x^2(4 - x - 2y).$$

Since $x = 0$ and $y = 0$ are the boundary points, then the stationary points can be found from the linear system of the equations:

$$\begin{cases} 3x+2y=8, \\ x+2y=4 \end{cases} \Rightarrow x=2;\ y=1;\ M(2;1).$$

Now we calculate the value of the function z at that point

$$z\big|_{N(2;1)} = 2^2 \cdot 1(4-2-1) = 4.$$

At the boundaries OB and OC of the domain the function $z=0$. Let us investigate the behavior of the function at the boundary BC where $y=6-x$. Then

$$z = x^2(6-x)(4-x-6+x) = 2x^2(x-6) = 2x^3 - 12x^2,$$
$$z_x' = 6x^2 - 24x = 6x(x-4).$$

Since $x \neq 0$, then $x-4=0 \rightarrow x=4$ and $y=6-x=6-4=2$. We have the stationary point $N(4;2)$. Compute the value of the function at that point

$$z\big|_{N(4;2)} = 4^2 \cdot 2(4-4-2) = -64.$$

Thus $z\big|_M = 4$ is the greatest value of the function and $z\big|_N = -64$ is the least value of the function.

We may conclude that the function attains the greatest value inside the domain and it reaches the least value at the boundary BC of the domain (at the point N).

Example 13. Find the extreme of the function

$$z = x^3 + 3xy^2 - 15x - 12y.$$

Solution. Now let us find the first partial derivatives:

$$\frac{\partial z}{\partial x} = 3x^2 + 3y^2 - 15;\quad \frac{\partial z}{\partial y} = 6xy - 12.$$

The next critical points can be found from the system:

$$\begin{cases} x^2+y^2=5, \\ xy-2=0 \end{cases} \Rightarrow P_1(1;2);\ P_2(2;1);\ P_3(-1;-2);\ P_4(-2;-1).$$

To establish the point of the extreme we should find the second derivatives:

$$\frac{\partial^2 z}{\partial x^2}=6x,\ \ \frac{\partial^2 z}{\partial x\partial y}=6y,\ \ \frac{\partial^2 z}{\partial y^2}=6x,$$

for the point P_1: $A=6;\ B=12;\ C=6;\ AC-B^2=36-144<0$ there is no extreme,

for the point P_2: $A=12;\ B=6;\ C=12;\ AC-B^2=144-36>0$ there is an extreme, $z_{\min}=z(2;1)=-28$,

for the point P_3: $A=-6;\ B=-12;\ C=-6;\ AC-B^2=36-144<0$ there is no extreme,

for the point P_4: $A=-12;\ B=-6;\ C=-12;\ AC-B^2=144-36>0$ there is an extreme, $z_{\max}=z(-2;-1)=28$.

9. A Conditional Extreme

Consider the problem of the extreme of the function of several variables, assuming that these variables are also subject to some constraint equations. We begin with the case of the function of two variables, because it is the most simple case.

Suppose it is required to find the extreme of the function $z=f(x,y)$, where the variables x and y are subject to the equation $\varphi(x,y)=0$. The last equation is called a constraint equation, or, simply a constraint (also a coupling or a subsidiary condition).

Definition 10. The function $z=f(x,y)$ of two variables is said to have a conditional or relative maximum (minimum) at the point (x_0, y_0) satisfying the constraint equation $\varphi(x, y) = 0$, if the inequality

$$f(x, y) \le f(x_0, y_0) \quad (f(x, y) \ge f(x_0, y_0))$$

holds in some neighbourhood of the point (x_0, y_0) for all the points (x, y) satisfying the constraint equation $\varphi(x, y) = 0$.

Note that the point of the unconditional extreme is always the point of the conditional extreme, but the converse is not true: the point of the conditional extreme is not necessarily the point of the ordinary extreme.

If the constraint equation $\varphi(x, y) = 0$ admits of the expressions of y as an explicit function of x, i. e. $y = \psi(x)$, we can substitute $\psi(x)$ for y into the function $z = f(x, y)$ to obtain the function of one variable

$$z = f(x, \psi(x)) = F(x).$$

On finding the values of x for which this function attains an extreme and determining the corresponding values of y from the equation $\varphi(x, y) = 0$ we obtain the desired points of the conditional extreme.

If the subsidiary condition is expressed by a complicated equation and if it is impossible to express explicitly one variable in terms of the other one the problem becomes more difficult. We can somewhat simplify the problem by considering the derivatives of the functions $z = f(x, y)$ and $\varphi(x, y)$, i. e. z'_x and φ'_x, bearing in mind that the variable y is the function of the variable x and at the point of the extreme $z'_x = 0$ and $\varphi'_x = 0$ from the constraint $\varphi(x, y) = 0$. Then

$$z'_x = \frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \varphi'_x = \frac{d\varphi}{dx} = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0.$$

Consider the method for obtaining the necessary conditions for a conditional extreme, using the so-called Lagrange's method of multipliers. We multiply the equality for the derivative φ'_x by some multiplier λ and add together the expressions for z'_x and φ'_x. As a result, we get

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}\right)+\lambda\left(\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial y}\frac{dy}{dx}\right)=0 \text{ or}$$

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}\right)+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}\right)\frac{dy}{dx}=0.$$

We choose the multiplier λ on the condition that at the point of the extreme

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}=0.$$

But for these values of x and y it follows that

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}=0.$$

Thus, for the points of the extreme we have three equations:

$$\begin{cases}\frac{\partial f}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}=0\\ \frac{\partial f}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}=0\\ \varphi(x,\ y)=0\end{cases} \qquad (1)$$

with three unknown x, y and λ.

