MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
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Guidelines
to practical tasks in introduction to mathematical analysis of the academic discipline "Higher Mathematics" for foreign and English-learning full-time students

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Kharkiv, S. Kuznets KhNUE, 2018

Затверджено на засіданні кафедри вищої математики й економіко-математичних методів.

Протокол № 1 від 27.08.2018 р.

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Guidelines to practical tasks in introduction to mathematical analysis of the educational discipline “Higher and Applied Mathematics” for foreign and English-learning full-time students / compiled by Ie. Misiura. – Kh. : Publishing House of S. Kuznets KhNUE, 2018. – 61 p. (English)

Methodical recommendations are intended for foreign and English-learning full-time students for practical classes on topic “Limits” of the discipline "Higher Mathematics". The sufficient theoretical material and typical examples are presented which give students the possibility to master the material on topic “Limits” and apply the obtained knowledge in practice. Individual tasks for self-study work and the list of theoretical questions which promote improving and extending of students’ knowledge on the theme are given.

Recommended for full-time students of training direction 6.030601 "Management".

Подано методичні рекомендації для студентів-іноземців та студентів, що навчаються англійською мовою, денної форми навчання для практичних занять з теми «Границя» навчальної дисципліни «Вища математика». Викладено необхідний теоретичний матеріал та наведено типові приклади, які сприяють найбільш повному засвоєнню матеріалу з теми «Границя» та застосуванню отриманих знань на практиці. Подано завдання для індивідуальної роботи та перелік теоретичних питань, що сприяють удосконаленню та поглибленню знань студентів з даної теми.

Рекомендовано для студентів денної форми навчання.

Introduction

Nowadays, it is impossible to conduct serious economic investigations without possessing powerful mathematical apparatus which is in an arsenal of mathematical analysis. Modern methods of mathematical analysis allow to solve the most difficult and miscellaneous economic problems, that is why mathematical analysis is the foundation of mathematical education. It requires the students to have considerable theoretical knowledge in mathematics whatever their speciality is, in particular, economic speciality.

The aim of this teaching aid is to present only the most basic, initial notions of mathematical analysis, such as: set, function, limit, continuity. These topics are illustrated with many examined examples. Students can apply the acquired knowledge and skills for solving many practical problems of economics and business.

These guidelines are intended for students of all specialities and can be used for studying under the guidance of a lecturer as well as independently.

Guidelines in Limits

1. Set

The notion of a set is one of the most fundamental and initial notions in mathematics and it cannot be defined in terms of other notions.

Let's present the definition of it.

Definition. A collection of things gathered according to a certain sign will be called a set. Objects entering into a set are called the elements of a set.

Sets are denoted by upper Latin letters $A, B, \ldots, X, Y$ and their elements by lower letters $a, b, \ldots, x, y$.

The belonging of the element x to the set X is symbolized in the following way: $x \in X$. If x does not enter in the set X it is written as $x \notin X$.

The symbol $X \subset Y$ means that the set X is included into the set Y. In this case the set X is called the subset of the set Y and the elements of the set X are, at the same time, the elements of the set Y, that is, if $x \in X$ then $x \in Y$. Symbols $\subset$ and $\supset$ are called the signs of including.

It is advisable to introduce into consideration a set containing not a single element. Such a set is called empty and it is denoted by the symbol $\varnothing$.

The following notation is also used. For instance, the set of real integer numbers is denoted by the expression $N=\{1,2,3,\dots\}$.

The sets X and Y are equal to each other if $X \subset Y$ and $Y \supset X$, i.e. they are being the subsets of themselves. It is written as $X = Y$.

2. Operations on Sets

There are arithmetic operations on sets possessing the properties analogous to the corresponding properties of the addition and multiplication of numbers.

Let's give their definitions.

Definition. The collection of all the elements of the sets X and Y is the set Z which is called the union (sum) of the sets X and Y. It is written as:

$$Z = X + Y \text{ or } Z = X \cup Y.$$

For example, the union of all rational and irrational numbers form the set of real numbers.

The addition of the set to itself does not change the set, i.e.

$$A + A = A \cup A = A.$$

Definition. The collection of elements belonging simultaneously to the sets X and Y is the set Z which is called the intersection of the sets X and Y. It is denoted as:

$$Z = XY \text{ or } Z = X \cap Y.$$

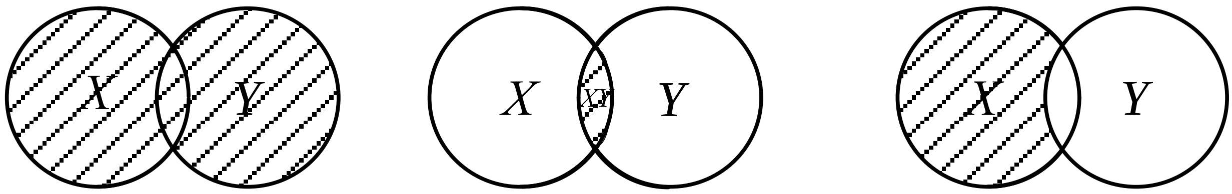
Let's note that $X \cap X = X$.

The expression $X \cap Y = \varnothing$ means that for any x belonging to the set X it follows that x does not belong to Y and for y from Y, y does not belong to X. Briefly it can be written as

$$\forall x \in X \Rightarrow x \notin Y \text{ and } \forall y \in Y \Rightarrow y \notin X .$$

The symbol "$\forall$" is read as *for any* or *for all* and the symbol "$\Rightarrow$" is read so: *it follows* or *whence*.

These sets are called not intersecting sets.



a) addition; b) intersection; c) difference

Fig. 1. Arithmetic operations on sets

Definition. The difference of the sets X and Y is called the set Z which is written as:

$$Z = X \quad \text{or } Z = X - Y .$$

In this case the set Z contains elements of the set X not entering in the set Y.

Foregoing operations are shown by the strokes part in Fig. 1.

3. Neighborhood of a Point

A necessity of using such a notion as a neighborhood of a point frequently appears in the mathematical analysis.

Let's present this notion.

The *real axis* is a straight line with a point O chosen as the *origin*, a positive direction, and a scale unit.

There is a one-to-one correspondence between the set of all real numbers R and the set of all points of the real axis, with each real x being represented by a point on the real axis separated from O by the distance $|x|$ and lying to the right of O for $x > 0$, or to the left of O for $x < 0$.

One often has to deal with the following number sets (sets of real numbers or sets on the real axis).

1. Sets of the form $(a,b),(-\infty,b),(a,+\infty)$ and $(-\infty,+\infty)$ consisting, respectively, of all $x \in R$ such that $a < x < b$, $x < b$, $x > a$, and x is arbitrary, are called *open intervals* (sometimes simply *intervals*).

2. Sets of the form $[a,b]$ consisting of all $x \in R$ such that $a \le x \le b$ are called *closed intervals* or *segments*.

3. Sets of the form $(a,b], [a,b), (-\infty,b], [a,+\infty)$ consisting of all x such that $a < x \le b$, $a \le x < b$, $x \le b$, $x \ge a$ are called *half-open intervals*.

A *neighborhood of a point* $x_0 \in R$ is defined as any open interval (a,b) containing x $(a < x < b)$. A neighborhood of the "point" $+\infty, -\infty$ or ∞ is defined, respectively, as any set of the form $(a,+\infty),(-\infty,a)$ or $(-\infty,a)\cup(a,+\infty)$ (here $a \ge 0$).

Let a and b be arbitrary points of a real axis (Fig. 2).

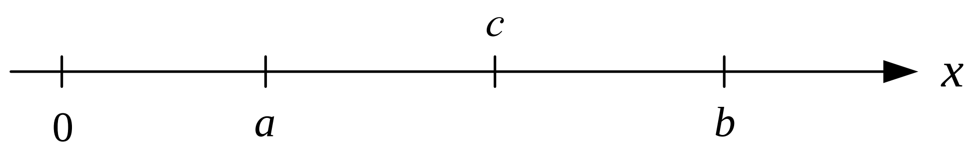


Fig. 2. An arbitrary interval (a,b) containing point c

If a two-dimensional point set is given on a plane with two arbitrary coordinates (x,y), then we may say about a neighborhood of a point in two-dimensional space R^2.

The number $|a-b|$ is called the distance between points a and b. An arbitrary interval (a,b) containing point c is called a neighborhood of point c. In particular the interval $(c-\varepsilon, c+\varepsilon)$, where $\varepsilon > 0$ is said to be ε-neighborhood of c (Fig. 3).

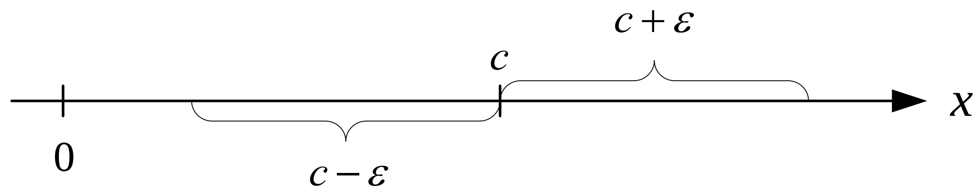


Fig. 3. A neighborhood of point c

Let there be given a point (x_0, y_0) in R^2. A set of points (x, y), in which coordinates satisfy the inequality

$$(x-x_0)^2+(y-y_0)^2<\varepsilon^2 \quad (\varepsilon>0)$$

is called an open circle of a radius ε with a center at the point (x_0, y_0).

A neighborhood of the point (x_0, y_0) on a plane is called an open circle of radius $\varepsilon>0$. If the radius of the circle is $\varepsilon>0$ then ε-neighborhood of this point will be this circle (Fig. 4).

A neighborhood of the three-dimensional point (x_0, y_0, z_0) will be called some open sphere of the radius $\varepsilon>0$.

In general case for n-dimensional point $(x_{10}, x_{20}, \ldots, x_{n0})$ its ε-neighborhood will be called an open n-dimensional sphere of the radius $\varepsilon>0$.

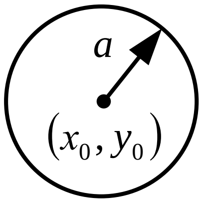


Fig. 4. A neighborhood of the point (x_0, y_0)

4. Closed and Bounded Sets

An imagination about a closed set would be more visual if we should define a boundary point of a set. A boundary point is called the set point, in which neighborhood contains points belonging to a set as well as not belonging to it. The point c is the boundary point of the set A (Fig. 5). Boundary points of a set form its boundary.

Let's give the definition of a closed set.

Definition 5. *A closed set* is called a set containing all its boundary points. Such a set can be bounded and unbounded.

Definition 6. Set $X=\{x\}$ is said to be *bounded above (below)* if there is such a number M that for all elements $x \in X$ the inequality $x \leq M$ ($x \geq M$) is fulfilled. The number M is called *the upper (lower) bound* of the set X.

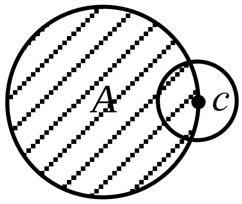


Fig. 5. The boundary point c of the set A

If a set is bounded both above and below it is said to be *bounded*.

Geometrically it means that there is a sphere, if a set is given in a space, or a circle, if a set is located on a plane of a finite radius R with a centre at any point of a set which entirely contains in itself all points of a given set (Fig. 6), otherwise a set is unbounded.

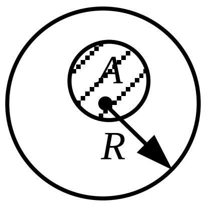


Fig. 6. A circle of a finite radius R with a centre at any point

5. Function

A function is one of the most basic mathematical notions as well.

Let there be given some number sets X and Y and a rule f according to which any number x from the set X ($x \in X$) is associated with some number y from Y ($y \in Y$). Then a function $y = f(x)$ is said to *be defined*.

The set X of points x for which a function $f(x)$ is defined is called the *domain of definition of the function* $f(x)$, while the set Y values of y is termed the *range of the function* $f(x)$.

A variable quantity x is called the *independent variable*, or the *argument* while y, which usually varies together with the independent variable, is termed the *dependent variable*.

For example, let there be given the function $y = \sin x$. Its domain of definition is the whole numerical axis $X=(-\infty;+\infty)$ and its range is the segment $Y=[-1;1]$.

The domain of definition of the function and its range may be also denoted by symbols D and E, respectively.

We may also consider a function of many or n-variables $y=(x_1,x_2,\ldots,x_n)$ when a number $y=(x_1,x_2,\ldots,x_n)$ corresponds to a collection of real numbers $x_1,x_2,\ldots,x_n$. The collection of numbers $x=(x_1,x_2,\ldots,x_n)$ may be conveniently regarded as a point of an n-dimensional space (briefly written as $y=f(x)$, where $x=(x_1,x_2,\ldots,x_n)$). Other notations are often used: $z=f(x,y)$ for a function of two variables and $u=f(x,y,z)$ for a function of three variables.

There are also another names of a function, it is just mapping, transformation and others. The most general used from them is mapping.

6. Methods of Representing Functions

Usually there are considered three methods of representing functions, namely: analytical, tabular and graphic methods.

Analytical method. In this case a formula is indicated by means of which it is possible to compute $f(x)$ for any $x\in X$. For instance, $y=6x^3$, x is the infinite interval $-\infty<x<+\infty$.

Tabular method. When specifying a function by means of a table, we simply write down a sequence of values of the independent variable and in the values of the function corresponding to them we also indicate the method of computing y for intermediate values of x, using the values given in table. Let's consider the table.

It is evident that here the function $y=x^3$ is given. We can see that the function analytically given can be represented in a table form. It means such functions may be tabulated.

Usually only analytically complicate functions frequently meeting in practice may be tabulated. This way of representing functions is widely used, for instance, everybody is undoubtedly familiar with tables of logarithms, tables of trigonometric functions and their logarithms, etc.

Table

Table of values of the independent and dependent variables

x	…	-3	-2	-1	0	1	2	3	…
y	…	-27	-8	-1	0	1	8	27	…

However there is no visual demonstration in a tabular and an analytical methods. But a graphic method has not this lack. A graphic method allows the accordance between the argument x and the function y to state with the aid of a graph.