So the necessary conditions for a local conditional extreme of the function $z=f(x,\ y)$ with the constraint equation $\varphi(x,\ y)=0$ can be obtained in the following way: consider Lagrange's function

$$L(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y),$$

where λ is some constant, the necessary conditions for a local conditional extreme of the function $L(x, y, \lambda)$ are fulfilled in the usual form:

$$\begin{cases} \dfrac{\partial L}{\partial x} = \dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial \varphi}{\partial x} = 0; \\ \dfrac{\partial L}{\partial y} = \dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial \varphi}{\partial y} = 0. \end{cases}$$

To determine the multiplier λ, we add to these conditions the constraint equation $\varphi(x, y) = 0$ that is $\partial L / \partial \lambda = 0.$ As a result we have the same system (1). Solving this system we can find the unknown x, y and λ which plays an auxiliary role and we don't need it in further investigations.

In the most general case the problem is posed as follows: given the function $u = f(x_1, x_2, x_3, ..., x_n)$ of n variables, it is required to find its extreme on condition that the variables are subject to $m(m < n)$ subsidiary conditions:

$$\begin{cases} \varphi_1(x_1, x_2, x_3, ..., x_n) = 0, \\ \varphi_2(x_1, x_2, x_3, ..., x_n) = 0, \\ .... \\ \varphi_m(x_1, x_2, x_3, ..., x_n) = 0. \end{cases}$$

In this case the auxiliary function of n variables involves additional unknown parameters (Lagrange's multipliers)

$$L(x_1, x_2, ..., x_n, \lambda_1, \lambda_2, ..., \lambda_m) = f(x_1, x_2, ..., x_n) + \sum_{i=1}^{m} \lambda_i \varphi_i(x_1, x_2, ..., x_n).$$

To find the points of the extreme of this function we form a corresponding system of $n + m$ equations from which the possible values of the coordinates $x_1, x_2, ..., x_n$ of the points of the conditional extreme are found.

Here we don't discuss sufficient conditions for the points of the conditional extreme; in a concrete problem the given conditions often make it possible to find out whether the point determined from the above equations is an extreme point without resorting to sufficient conditions.

Note that for the function of two variables, if the found point (x_0, y_0) is the critical point, the sufficient condition can be written as

$$\Delta = \begin{vmatrix} L''_{xx} & L''_{xy} \\ L''_{yx} & L''_{yy} \end{vmatrix} > 0.$$

At that point there is the extreme.

It will be maximum if $L''_{xx} < 0\,(L''_{yy} < 0)$ at that point and minimum, when $L''_{xx} > 0\,(L''_{yy} > 0)$ at the indicated point.

If $\Delta < 0$ there is no extreme at that point and for $\Delta = 0$ an additional investigation is required.

And now let us consider some examples.

Example 14. Find an absolute and a conditional extreme of the function $z = x^2 + y^2$ with the constraint $x + y - 1 = 0.$

Solution.

1) The absolute extreme.

First of all we have to find the critical points of the function under consideration using a necessary condition

$$\begin{cases} \dfrac{\partial z}{\partial x} = 2x \\ \dfrac{\partial z}{\partial y} = 2y \end{cases} \Rightarrow 2x = 0, 2y = 0, \text{ i.e. } x = 0; \ y = 0.$$

The point $(0, 0)$ is the critical point. Since the function z is the unbounded function for $\forall x \in (-\infty; +\infty)$, $\forall y \in (-\infty; +\infty)$, then $(0, 0)$ is the point of minimum and $z_{\min} = z(0; 0) = 0$.

2) The conditional extreme.

Note that we can seek the points of the conditional extreme for the values x and y, satisfying the constraint $x + y - 1 = 0$. By means of the constraint equation we can write y in terms of x, therefore this problem can be resolved without using Lagrange's function.

Resolve the constraint with respect to y, i. e. $y=1-x$ and substitute it into the expression for z.

We obtain $z=x^2+(1-x)^2$ as a function of one variable x. Research it for the extreme

$$\frac{dz}{dx}=2x+2(1-x)(-1)=0 \quad \Rightarrow \quad x=\frac{1}{2}.$$

If $x=1/2$, then $y=1-1/2=1/2$, i. e. the conditional extreme is attained at the point $P(1/2, 1/2)$.

At that point

$$z=(1/2)^2+(1-1/2)^2=1/2$$

and has a conditional extreme, since z is the unbounded function.

The point $M(1/2, 1/2, 1/2)$ is the vertex of the parabola obtained as a result of intersection of the paraboloid $z=x^2+y^2$ with the plane $x+y-1=0$.

In Fig. 6 the point $O(0, 0, 0)$ is the point of the absolute extreme for z and the point $M(1/2, 1/2, 1/2)$ is the point of the conditional extreme of the function z with the constraint $x+y-1=0$.

Example 15. Find a conditional extreme of the function $z=cos^2 x+cos^2 y$ on the sabsidiary condition $y-x=\frac{\pi}{4}$.

Solution. We can solve this problem by two methods.

Method 1. Let us construct a Lagrange's function

$$L(x,y,\lambda)=cos^2 x+cos^2 y+\lambda(y-x-\pi/4)$$

and write down the system of equations to determine the critical points and the parameter λ

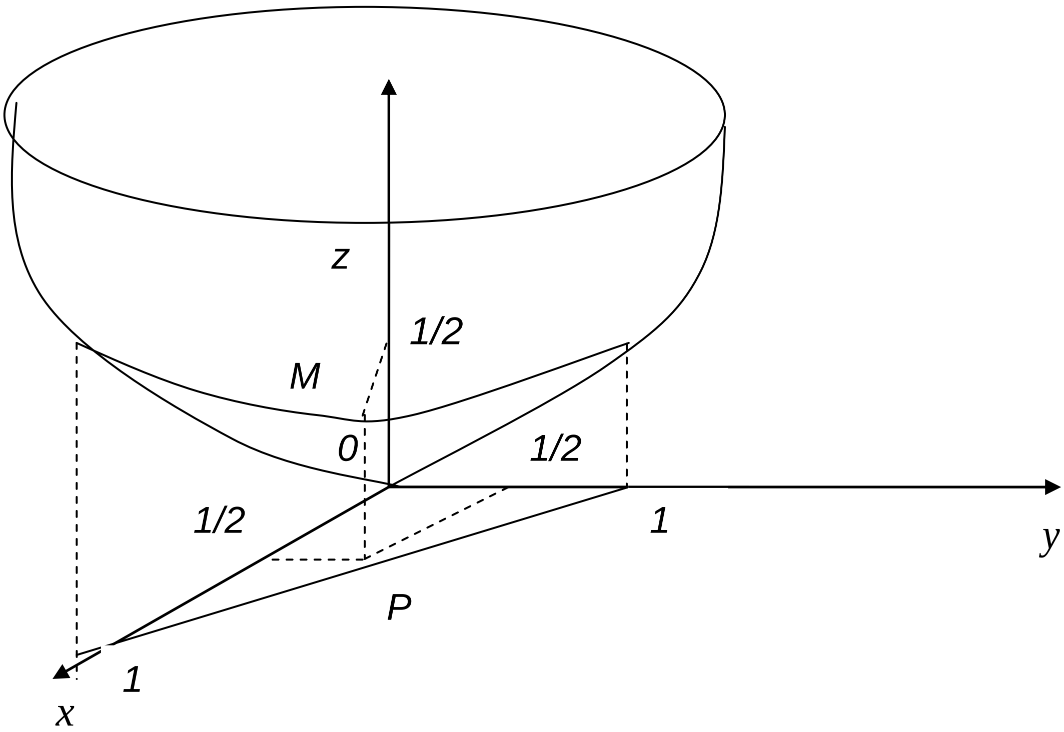

Fig. 6. **Intersection of the paraboloid with the plane**

$$\begin{cases} \dfrac{\partial L}{\partial x} = -2\cos x \sin x - \lambda = 0; \\ \dfrac{\partial L}{\partial y} = -2\cos y \sin y + \lambda = 0; \\ y - x - \dfrac{\pi}{4} = 0. \end{cases} \Rightarrow \begin{cases} \sin 2x = -\lambda; \\ \sin 2y = \lambda; \\ y - x = \dfrac{\pi}{4}. \end{cases}$$

Since $\sin 2x + \sin 2y = 2\sin(x+y)\cos(x-y) = 0 \quad (-\lambda + \lambda = 0)$ and $\cos(x-y) \neq 0$, then $\sin(x+y) = 0$, whence $x + y = \pi k, k \in Z$.

Consider the following system

$$\begin{cases} y - x = \pi / 4; \\ y + x = \pi k. \end{cases}$$

Its solution has the form

$$\begin{cases} x_k = \dfrac{\pi k}{2} - \dfrac{\pi}{8}; \\ y_k = \dfrac{\pi k}{2} + \dfrac{\pi}{8}, \text{ where } k \in Z. \end{cases}$$

10. Applying the Differential to Approximate Calculations

And now we have to find the second derivatives of the function L and use the sufficient condition of the conditional extreme

$$L''_{xx} = \frac{\partial^2 L}{\partial x^2} = -2\cos 2x; \qquad L''_{yy} = \frac{\partial^2 L}{\partial y^2} = -2\cos 2y;$$

$$L''_{xy} = L''_{yx} = \frac{\partial^2 L}{\partial x \partial y} = \frac{\partial^2 L}{\partial y \partial x} = 0,$$

then

$$\Delta = \begin{vmatrix} L''_{xx} & L''_{xy} \\ L''_{yx} & L''_{yy} \end{vmatrix} = \begin{vmatrix} -2\cos 2x & 0 \\ 0 & -2\cos 2y \end{vmatrix} = 4\cos 2x \cos 2y.$$

At the point (x_k, y_k) we have

$$\Delta = 4\cos(\pi k - \pi/4)\cos(\pi k + \pi/4) = 2\cos 2\pi k = 2 > 0 \text{ and}$$

$$\text{for } k = 2n, \; L''_{xx}\Big|_{(x_k, y_k)} = -2\cos(\pi k - \pi/4) =$$

$$= -2\cos(2\pi n - \pi/4) = -\sqrt{2} < 0,$$

$$\text{for } k = 2n+1, \; L''_{xx}\Big|_{(x_k, y_k)} = -2\cos(2\pi n + \pi - \pi/4) =$$

$$= -2\cos(2\pi n + 3\pi/4) = \sqrt{2} > 0$$

It means there is a conditional maximum

$$z_{max} = \cos^2\left(\pi n - \frac{\pi}{8}\right) + \cos^2\left(\pi n + \frac{\pi}{8}\right) = \frac{1}{2}\left(1 + \cos\left(2\pi n - \frac{\pi}{4}\right)\right) +$$

$$+ \frac{1}{2}\left(1 + \cos\left(2\pi n + \frac{\pi}{4}\right)\right) = \frac{1}{2} + \frac{\sqrt{2}}{4} + \frac{1}{2} + \frac{\sqrt{2}}{4} = 1 + \frac{\sqrt{2}}{2}$$

at the points (x_{2n}, y_{2n}), and there is a conditional minimum

$$z_{min} = \cos^2\left(\pi n + \frac{3\pi}{8}\right) + \cos^2\left(\pi n + \frac{5\pi}{8}\right) =$$

$$= \frac{1}{2}\left(1 + \cos\left(2\pi n + 3\pi/4\right)\right) + \frac{1}{2}\left(1 + \cos\left(2\pi n + 5\pi/4\right)\right) =$$

$$= \frac{1}{2} - \frac{\sqrt{2}}{4} + \frac{1}{2} - \frac{\sqrt{2}}{4} = 1 - \frac{\sqrt{2}}{2}.$$

at the points (x_{2n+1}, y_{2n+1}).