Graphic method. We begin with the following definition.

Definition 7. The graph of a function (in rectangular Cartesian coordinates) is the locus of all points in which abscissas are values of the independent variable and ordinates are the corresponding values of the function.

In other words, if we take the abscissa equal to the value of the independent variable and the ordinate of the corresponding point of the graph is equal to the value of the function corresponding to that the value of the independent variable (Fig. 7). When plotting a graph we can take similar or different scales along the coordinate axes.

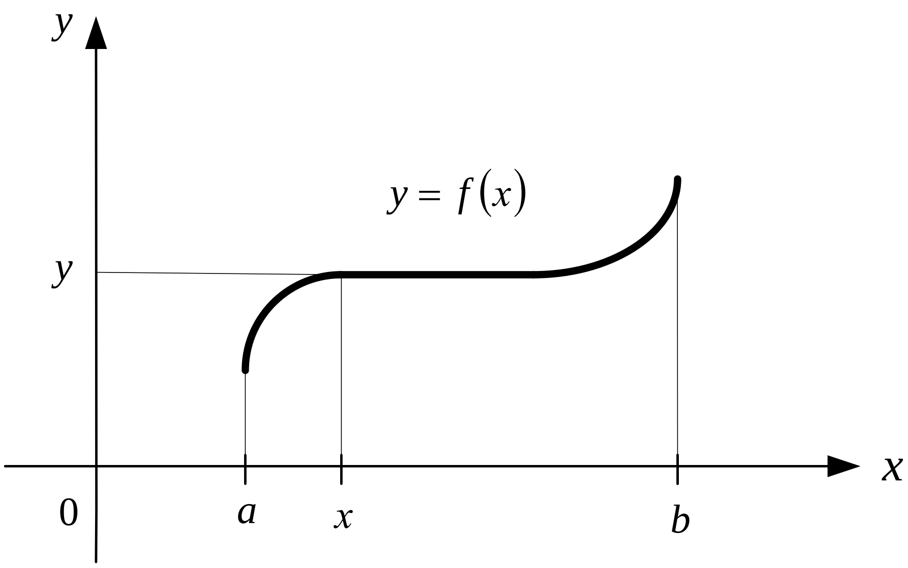


Fig. 7. The graph of a function $y = f(x)$

Usually the graph of a function is a curve.

For plotting the graph of the analytically given function $y = f(x)$ first of all it is necessary to compile the table of the values x and y and then to construct the system of points on a plane considering x as an abscissa and y as an ordinate. Joining these points with a line we obtain an approximate graphic picture of the function.

So in Fig. 8 the graph of the function $y = x^3$ is plotted in which values at some points are represented in table.

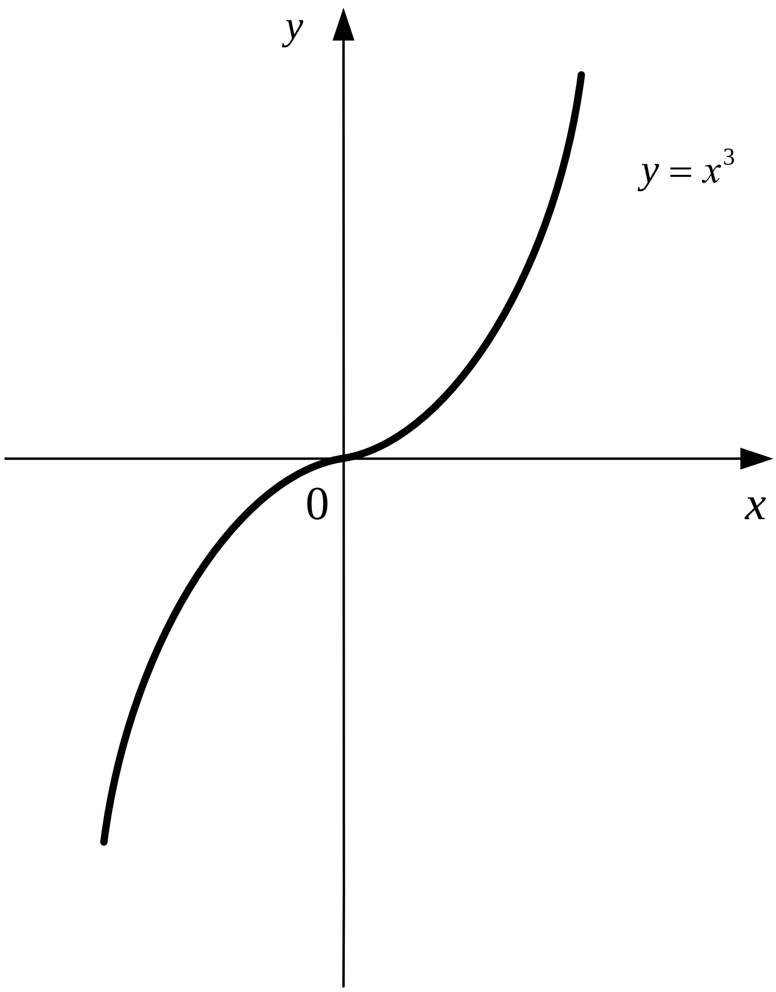


Fig. 8. The graph of the function $y = x^3$

Expanding our information of analytically represented functions we introduce the additional notions such as: composite, implicit and inverse functions.

7. Composite Function

Let a function $u=\varphi(x)$ of the argument x be defined on a set D being the range of that function u.

Furthermore, let another function $y=f(u)$ be defined on the set G. Then to each value x belonging to the set D there corresponds a definite value of u belonging to the set G, and to u, in its turn there corresponds a definite value of y. Therefore, ultimately, to each of x from the set D there corresponds a definite value of y and hence y is a function of x. Denoting this new function by $y=F(x)$ we can write down the expression of the function $F(x)$ in terms of the functions f and φ:

$$y=F(x)=f(\varphi(x)).$$

It is said that $F(x)$ as a function of x is a composite function ("a function of a function") formed of the function f and φ. The function $u=\varphi(x)$ entering the expression $f(\varphi(x))$ is referred to as the intermediate variable.

8. Implicit Function

Up to now we have confined ourselves to those functions specified analytically for which the left term of the equality defining the function is the variable y alone while the right term is an expression only involving x. We will call such functions explicit.

But a more general equation connecting two variables and not resolved in one of them can also specify one variable as a function of the others. For example, the equation of the circle

$$x^2+y^2=1$$

represents not one but two functions $y=\sqrt{1-x^2}$ and $y=-\sqrt{1-x^2}$ each of which is defined on the interval $[-1;1]$. This equation defines y as a multiple-valued (two-valued) function of x.

In this approach we can say that an implicit function y (which can be multiple-valued) of an independent variable x is a function in which values are found from an equation connecting x and y not solved in y.

If we take the equation

$$y = \frac{1}{2}\sin y + x,$$

which also determines y as a function of x we see that it cannot be solved algebraically, and y cannot be expressed explicitly in terms of x. Equations of this type can be solved numerically.

At last we remark that general notation of the implicit function connecting two variables x and y has a form $F(x, y) = 0$.