Method 2. Using the constraint equation $y = x + \pi/4$ we can find the conditional extreme without Lagrange’s function. Substitute $y = x + \pi/4$ in the equation for z, then $z = \cos^2 x + \cos^2 y = \cos^2 x + \cos^2(x + \pi/4)$.

We have the function of one variable x and investigate it for the extreme. The necessary condition is:

$$\frac{dz}{dx} = -2\sin x \cos x - 2\sin(x + \pi/4)\cos(x + \pi/4) = -\sin 2x -$$

$$- \sin(2x + \pi/2) = -(\sin 2x + \sin(2x + \pi/2)) = 0,$$

but

$$\sin 2x + \sin(2x + \pi/2) = 2\sin(2x + \pi/4)\cos(\pi/4) =$$

$$= 2\sin(2x + \pi/4) \cdot \frac{\sqrt{2}}{2} = \sqrt{2}\sin(2x + \pi/4) \text{ and}$$

$$\sin(2x + \pi/4) = 0,\ 2x + \pi/4 = k\pi,\ k \in \mathbb{Z} \text{ and}$$

$x_k = \frac{k\pi}{2} - \frac{\pi}{8},\ y_k = \frac{k\pi}{2} + \frac{\pi}{8}$ are the points of a possible extreme. The second sufficient condition for the extreme is:

[illegible]

For $k = 2n$ $z''_{xx} = -2\sqrt{2} < 0$ and at the point (x_{2n}, y_{2n}) we have a conditional maximum $z_{\max} = 1 + 2\sqrt{2}$, for $k = 2n+1$ $z''_{xx} = 2\sqrt{2} > 0$ and at the point (x_{2n+1}, y_{2n+1}) there is a conditional minimum $z_{\min} = 1 - 2\sqrt{2}$. We see that the answers obtained by two methods coincide.

As an application of the theory of conditional extreme the question on the greatest and the least value of the function of several variables on a bounded closed set can serve, we have considered how local extremes are found at the interior points of a set when certain constraints are imposed on a function $f(x)$.

Let us now show how local extremes are found at boundary points. For the sake of simplicity, let us confine ourselves to the case of three variables. Let the surface defined by the equation $\varphi(x, y, z) = 0$ be the boundary of the domain of change of the variables x, y and z in which the function $u = f(x, y, z)$ is defined and let the functions $f(x, y, z)$ and $\varphi(x, y, z)$ have continuous partial derivatives of the second order.

Then we arrive at the following problem: find the points of the maximum or the minimum of the function $u = f(x, y, z)$ under the condition that $\varphi(x, y, z) = 0$. This is just a problem of a conditional extreme.

Example 16. The canal section has a form of the isosceles trapezoid of a given area S. How can we choose its dimensions the washed surface to be the least?

Solution. Denote as l and a the lateral side and the lesser base of the trapezoid respectively. Let α be the inclined angle of the lateral side. Then there are the following relationships between the altitude h and the greater b base of the trapezoid (Fig. 7).

Here

$$h = l\sin\alpha,\ l = h/\sin\alpha,\ b = a + 2l\cos\alpha.$$

Denote the least washed surface as u. It is obvious that $u = 2l + a = a + 2h/\sin\alpha$, where u is the function of three variables h, a, α. Reduce u to the function of two variables. For this purpose express the known area S in term of h, a and α.

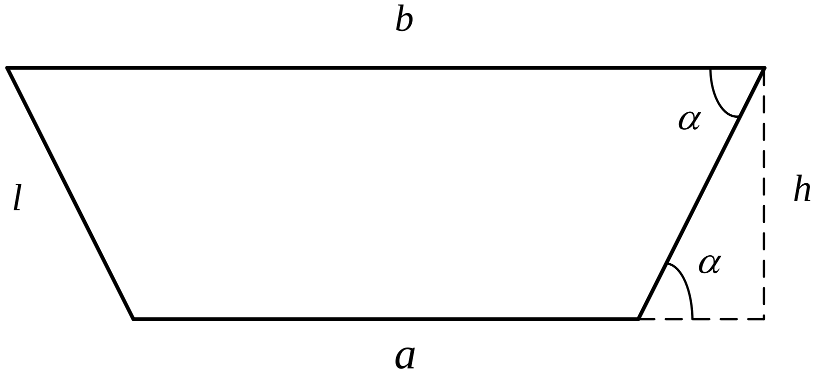

Fig. 7. **The isosceles trapezoid of the given area S**

By this expression we can write $a = S/h - h\,ctg\,\alpha$. Now u can be written as a function of two variables h and α, i. e.

$$u = \frac{2h}{\sin\alpha} + a = \frac{2h}{\sin\alpha} + \frac{S}{h} - h\,ctg\,\alpha.$$

Use the necessary condition for the extreme

$$\begin{cases} \dfrac{\partial u}{\partial h} = \dfrac{2}{\sin\alpha} - \dfrac{S}{h^2} - ctg\,\alpha \\ \dfrac{\partial u}{\partial \alpha} = -\dfrac{2h\cos\alpha}{\sin^2\alpha} + \dfrac{h}{\sin^2\alpha}, \end{cases} \Rightarrow \begin{cases} \dfrac{2}{\sin\alpha} - \dfrac{S}{h^2} - ctg\,\alpha = 0 \\ \dfrac{h}{\sin^2\alpha}(1 - 2\cos\alpha) = 0 \end{cases}$$

The second equation of this system can be fulfilled when $h = 0$ or $1 - 2\cos\alpha = 0$. But $h \neq 0$ since h is the depth of the canal, then $\cos\alpha = 1/2$ and $\alpha = \pi/3 = 60^o$; $\sin\alpha = \sqrt{3}/2$. By the first equation of the system we have:

$$\frac{4}{\sqrt{3}} - \frac{S}{h^2} - \frac{1}{\sqrt{3}} = 0 \Rightarrow h^2 = \frac{S}{\sqrt{3}} \Rightarrow h = \sqrt{\frac{S\sqrt{3}}{3}}$$

It follows that the critical values of $\alpha=\frac{\pi}{3}$ and $h=\frac{\sqrt{S}}{\sqrt[4]{3}}$. Let us define the values of the second order derivatives for the found quantities of α and h:

$$\begin{cases}\frac{\partial^2 u}{\partial h^2}=\frac{2S}{h^3}\Big|_{h=\sqrt{S}/\sqrt[4]{3}}=\frac{2S}{\left(\sqrt{S}/\sqrt[4]{3}\right)^3}=\frac{6\sqrt[4]{3}}{\sqrt{S}}>0\\ \frac{\partial^2 u}{\partial \alpha^2}=\left(\frac{2h}{\sin^3\alpha}(1-2\cos\alpha)+\frac{2h\sin\alpha}{\sin^2\alpha}\right)\Big|_{h=\sqrt{S}/\sqrt[4]{3},\ \alpha=\pi/3}=\end{cases}$$

$$\begin{cases}=\frac{2\sqrt{S}/\sqrt[4]{3}}{3/4}=\frac{8\sqrt{S}}{3\sqrt[4]{3}}>0\\ \frac{\partial^2 u}{\partial h\,\partial\alpha}=\frac{\partial^2 u}{\partial\alpha\,\partial h}=\left(-\frac{2\cos\alpha}{\sin^2\alpha}+\frac{1}{\sin^2\alpha}\right)\Big|_{\alpha=\pi/3}=0\end{cases}$$

$$\Delta=\begin{vmatrix}\frac{\partial^2 u}{\partial h^2} & \frac{\partial^2 u}{\partial h\,\partial\alpha}\\ \frac{\partial^2 u}{\partial\alpha\,\partial h} & \frac{\partial^2 u}{\partial\alpha^2}\end{vmatrix}=\begin{vmatrix}\frac{6\sqrt[4]{3}}{\sqrt{S}} & 0\\ 0 & \frac{8\sqrt{S}}{3\sqrt[4]{3}}\end{vmatrix}=\frac{16\sqrt[4]{3}}{\sqrt[4]{3}}>0$$

There is an extreme, in particular a minimum, since $\Delta>0$ and $\frac{\partial^2 u}{\partial h^2}>0$. As a result

$$u_{min}=2\sqrt{S}\sqrt[4]{3}.$$

Now consider the same problem by means of Lagrange's multipliers method on admitting

$$u_1=\frac{2h}{\sin\alpha}\ \text{[illegible]}$$

For that purpose construct the Lagrange's function:

$$L(h,a,\alpha,\lambda)=\frac{2h}{\sin\alpha}+a+\lambda\left(\left(a+h\,ctg\alpha\right)h-S\right)=$$
$$=\frac{2h}{\sin\alpha}+a+\lambda\left(ah+h^2ctg\alpha-S\right)$$

and write down a corresponding system for the determination of λ and the coordinates of the critical points

$$\begin{cases}\dfrac{\partial L}{\partial h}=\dfrac{2}{\sin\alpha}+\lambda a+2\lambda h\,ctg\alpha=0;\\[2mm] \dfrac{\partial L}{\partial \alpha}=-\dfrac{2h\cos\alpha}{\sin^2\alpha}-\dfrac{\lambda h^2}{\sin^2\alpha}=0;\end{cases}$$

$$\begin{cases}\dfrac{\partial L}{\partial a}=1+\lambda h=0;\\[2mm] \dfrac{\partial L}{\partial \lambda}=ah+h^2ctg\alpha-S=0.\end{cases}$$

By the third equation we obtain $\lambda=-\dfrac{1}{h}$. On substituting its value into the rest of the equations we have:

$$\begin{cases}\dfrac{2}{\sin\alpha}-\dfrac{a}{h}-2ctg\alpha=0;\\[2mm] -\dfrac{2h\cos\alpha}{\sin^2\alpha}+\dfrac{h}{\sin^2\alpha}=0;\\[2mm] ah+h^2ctg\alpha-S=0.\end{cases}$$

From the second equation we get $2\cos\alpha=1;\ \cos\alpha=1/2;\ \alpha=\dfrac{\pi}{3}=60^o$.

So the system gains the form:

$$\begin{cases} \frac{4}{\sqrt{3}} - \frac{a}{h} - \frac{2}{\sqrt{3}} = 0; \\ ah + \frac{h^2}{\sqrt{3}} - S = 0. \end{cases}$$

By the first equation $h = \frac{a\sqrt{3}}{2}$. Substitute it into the second equation and we obtain

[illegible]

We receive the dimension coinciding with previous values. As an addition we have to note what the economic sense of Lagrange’s multipliers is. The economic sense of Lagrange’s multipliers is the following: the number of Lagrange’s multipliers has an influence on the supplies of resources of some production and in the end on the profit. Non-zero multipliers indicate that the corresponding resources are in short supply and they should be increased.

Zero multiplies say that the corresponding resources are in plenty and they may be decreased. The reader can know more details about it in the Course of mathematical programming.