9. Inverse Function

One of the methods of implicit representation of a function is representing the function $y = f(x)$ by means of the relationship $x = \varphi(y)$.

Let for all y from the domain of definition of the function $\varphi(y)$ the following condition be met: if $y_1 \neq y_2$ then $x_1 \neq x_2$ where $x_1 = \varphi(y_1)$ and $x_2 = \varphi(y_2)$. Then we can assign a definite value of y to every value of x, using the relationship $x = \varphi(y)$, that is, the relationship $x = \varphi(y)$ specifies some function $y = f(x)$.

In this case, the function $f(x)$ is called the inverse of the function $\varphi(y)$. For example, the function $y = x^n$ has the inverse function $x = \sqrt[n]{y}$, where $x \geq 0$ and n is a natural number ($n \geq 1$). Then for this function replacing x upon y we obtain the function $y = \sqrt[n]{x}$. The graphs of the functions $y = x^n$ and $y = \sqrt[n]{x}$ are represented in Fig. 9.

Here we must remark that graphs of mutually inverse functions are symmetric about the straight line $y = x$.

10. Classification of one Argument Functions

A function of the form

$$P_m(x) = a_0x^m + a_1x^{m-1} + \ldots + a_m,$$

where $m \geq 0$ is an integer and coefficients $a_0, a_1, \ldots, a_m$ are constants $(a_0 \neq 0)$, is called entire rational function or polynomial of the m^{th}-degree.

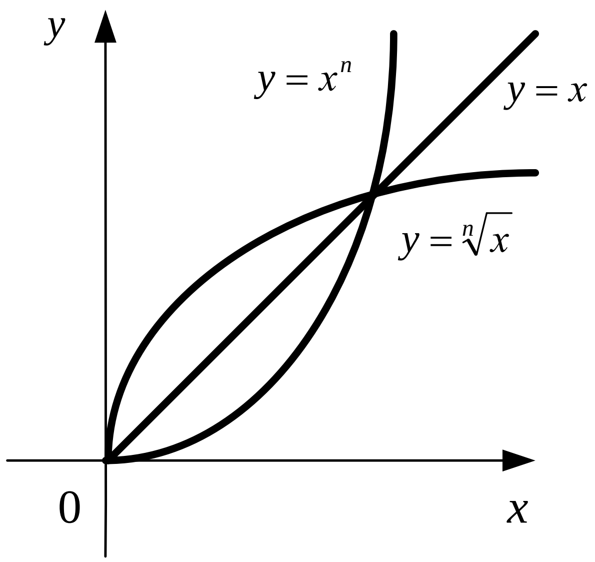


Fig. 9. The graphs of the functions $y = x^n$**,** $y = \sqrt[n]{x}$ **and** $y = x$

A function represented as a quotient of two polynomials

$$R(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_0x^n + a_1x^{n-1} + \ldots + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_m}$$

is also *a rational function*.

The domain of definition of a rational function is the whole axis of abscissas with the exception of the points (the roots of the denominator) in which the denominator of the fraction vanishes.

A wide class of functions is obtained if, besides the arithmetical operations, the extraction of roots is allowed.

For instance, the function

$$f(x)=\sqrt{R(x)}$$

where $R(x)$ is a rational function is called the irrational function.

The function of the form

$$f(x)=\sqrt[3]{\frac{\sqrt{5x^2+1}+\sqrt[5]{2x^2+3}}{x^4+1}}$$

can serve the example of the irrational function.

The collection of entire rational, rational and irrational functions form the class of explicit algebraic functions.

In general case an algebraic function is a function $y=y(x)$ specified by an algebraic equation

$$P_0(x)y^n+P_1(x)y^{n-1}+\ldots+P_n(x)=0$$

where $P_j(x)$ $(j=0,1,\ldots,n)$ are polynomials and $P_0(x)\neq 0$.

Every function belonging not to the class of algebraic function is called *a transcendental function*.

All logarithmic, trigonometric and inverse trigonometric functions belong to the class of elementary transcendental functions. For example, such functions as $a^x, \log_a x, \cos x, \operatorname{arctg} x$ are transcendental functions.

Algebraic, elementary transcendental functions and their finite combinations are called *elementary functions*.

11. Sequence. Limit of the Sequence

Let's define the sequence.

Definition 8. Let every positive $n\left(n=1,2,\ldots\right)$ by some rule f be associated with some real number $x_n = f\left(n\right)$ then it is said *the sequence* to be defined.

Briefly it can be written $\left\{x_n\right\}=\left\{x_1,x_2,\ldots\right\}$. Separate numbers x_n of the sequence $\left\{x_n\right\}$ are called its *elements*.

Elements of arithmetic and geometric progressions where every next element a_n can be represented in term of a preceding element a_{n-1} by means of recurrent relationships can serve well-known examples of a sequence.

So for *arithmetic progression* these relationships have the form:

$$a_n=a_{n-1}+d, \quad n=1,2,\ldots$$

where a number d is *a difference* and for *geometric progression*

$$a_n=a_{n-1}\cdot q, \quad n=1,2,\ldots$$

where a number q is called *denominator*.

Now we define the limit of the sequence.

Definition 9. A number a is called the limit of sequence $\left\{x_n\right\}$ if for any $\varepsilon>0$ there can be found a number $n>N\left(\varepsilon\right)$ such that the inequality $\left|x_n-a\right|<\varepsilon$ holds for all $n>N\left(\varepsilon\right)$. Briefly it can be written

$$\lim_{n\to\infty} x_n=a,$$

i.e. the sequence has a limit a.

Let's consider the examples which would illustrate the foregoing words.

Example 1. It is required to show that $\lim\limits_{n\to\infty} x_n=\dfrac{3}{7}$ if $x_n=\dfrac{3n+5}{7n}$.

Solution. Let's attempt for any $\varepsilon>0$ to find such a natural $N\left(\varepsilon\right)$ that for any $n>N_\varepsilon$ the inequality

$$\left|\frac{3n+5}{7n}-\frac{3}{7}\right|<\varepsilon$$

would be fulfilled. Reducing the expression of module to a common denominator we get

$$\left|\frac{5}{7n}\right|<\varepsilon$$

whence $n>\dfrac{7}{5\varepsilon}$.

Let's take

$$N(\varepsilon)=\left[\frac{5}{7\varepsilon}\right]$$

where by square brackets the entire part of $\dfrac{5}{7\varepsilon}$ is denoted then it follows by definition 9 for $n>\left[\dfrac{5}{7\varepsilon}\right]$ the inequality

$$\left|\frac{3n+5}{7n}-\frac{3}{7}\right|<\varepsilon$$

is fulfilled, i.e.

$$\lim_{n\to\infty}\frac{3n+5}{7n}=\frac{3}{7}.$$

Example 2. Prove that the limit $\lim\limits_{n\to\infty}3^{\sqrt[3]{n}}=\infty$.

Solution. For a number $M>0$ taking a logarithm of the inequality $3^{\sqrt[3]{n}}>M$ we obtain

$$\sqrt[3]{n}>\log_3 M\ ,$$

where from $n > \left(\log_3 M\right)^3$.