11. Economic interpretation

Marginal products. For a function of one variable $y = f(x)$ the derivative $f'(x)$ measures (infinitesimally) how a Δx-change in x to y:

$$\Delta y \approx f'(x)\Delta x.$$

The same interpretation applies to functions of several variables. For example, let $Q = F(K, L)$ be a production function, which relates the output Q to amounts of capital input K and labor input L. If the firm is presently using K^* units of capital and L^* units of labor to produce $Q^* = F(K^*, L^*)$ units of output, then the partial derivative

$$\frac{\partial F}{\partial K}\left(K^*, L^*\right)$$

is the rate at which output changes with respect to capital K, keeping L fixed at L^*. If capital increases by ΔK, then output will increase by

$$\Delta Q \approx \frac{\partial F}{\partial K}\left(K^*, L^*\right)\cdot \Delta K .$$

Setting $\Delta K = 1$, we see that $\frac{\partial F}{\partial K}\left(K^*, L^*\right)$ estimates the change in output due to a one unit increase in capital (with L fixed). Hence, $\frac{\partial F}{\partial K}\left(K^*, L^*\right)$ is called *the marginal product of capital* or MPK. Similarly, $\frac{\partial F}{\partial L}\left(K^*, L^*\right)$ is the rate at which output changes with respect to labor, with capital held fixed at K^*. Since it is a good estimate of the change in output for a one unit increase in labor input, $\frac{\partial F}{\partial L}\left(K^*, L^*\right)$ is called *the marginal product of labor* (often abbreviated as MPL).

Example 17. Consider the Cobb-Douglas production function $Q = 4K^{3/4}L^{1/4}$. When $K = 10000$ and $L = 625$, output Q is

$$Q(10000;625) = 4\cdot 10000^{3/4}625^{1/4} = 4\cdot 10^3\cdot 5 = 20000 .$$

Computing partial derivatives,

$$\frac{\partial Q}{\partial K} = 4L^{1/4}\cdot \frac{3}{4}K^{-1/4} = 3L^{1/4}K^{-1/4}$$

(remember to treat L as constant) and

$$\frac{\partial Q}{\partial L} = 4K^{3/4} \cdot \frac{1}{4} L^{-3/4} = K^{3/4} L^{-3/4}$$

(treating K as constant). Furthermore,

$$\left.\frac{\partial Q}{\partial K}\right|_{(10000;\,625)} = 3L^{1/4}K^{-1/4}\Big|_{(10000;\,625)} = 3 \cdot 625^{1/4} \cdot 10000^{-1/4} = \\ = \frac{3 \cdot 625^{1/4}}{10000^{1/4}} = \frac{3 \cdot 5}{10} = 1.5 \; ; \qquad (2)$$

$$\left.\frac{\partial Q}{\partial L}\right|_{(10000;\,625)} = K^{3/4}L^{-3/4}\Big|_{(10000;\,625)} = 10000^{3/4} \cdot 625^{-3/4} = \\ = \frac{10000^{3/4}}{625^{3/4}} = \frac{10^3}{5^3} = 8 \; . \qquad (3)$$

If L is held constant and increased by ΔK, Q will increase by approximately $1.5 \cdot \Delta K$. For an increase in K of 10 units, use (2) to estimate $Q(10010;625)$ to be

$$Q(10010;625) = Q(10000;625) + 1.5 \cdot \Delta K = 20000 + 1.5 \cdot 10 = 20015.$$

Similarly, because of (3), a 2-unit decrease in L should induce a $8 \cdot \Delta Q = 8 \cdot 2 = 16$-unit decrease in Q. Consequently, we estimate $Q(10000;623)$ to be

$$Q(10000;623) = Q(10000;625) + 8 \cdot (-2) = 20000 - 16 = 19984.$$

Elasticity. If $Q_1 = Q_1(P_1, P_2, I)$ represents the demand for good 1 in terms of the prices of goods 1 and 2 and income, then $\frac{\partial Q_1}{\partial P_1}$ is the rate of change of demand with respect to own price. If the price of good 1 rises by a small amount ΔP_1, the demand for good 1 will change roughly by

$$\Delta Q_1 \approx \frac{\partial Q_1}{\partial P_1} \cdot \Delta P_1. \qquad (4)$$

In general, we would expect $\frac{\partial Q_1}{\partial P_1}$ to be negative. The quantity $\frac{\partial Q_1}{\partial P_1}$ is unsatisfactory as a measure of price sensitivity because it depends too heavily on the units used. To remove this dependency on units, economists measure the sensitivity of demand in percentage terms. More precisely, they define the own price elasticity of demand as

$$\varepsilon_1 = \frac{change\ in\ demand\ \%}{change\ in\ own\ price\ \%} = \frac{\Delta Q_1 / Q_1}{\Delta P_1 / P_1} = \frac{P_1}{Q_1} \cdot \frac{\Delta Q_1}{\Delta P_1}.$$

Since

$$\frac{\Delta Q_1}{\Delta P_1} = \frac{Q_1(P_1 + \Delta P_1) - Q_1(P_1)}{\Delta P_1} \approx \frac{\partial Q_1}{\partial P_1}$$

for small ΔP_1 by (4), this elasticity in calculus terms is

$$\varepsilon_1 = \frac{P_1^* \cdot \frac{\partial Q_1}{\partial P_1}\left(P_1^*, P_2^*, I^*\right)}{Q_1\left(P_1^*, P_2^*, I^*\right)}.$$

It is usually negative. If it lies between -1 and 0, good 1 is called inelastic. If this elasticity lies between $-\infty$ and -1, good 1 is called elastic – a small percentage change in price results in a large percentage change in quantity demanded.

To study the sensitivity in demand of one good to price changes in other goods, economists use the cross price elasticity of demand

$$\varepsilon_{Q_1,\, P_2} = \frac{change\ in\ demand\ \ for\ good\ 1\,\%}{change\ in\ price\ of\ \ good\ 2\,\%} = \frac{\Delta Q_1 / Q_1}{\Delta P_2 / P_2} \cdot$$

Individual tasks

1. Find and plot the domain of the definition of the function.
2. Find partial derivatives of the first and the second orders, check the equality of the mixed partial derivatives z''_{xy} and z''_{yx}, write down a differential of the first and the second orders of the function.
3. The function $z = f(x, y)$ and the points M_1 and M_2 are given. Find a derivative of this function at the point M_1 in the direction $\overrightarrow{M_1M_2}$ and $grad\, z(M_1)$.
4. Find a local extreme of a function.