Then if $0 < M \le N$, where N is any natural number we may choose $M > 1$. Thus, it is necessary to take $N = (\log_3 M)^3$.

Consequently, for $n > N = [(\log_3 M)^3]$ the condition $3^{\sqrt[3]{n}} > M$ is fulfilled. Since $M > 0$ is an arbitrary number, then $\lim\limits_{n\to\infty} 3^{\sqrt[3]{n}} = \infty$.

12. Limit of a Function

Passing to one of the most important topics of mathematical analysis which differential equations are preliminary it is necessary to meet with the notion of a function limit and to master a technique of evaluating limit of function.

Definition 10. A number A is said to be the limit of the function $y = f\left(x\right)$ as $x \to a$ if $f\left(x\right)$ being defined in any neighborhood U_a of this point a, excluding may be this point and for any $\varepsilon > 0$ there is a number $\delta > 0$ dependent on $\varepsilon\ \left(\delta = \delta(\varepsilon)\right)$ such that for all x satisfying the condition $0 < |x - a| < \delta$ the inequality $\left|f\left(x\right) - A\right| < \varepsilon$ is fulfilled.

Usually this assertion is written in the following way:

$$\lim_{x\to a} f\left(x\right) = A.$$

Now we give the definition of the function limit as x tends to infinity.

Let's suppose that the function $f\left(x\right)$ is defined on the whole numerical axis or for all x module of which is greater than some positive number M.

Definition 11. A number A is said to be the limit of the function $y = f\left(x\right)$ as $x \to +\infty$ if for all sufficiently large values of x the corresponding values of the function $f\left(x\right)$ become arbitrary close to the number A. If *A* is the limit of the function $f\left(x\right)$ as $x \to +\infty$ we write as

$$\lim_{x\to+\infty} f\left(x\right) = A.$$

In a similar way we can determine the limit of the function as $x \to -\infty$. Let's consider the example.

Example 3. It is required to prove that $\lim\limits_{x \to +\infty} \frac{3x-5}{x} = 3$.

Solution. Here $f(x) = \frac{3x-5}{x}$. Let ε be 0,01 and we consider any difference

$$|f(x)-3| = \left|\frac{3x-5}{x} - 3\right| = \left|-\frac{5}{x}\right| = \frac{5}{|x|}.$$

Thus, $\frac{5}{|x|} < \varepsilon \Rightarrow |x| > \frac{5}{\varepsilon}$. Since $x \to +\infty$ then we can consider x as a positive value. Therefore, we can write down that $x > \frac{5}{\varepsilon}$.

Whence we see that sufficiently large number M is equal to $M = \frac{5}{\varepsilon} = \frac{5}{0{,}01} = 500$.

So if $x > 500$ then $\left|\frac{3x-5}{x} - 3\right| < \varepsilon = 0{,}01$. It means that $\lim\limits_{x \to +\infty} \frac{3x-5}{x} = 3$.

The notion of one-sided limits are also useful for studying functions of one variable. But preliminarily we introduce the notion of right (left)–hand side neighborhood of the point a (a is a number).

Definition 12. Any interval $U_a^- = (a-\delta; a)$ in which right side edge is a point a is called a left-hand side neighborhood U_a^- of the point a.

The notation $x \to a-0$ means that x assumes (appropriates) all values belonging to some neighborhood of the point a, i.e. $x \to a$ remaining all the time less than $a\,(x<a)$. Analogously the notation $x \to a+0\,(x>a)$ is defined.

And now we define the right (left)-hand side limit of the function $f(x)$.

Definition 13. The number A is said to be the right (left)-hand side limit of the function $f(x)$ as $x \to a$ if for any $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for all $x \in U_a^+ \left(x \in U_a^-\right)$ that is satisfying the inequality $0 < x - a < \delta \; (-\delta < x - a < 0)$.

We must remark that the function $f(x)$ has a limit at the point a if and only if

$$\lim_{x \to a-0} f(x) = \lim_{x \to a+0} f(x) = A.$$

13. Infinitesimals and Infinitely Large Functions or Magnitudes

An important element of limit theory is the notion of the limit of an infinitesimal and an infinity large function or magnitude.

We give their definitions.

Definition 14. A function $\alpha(x)$ tending to zero as $x \to a$ is called an infinitesimal, as $x \to a$. According to what was indicated this means that, if there is given any $\varepsilon > 0$ (however small), there is $\delta > 0$ such that the conditions $x \neq a$ and $|x - a| < \delta$ imply the inequality $|\alpha(x)| < \varepsilon$. It is equivalent to the notation

$$\lim_{x \to a} \alpha(x) = 0,$$

that is the limit of an infinitesimal.

Analogously, an infinitesimal is defined as $x \to 0 - 0$ and $x \to 0 + 0$ as well as $x \to -\infty$ or $x \to +\infty$.

If $\lim\limits_{x \to +\infty} f(x) = A$ it means definition of the function limit that the difference

$$f(x) - A = \alpha(x) \quad \text{or} \quad f(x) = A + \alpha(x),$$

where $\alpha(x) \to 0$, as $x \to a$.

Definition 15. A function $y = f(x)$ is said to be *an infinitely large* *magnitude* (or is said to become infinite or to approach infinity) as $x \to a$ if for all the values x lying sufficiently close to a the module of the corresponding values of the function $f(x)$ become greater than any given arbitrary large positive number.

If the function $f(x)$ approaches infinity as $x \to a$ we write

$$\lim_{x \to +\infty} f(x) = +\infty .$$

It is important to remark that if $f(x) \to \infty$ as $x \to a$ then $\frac{1}{f(x)} \to 0$, as $x \to a$ and if $\alpha(x) \to 0$, as $x \to a$ ($\alpha(x) \neq 0$ for $x \neq a$), then $\frac{1}{\alpha(x)} \to \infty$ as $x \to a$. It means that an infinitesimal and an infinitely large magnitude are mutually inverse functions.

14. Basic Theorems about Infinitesimals

In order to evaluate limits it is necessary to know basic theorems concerning infinitesimals. These theorems are formulated in the following way:

1. An algebraic sum of two, three and, generally, of a finite number of infinitesimals is again an infinitesimal. It means, if $\alpha(x), \beta(x), \gamma(x)$ are infinitesimals as $x \to a$, then

$$\lim_{x \to a} (\alpha(x) + \beta(x) - \gamma(x)) = 0 .$$

2. A product of a constant number as well as a bounded function by an infinitesimals as $x \to a$ is an infinitesimal. It means if $f(x) = C$ and $\alpha(x)$ are infinitesimals as $x \to a$, then $\lim\limits_{x \to a} C\alpha(x) = 0$ and if $|f(x)| < M$ ($f(x)$ is a bounded function, where a number $M > 0$) as $x \in U_a$, where U_a is some neighborhood of the point a and $\alpha(x) \to 0$ as $x \to a$, then

$$\lim_{x \to a} f(x) \cdot \alpha(x) = 0.$$

3. A product of finite infinitesimals as $x \to a$ is an infinitesimal. It is written down

$$\lim_{x \to a} \alpha(x) \cdot \beta(x) = 0.$$

where $\alpha(x), \beta(x)$ are infinitesimals as $x \to a$.

And now we write the corollary following from this theorem.

Corollary. If n is a positive integer power of the infinitesimal $(\alpha(x))^n$, as $x \to a$, then $\lim_{x \to a} (\alpha(x))^n = 0$, where $\alpha(x) \to 0$ as $x \to a$.

Regarding the limit of the quotient of infinitesimals we remark this limit is a so-called indeterminate form $\left\{\frac{0}{0}\right\}$ and requires an additional transformation or a special technique. We will speak about such limits later.

15. Comparison of Infinitesimals

There are following definitions for comparison of two infinitesimals.

Definition 16. Two infinitesimals are said to be one and the same order if the limit of their quotient is equal to a finite non-zero number as $x \to a$, i.e.

$$\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = k \neq 0.$$

Definition 17. If $\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = 0$, then $\alpha(x)$ is termed an infinitesimal quantity of higher order relative to $\beta(x)$, and $\beta(x)$ is a quantity of lower order with respect to $\alpha(x)$.

Definition 18. If $\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = k$ (finite and not equal to zero), then $\alpha(x)$

is called an infinitesimal of the n^{th} order with respect to $\beta(x)$.

Definition 19. If $\lim\limits_{x \to a} \dfrac{\alpha(x)}{\beta(x)} = 1$, then $\alpha(x)$ and $\beta(x)$ are termed the equivalent infinitesimals. The equivalence of infinitesimals is denoted by the same symbol as approximate equality $(\approx)$.

Thus $\alpha(x) \approx \beta(x)$.

The equivalence of infinitesimals is frequently used on evaluating function limits because the infinitesimal can substitute the equivalent function.

For instance, it may be shown the functions $\sin\alpha(x)$ and $\alpha(x)$ are the equivalent infinitesimals, as $x \to 0$, i.e.

$$\lim_{\alpha(x) \to 0} \frac{\sin\alpha(x)}{\alpha(x)} = 1.$$

This is a so-called first remarkable limit which is necessary to remember. The following limits are analogous to the first remarkable limit

$$\lim_{\alpha(x) \to 0} \frac{\mathrm{tg}\,\alpha(x)}{\alpha(x)} = 1;$$

$$\lim_{\alpha(x) \to 0} \frac{\arcsin\alpha(x)}{\alpha(x)} = 1; \qquad \lim_{\alpha(x) \to 0} \frac{\mathrm{arctg}\,\alpha(x)}{\alpha(x)} = 1.$$

16. Basic Theorems about Limits

In order to master the technique of evaluating limits it is necessary to study the following theorems about limits:

Theorem 1. Let $\lim\limits_{x \to a} f(x) = A$ and $\lim\limits_{x \to a} \varphi(x) = B$ exist for the functions $f(x)$ and $\varphi(x)$ defined for the same values of x in some neighborhood of the point a. Then

$$\lim_{x \to a} (C_1 f(x) + C_2 \varphi(x)) = C_1 \lim_{x \to a} f(x) + C_2 \lim_{x \to a} \varphi(x) = AC_1 + BC_2,$$

$$\lim_{x\to a} f(x)\varphi(x) = \lim_{x\to a} f(x) \lim_{x\to a} \varphi(x) = AB,$$

$$\lim_{x\to a} f(x)/\varphi(x) = \lim_{x\to a} f(x)/\lim_{x\to a} \varphi(x) = A/B, \text{ for } \lim_{x\to a} \varphi(x) =$$
$$= B \neq 0,$$

where C_1 and C_2 are constants.

In particular, setting $f(x)=\varphi(x)$, $C_1 = C$, $C_2 = 0$, we have

$$\lim_{x\to a} (f(x))^2 = (\lim_{x\to a} f(x))^2 = A^2, \ \lim_{x\to a} Cf(x) = C\lim_{x\to a} f(x) = CA.$$

Theorem 2. If a function $f(x)$ has a finite limit A as $x \to a$ then the limit of any rational degree p of $f(x)$ is equal to the same degree of the function limit, i.e.

$$\lim_{x\to a} (f(x))^p = (\lim_{x\to a} f(x))^p = A^p.$$

Besides the first remarkable the second remarkable limit can often be used for solving examples of limits.

This limit has a form:

$$\lim_{\alpha(x)\to\infty} \left(1 + \frac{1}{\alpha(x)}\right)^{\alpha(x)} = e \text{ or } \lim_{\alpha(x)\to 0} (1+\alpha(x))^{\frac{1}{\alpha(x)}} = e,$$

where $e = 2{,}7182\ldots.$

Remark: for evaluating limits containing logarithms it is necessary to know that if $\lim_{x\to a} f(x)$ exists and it is positive, then

$$\lim_{x\to a} \ln f(x) = \ln \lim_{x\to a} f(x) \quad \text{and if} \quad \lim_{x\to a} \ln f(x) = \ln \lim_{x\to a} f(x) = A,$$

(A is a number) then $\lim_{x\to a} f(x) = e^A$. It follows from properties of continuous functions which will be considered in sec. 18.

Let us consider two examples using this property.

Example 4. Compute $\lim\limits_{x\to 0}\left(1+\mathrm{tg}x\right)^{\frac{3}{x}}$.

Solution. Let

$$y=\left(1+tgx\right)^{\frac{3}{x}}, \text{ then } \ln y=\ln\left(1+tgx\right)^{\frac{3}{x}}=\frac{3}{x}\ln\left(1+tgx\right).$$

Passing to the limit we obtain

$$\lim_{x\to 0}\frac{3}{x}\ln\left(1+\mathrm{tg}x\right)=3\lim_{x\to 0}\frac{\ln\left(1+\mathrm{tg}x\right)}{x}=3\lim_{x\to 0}\frac{\ln\left(1+\mathrm{tg}x\right)}{\mathrm{tg}x}\lim_{x\to 0}\frac{\mathrm{tg}x}{x}=3, \text{ so}$$

$$\lim_{x\to 0}\left(1+\mathrm{tg}x\right)^{\frac{3}{x}}=e^{3}.$$

Example 5. Find $\lim\limits_{x\to 0}\frac{e^{x}-1}{x}$.

Solution. Let $y=e^{x}-1\to e^{x}=y+1\to x=\ln(y+1)$. It is evident that $y\to 0$ if $x\to 0$ thus $\lim\limits_{x\to 0}\frac{e^{x}-1}{x}=\lim\limits_{y\to 0}\frac{y}{\ln(1+y)}=1$.

17. Examples of Evaluating Limits of functions

Let us consider examples of few types of evaluating limits using the theorems about the limits of functions and infinitesimals as well as the first and second remarkable limits.

We'll show some nondifficult methods in technique of computing of certain kinds of limits.

Example 6. Find $\lim\limits_{x\to\infty}\frac{(2x+3)^{3}(3x-2)^{2}}{x^{5}+5}$.