Variant 1

1) $z = \dfrac{3xy}{2x-5y}$;

2) $z = e^{xy} + y^2$;

3) $z = x^2y + y^2x$; $M_1(1, -1)$; $M_2(3, 4)$;

4) $z = x^2 + xy + y^2 - 6x - 9y$;

Variant 2

1) $z = \arcsin(x - y)$;

2) $z = e^{x^2y} + 2y$;

3) $z = 5xy^3$; $M_1(2, 1)$; $M_2(4, -3)$;

4) $z = (x-2)^2 + 2y^2 - 10$;

Variant 3

1) $z=\sqrt{y^2-x^2}$;

2) $z=e^{xy^2}+2x$;

3) $z=\ln\left(x^2+y^2\right)$; $M_1(-1,2)$; $M_1(0,-2)$;

4) $z=(x-5)^2+y^2+1$;

Variant 4

1) $z=\ln\left(4-x^2-y^2\right)$;

2) $z=e^{x^2y^3}+y^2$;

3) $z=e^{x^2+y^2}$; $M_1(0,0)$; $M_2(3,-4)$;

4) $z=x^3+y^3-3xy$;

Variant 5

1) $z=\dfrac{2}{6-x^2-y^2}$;

2) $z=e^{x^2y^3}+x^2$;

3) $z=\ln(xy+y)$; $M_1(-2,3)$; $M_2(2,1)$;

4) $z=2xy-2x^2-4y^2$;

Variant 6

1) $z=\sqrt{x^2+y^2-5}$;

2) $z=e^{x^2y^3}+2y$;

3) $z=\sqrt{1+x^2+y^2}$; $M_1(1,1)$; $M_2(3,2)$;

4) $z=x\sqrt{y}-x^2-y+6x+3$;

Variant 7

1) $z=\arccos(x+y)$;

2) $z=e^{x^2y^3}+3y^2$;

3) $z=x^2y+x-2$; $M_1(1,1)$; $M_2(2,-1)$;

4) $z=2xy-5x^2-3y^2+2$;

Variant 8

1) $z=\dfrac{3x+y}{2-x+y}$;

2) $z=e^{xy^3}+2x$;

3) $z=xe^y+ye^x$; $M_1(1,0)$; $M_2(4,1)$;

4) $z=xy-x^2-y^2+9$;

Variant 9

1) $z=\sqrt{9-x^2-y^2}$;

2) $z=e^{xy^3}+2y$;

3) $z=3xy^2-yx$; $M_1(1,1)$; $M_2(3,-1)$;

4) $z=xy(12-x-y)$;

Variant 10

1) $z=\ln\left(x^2+y^2-3\right)$;

2) $z=e^{xy^3}+2y$;

3) $z=5x^2y-y^2x$; $M_1(1,1)$; $M_2(9,-3)$;

4) $z=2xy-3x^2-2y^2+10$;

Variant 11

1) $z=\sqrt{2x^2-y^2}$;

2) $z=e^{xy^3}-2^2$;

3) $z=\frac{x}{x^2+y^2}$; $M_1(1,2)$; $M_2(-3,2)$;

4) $z=x^3+8y^3-6xy+1$;

Variant 12

1) $z = \frac{4xy}{x - 3y + 1}$;

2) $z = e^{x^3 y^2} + y^2$;

3) $z = y^2 - 2xy$; $M_1(3, 1)$; $M_2(-2, 1)$;

4) $z = y\sqrt{x} - y^2 - x + 6y$;

Variant 13

1) $z = \frac{\sqrt{xy}}{x^2 - y^2}$;

2) $z = e^{x^3 y^2} + 2y$;

3) $z = x^2 + y^2 - 2xy$; $M_1(1, -1)$; $M_2(5, -1)$;

4) $z = x^2 - xy + y^2 + 9x - 6y + 20$;

Variant 14

1) $z = \ln\left(y^2 - x^2\right)$;

2) $z = e^{x^3 y^2} + 2x$;

3) $z = \ln\left(1 + x + y^2\right)$; $M_1(1, 1)$; $M_2(3, -5)$;

4) $z = xy(6 - x - y)$;

Variant 15

1) $z = \arcsin \frac{x}{y}$;

2) $z = e^{x^3 y^2} - 2y^2$;

3) $z = x^2 + 2y^2 - 5$; $M_1(1, 2)$; $M_2(-3, -2)$;

4) $z = x^2 + y^2 - xy + x + y$;

Variant 16

1) $z = \frac{x^3 y}{3 + x - y}$;

2) $z = e^{x^4 y} + y^2$;

3) $z = \ln\left(x^3 + y^3 + 1\right)$; $M_1(1, 3)$; $M_2(-4, 1)$;

4) $z = x^2 + xy + y^2 - 2x - y$;

Variant 17

1) $z = \arccos(x + 2y)$;

2) $z = e^{x^4 y} + 2y$;

3) $z = x - 2y$; $M_1(-4, -5)$; $M_2(2, 3)$;

4) $z = (x - 1)^2 + 2y^2$;

Variant 18

1) $z = \dfrac{\sqrt{3x - 2y}}{x^2 + y^2 - 4}$;

2) $z = e^{x^4 y} - 2y$;

3) $z = \left(x - y^2\right)^2$; $M_1(1, 5)$; $M_2(3, 7)$;

4) $z = xy - 3x^2 - 2y^2$;

Variant 19

1) $z = \ln\left(9 - x^2 - y^2\right)$;

2) $z = e^{x^4 y} + x^2$;

3) $z = 3x^2 y$; $M_1(-2, -3)$; $M_2(5, -2)$;