Solution. Here the entire rational polynomials of the fifth degree are found in the numerator and denominator. Seeking the limit of the ratio of two entire polynomials to x, as $x\to\infty$, the both members of the ratio preliminarily it is useful to divide by x^n where n is the highest polynomial degree. In this case $n=5$. Then we get

$$\lim_{x\to\infty} \frac{(2+3/x)^3(3-2/x)^2}{1+5/x^5} = \frac{\lim\limits_{x\to\infty}(2+3/x)^3 \lim\limits_{x\to\infty}(3-2/x)^2}{\lim\limits_{x\to\infty}(1+5/x^5)} =$$

$$= \frac{(\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty}(3/x))^3 (\lim\limits_{x\to\infty} 3 - \lim\limits_{x\to\infty}(2/x))^2}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty}(5/x^5)}.$$

The limit of constant is equal to this constant. That is

$$\lim_{x\to\infty} 2 = 2, \lim_{x\to\infty} 3 = 3, \lim_{x\to\infty} 1 = 1,$$

and fractions $\frac{3}{x}$, $\frac{2}{x}$, $\frac{5}{x^5}$ are infinitesimals, as $x\to\infty$ thus

$$\lim_{x\to\infty} 2/x = \lim_{x\to\infty} 3/x = \lim_{x\to\infty} 1/x = 0.$$

Hence our limit is equal to

$$\lim_{x\to\infty} \frac{(2x+3)^3(3x-2)^2}{x^5+5} = \frac{2^3 3^2}{1^5} = 72.$$

Example 7. Find $\lim\limits_{x\to\infty} \dfrac{7x^4+3x^2-12}{2x^4-6}$.

Solution. Acting in the same manner let us divide the numerator and denominator by x^4:

$$\lim_{x\to\infty} \frac{7+3/x^2-12/x^4}{2-6/x^4} = \frac{\lim\limits_{x\to\infty} 7 + \lim\limits_{x\to\infty} 3/x^2 - \lim\limits_{x\to\infty} 12/x^4}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} 6/x^4\,)},$$

since $\lim\limits_{x\to\infty} 7=7$, $\lim\limits_{x\to\infty} 2=2$, $\lim\limits_{x\to\infty} 3/x^2=\lim\limits_{x\to\infty} 12/x^4=\lim\limits_{x\to\infty} 6/x^4=0$, then the sought for limit will be equal to

$$\lim_{x\to\infty}\frac{7x^4+3x^2-12}{2x^4-6}=\frac{7}{2}.$$

Grounding in the foregoing examples we can do the following conclusion: if there are polynomials of the same degree in numerator and denominator the limit of their quotient as $x\to\infty$ will be equal to the ratio of coefficients of the highest degree of x.

The analogous manner we can use in many cases on fractions containing irrationalities,

Example 8. Find $\lim\limits_{x\to\infty}\dfrac{2x^2-3x-4}{\sqrt{x^4+1}}$.

Solution. Here there is irrationality in the denominator. Then we divide the nominator and denominator by x^2:

$$\lim_{x\to\infty}\frac{2-3/x-4/x^2}{\sqrt{1+1/x^4}}=\frac{\lim\limits_{x\to\infty}2-\lim\limits_{x\to\infty}3/x-\lim\limits_{x\to\infty}4/x^2}{\sqrt{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}1/x^4}}=2.$$

In order to show the methods of finding limits of the quotient of two functions as $x\to a$ where a is some number we consider the following examples.

Let there be given two entire rational polynomials $P(x)$ in the numerator and $Q(x)$ in the denominator and let $P(x)\neq 0$ and $Q(x)\neq 0$ then the limit of the fraction $\lim\limits_{x\to a}P(x)/Q(x)$ can be found by direct substitution $x=a$.

Example 9. Find $\lim\limits_{x\to 2}\dfrac{2x^3+1}{x^2-3}$.

Solution. Let's substitute the limiting value $x=2$ and get:

$$\lim_{x\to 2}\frac{2x^3+1}{x^2-3}=\frac{2\cdot 2^3+1}{2^2-3}=17.$$

The indeterminate form we have when $P(a)=Q(a)=0$ then it is necessary to divide the denominator of the fraction one or several times by $(x-a)$, because in this case the number a will be the root of these polynomials.

Example 10. Find $\lim\limits_{x\to 1}\frac{x^2-3x+2}{x^2-4x+3}$.

Solution. Substitution $x=1$ yields zero in the numerator and denominator. Then we are obtain the indeterminate form of the kind $\left\{\frac{0}{0}\right\}$. Dividing the numerator and denominator by $(x-1)$ we get

$$\lim_{x\to 1}\frac{x^2-3x+2}{x^2-4x+3}=\lim_{x\to 1}\frac{(x-1)(x-2)}{(x-1)(x-3)}=\frac{1-2}{1-3}=\frac{1}{2}.$$

Example 11. Find $\lim\limits_{x\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right)$.

Solution. Substituting $x=1$ we get the indeterminate form $\{\infty-\infty\}$. Reducing the difference in parenthesis to the common denominator by $(1-x)$, we obtain

$$\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right)=\lim_{x\to 1}\frac{1+x-2}{1-x^2}=-\lim_{x\to 1}\frac{1-x}{(1-x)(1+x)}=-\frac{1}{2}.$$

We can reduce the expression containing irrational form either by change of variables or by transposition the irrationality from the numerator into the denominator or otherwise from the denominator into the numerator simultaneously multiplying and dividing irrationalities by the conjugate expression.

Let us solve the following examples using both these methods.

Example 12. Find $\lim\limits_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$.

Solution. The numerator and the denominator of this limit contain irrationalities. Let us bring in the fraction the variable $y^{12} = x$. We may assume the power of this new variable is equal to the least common multiple of the radical indices 3 and 4. The least common multiple equals 12 and if $x \to 1$ then $y \to 1$ as well.

Thus

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1} = \lim_{y\to 1}\frac{\sqrt[3]{y^{12}}-1}{\sqrt[4]{y^{12}}-1} = \lim_{y\to 1}\frac{y^4-1}{y^3-1} = \lim_{y\to 1}\frac{(y-1)(y+1)(y^2+1)}{(y-1)(y^2+y+1)} =$$

$$= \frac{2\cdot 2}{1+1+1} = \frac{4}{3}.$$

Example 13. Find $\lim\limits_{x\to 7}\dfrac{2-\sqrt{x-3}}{x^2-49}$.

Solution. Multiplying the irrationality in the numerator by the conjugate expression $(2+\sqrt{x-3})$ and writing down this expression in the denominator we obtain

$$\lim_{x\to 7}\frac{(2-\sqrt{x-3})(2+\sqrt{x-3})}{(x^2-49)(2+\sqrt{x-3})} = -\lim_{x\to 7}\frac{(x-7)}{(x-7)(x+7)(2+\sqrt{x-3})} =$$

$$= -\lim_{x\to 7}\frac{1}{(x+7)(2+\sqrt{x-3})} = -\frac{1}{56}.$$

Example 14. Find $\lim\limits_{x\to 4}\dfrac{3-\sqrt{x+5}}{1-\sqrt{5-x}}$.