4) $z = x^2 + 3(y + 2)^2$;

Variant 20

1) $z = \dfrac{1}{\sqrt{x^2 + y^2 - 5}}$;

2) $z = e^{x^4 y} + 2x$;

3) $z = x^y$; $M_1(3, 1)$; $M_2(1, -1)$;

4) $z = 2(x + y) - x^2 - y^2$;

Variant 21

1) $z=\sqrt{3-x^2-y^2}$;

2) $z=e^{x^4 y}+2x^2$;

3) $z=e^{xy}$; $M_1(-5,0)$; $M_2(2,4)$;

4) $z=y\sqrt{x}-2y^2-x+14y$;

Variant 22

1) $z=\dfrac{4x+y}{2x-5y}$;

2) $z=e^{xy^4}+y^2$;

3) $z=\left(x^2+y^2\right)^3$; $M_1(1,2)$; $M_2(0,-1)$;

4) $z=x^3+8y^3-6xy+5$;

Variant 23

1) $z=\arcsin(2x-y)$;

2) $z=e^{xy^4}+2y$;

3) $z=x^y-3yx$; $M_1(2,2)$; $M_2(1,0)$;

4) $z=1+15x-2x^2-xy-2y^2$;

Variant 24

1) $z = \dfrac{5}{4 - x^2 - y^2}$;

2) $z = e^{xy^4} + x^2$;

3) $z = x^2 y + y^2$; $M_1(0, -2)$; $M_2(12, -5)$;

4) $z = 1 + 6x - x^2 - xy - y^2$;

Variant 25

1) $z = \ln(2x - y)$;

2) $z = e^{xy^4} - 2y^2$;

3) $z = \dfrac{10}{x^2 + y^2 + 1}$; $M_1(-1, 2)$; $M_2(2, 0)$;

4) $z = x^3 + y^2 - 6xy - 39x + 18y + 20$;

Variant 26

1) $z = \dfrac{7x^3 y}{x - 4y}$;

2) $z = e^{x^3 y^2} - 3x^2$;

3) $z = \ln(1 + x^2 - y^2)$; $M_1(1, 1)$; $M_2(5, -4)$;

4) $z = 2x^3 + 2y^3 - 6xy + 5$;

Variant 27

1) $z = \sqrt{1 - x - y}$;

2) $z = e^{x^3 y^2} + x^3$;

3) $z = \frac{x}{y} + \frac{y}{x}$; $M_1(-1, 1)$; $M_2(2, 3)$;

4) $z = 3x^3 - 3y^3 - 9xy + 10$;

Variant 28

1) $z = e^{\sqrt{x^2 + y^2 - 1}}$;

2) $z = e^{x^3 y} + y^2$;

3) $z = x^3 + y^2 x - 6xy$; $M_1(1, 3)$; $M_2(4, 2)$;

4) $z = x^2 + xy + y^2 + x - y + 1$;

Variant 29

1) $z = \frac{1}{x^2 + y^2 - 6}$;

2) $z = e^{x^3 y} + 3y$;

3) $z = \frac{x}{y} - \frac{y}{x}$; $M_1(2, 2)$; $M_2(-3, 4)$;

4) $z = 4(x - y) - x^2 - y^2$;

Variant 30

1) $z = \arccos(x - y);$

2) $z = e^{x^3 y} + 2y^2;$

3) $z = e^{x-y};\ M_1(1, 0);\ M_2(2, -4);$

4) $z = 6(x - y) - 3x^2 - 3y^2;$

Theoretical questions

1. A set of points.
2. A range of a function.
3. The domain of the definition of the function.
4. The function of two variables.
5. A geometrical interpretation of the function of two variables.
6. The surface and the plane.
7. A tangent line to a section of a surface.
8. An implicit function.
9. An explicit function.
10. The xOy-plane.
11. A limit of a function.
12. The δ-neighbourhood of a point.
13. A continuous function.
14. A continuity at a point (x_0, y_0).
15. A total increment of the function of two variables.
16. A linear function of the increment.
17. The principal part of the total increment.
18. A partial increment of the function of two variables.
19. A partial derivative of the function of two variables.
20. A sufficient condition for differentiability.
21. An independent variable and a dependent variable.
22. An argument of a function.
23. An original generalization of the partial derivatives.
24. The function of three variables.
25. A geometrical meaning of partial derivatives.
26. A function differentiable at a point.
27. A total differential of a function.
28. A partial differential of a function.
29. An approximate calculations of a function.
30. Applying the total differential to approximate calculation.
31. The differentiation of the function of two variables.
32. An infinitesimal value.
33. A composite function of two independent variables.
34. An intermediate argument.

35. A definiteness.
36. A composite function.
37. A directional derivative.
38. The unit vector.
39. Direction cosines.
40. A gradient of a function.
41. Projections of a vector.
42. Projections of a gradient.
43. A scalar product of vectors.
44. The point of extreme of the function of two variables.
45. The point of maximum or the point of minimum.
46. A necessary condition for the extremum.
47. A cuspidal point of a surface.
48. A stationary or critical point of a function.
49. A sufficient condition for the extremum.
50. A second derivative of a function.
51. The greatest and the least values of a function.
52. A closed domain.
53. A boundary of a domain.
54. An interior point of a domain.
55. A boundary point.
56. A magnitude.
57. A coupling or a subsidiary condition.
58. An unconditional extreme.
59. A point of a conditional extreme.
60. The necessary conditions for a conditional extreme.
61. Lagrange’s method of multipliers.
62. Necessary conditions for a local conditional extreme of a function.
63. A Lagrange’s function.
64. Lagrange’s multipliers.
65. Sufficient conditions for the points of a conditional extreme.
66. An absolute and a conditional extreme of the function.
67. A constraint equation or a constraint.
68. A canal section, an isosceles trapezoid.

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