Solution. In order to find this limit we also apply the foregoing method then

$$\lim_{x\to 4}\frac{3-\sqrt{x+5}}{1-\sqrt{5-x}} = \lim_{x\to 4}\frac{(3-\sqrt{x+5})(3+\sqrt{x+5})(1+\sqrt{5-x})}{(1-\sqrt{5-x})(3+\sqrt{x+5})(1+\sqrt{5-x})} =$$

$$= -\lim_{x\to 4}\frac{(x-4)(1+\sqrt{5-x})}{(x-4)(3+\sqrt{x+5})} = -\frac{1+\sqrt{5-4}}{3+\sqrt{5+4}} = -\frac{1}{3}.$$

Example 15. Find $\lim\limits_{x\to 4}\ln\dfrac{x-4}{\sqrt{x+4}-\sqrt{8}}$.

Solution. Preliminary let us recall the remark of the preceding section then this limit is equal to

$$\lim_{x\to 4}\ln\frac{x-4}{\sqrt{x+4}-\sqrt{8}}=\ln\lim_{x\to 4}\frac{x-4}{\sqrt{x+4}-\sqrt{8}}=\ln\lim_{x\to 4}\frac{(x-4)(\sqrt{x+4}+\sqrt{8})}{(\sqrt{x+4}-\sqrt{8})(\sqrt{x+4}+\sqrt{8})}=$$

$$=\ln\lim_{x\to 4}\frac{(x-4)(\sqrt{x+4}+\sqrt{8})}{(x+4-8)}=\ln\lim_{x\to 4}(\sqrt{x+4}+\sqrt{8})=\ln 2\sqrt{8}=\ln 4\sqrt{2}.$$

The first remarkable limit ($\lim\limits_{\alpha(x)\to 0}\dfrac{\sin\alpha(x)}{\alpha(x)}=1$) is often used as well as the various trigonometric formulas of transformation for evaluating limits containing trigonometric functions.

We suppose that $\lim\limits_{x\to a}\sin x=\sin a$, $\lim\limits_{x\to a}\cos x=\cos a$ are known. Sometimes it is useful to apply the property of equivalence of infinitesimals. Changing an infinitesimal with their equivalent functions.

Let us show that the function $(1-\cos x)$ is equivalent to $\dfrac{x^2}{2}$ as $x\to 0$. It needs to show that the limit of their ratio equals to unity. We consider

$$\lim_{x\to 0}\frac{1-\cos x}{x^2/2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x\cdot x/2}=\lim_{x\to 0}\frac{\sin\frac{x}{2}\sin\frac{x}{2}}{x/2\cdot x/2}=\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x/2}=\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x/2}=1.$$

So $1-\cos x\approx\dfrac{x^2}{2}$.

Example 16. Find $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$.

Solution. We have $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}=\lim\limits_{x\to 0}\dfrac{3x}{x}=\lim\limits_{x\to 0}3=3.$

Example 17. Find $\lim\limits_{x\to 0}\dfrac{\mathrm{tg}4x}{\mathrm{tg}5x}$.

Solution. We get $\lim\limits_{x\to 0}\dfrac{\mathrm{tg}4x}{\mathrm{tg}5x}=\lim\limits_{x\to 0}\dfrac{4x}{5x}=\lim\limits_{x\to 0}\dfrac{4}{5}=\dfrac{4}{5}$.

In a similar way we can show that $\mathrm{arctg}x \approx x$.

Example 18. Find $\lim\limits_{x\to 0}\dfrac{\mathrm{arctg}x}{x}$.

Solution. We change $y=\mathrm{arctg}x$ then $x=\mathrm{tg}y$ and now as $x\to 0$ then $y\to 0$.

$$\lim_{x\to 0}\frac{\mathrm{arctg}x}{x}=\lim_{y\to 0}\frac{y}{\mathrm{tg}y}=1.$$

Example 19. Find $\lim\limits_{x\to 0}\dfrac{\mathrm{arctg}2x}{\sin 3x}$.

Solution. Using the equivalence to $\mathrm{arctg}\,2x$ and $\sin 3x$ functions $2x$ and $3x$, we get

$$\lim_{x\to 0}\frac{\mathrm{arctg}2x}{\sin 3x}=\lim_{x\to 0}\frac{2x}{3x}=\lim_{x\to 0}\frac{2}{3}=\frac{2}{3}.$$

Example 20. Find $\lim\limits_{x\to \pi}\dfrac{1-\sin x/2}{\pi-x}$.

Solution. We substitute $\sin\dfrac{x}{2}=\cos\left(\dfrac{\pi}{2}-\dfrac{x}{2}\right)$ then

$$\lim_{x\to\pi}\frac{1-\sin x/2}{\pi-x}=\lim_{x\to\pi}\frac{1-\cos(\pi/2-x/2)}{\pi-x}=\lim_{x\to\pi}\frac{2\sin^2(\pi/4-x/4)}{4(\pi/4-x/4)}=$$

$$=\frac{1}{2}\lim_{x\to\pi}\frac{\sin(\pi/4-x/4)}{(\pi/4-x/4)}\cdot\lim_{x\to\pi}\sin(\pi/4-x/4)=\frac{1}{2}\cdot 1\cdot 0=0,$$

here $\lim\limits_{x\to\pi}\dfrac{\sin(\pi/4-x/4)}{(\pi/4-x/4)}=1,$ since $\dfrac{\pi}{4}-\dfrac{x}{4}\to 0$ and $\sin\left(\dfrac{\pi}{4}-\dfrac{x}{4}\right)\to 0$ as $x\to\pi$.

Remark. When we deal with limits of the kind $\lim\limits_{x\to a}\varphi(x)^{\psi(x)}$ it is necessary to take into account that:

a) in the case if there are finite limits $\lim\limits_{x\to a}\varphi(x)=A$ and $\lim\limits_{x\to a}\psi(x)=B$ then $C=A^B$;

b) if $\lim\limits_{x\to a}\varphi(x)=A\neq 1$ and $\lim\limits_{x\to a}\psi(x)\neq\infty$ then $C=\lim\limits_{x\to a}A^{\psi(x)}=A^{\lim\limits_{x\to a}\psi(x)}$.

Consequently this example as well as the preceding example we can solve directly.

c) if $\lim\limits_{x\to a}\varphi(x)=1$ and $\lim\limits_{x\to a}\psi(x)=\infty$ then we suppose $\varphi(x)=1+\alpha(x)$ (where $\alpha(x)$ is infinitesimal and $\alpha(x)=\varphi(x)-1$) as $x\to a$ then

$$C=\lim_{x\to a}\left\{[1+\alpha(x)]^{1/\alpha(x)}\right\}^{\alpha(x)\cdot\psi(x)}$$

thus $\lim\limits_{x\to a}[1+\alpha(x)]^{1/\alpha(x)}=e$ that is the second remarkable limit and $C=e^{\lim\limits_{x\to a}\alpha(x)\psi(x)}$.

Example 21. Find $\lim\limits_{x\to 0}\left(\dfrac{2+x}{5-x}\right)^x$.

Solution. Since $\lim\limits_{x\to 0}\left(\dfrac{2+x}{5-x}\right)=\dfrac{2}{5}$ and $\lim\limits_{x\to 0}x=0$ then it is equal to

$$\lim_{x\to 0}\left(\frac{2+x}{5-x}\right)^x=(2/5)^0=1.$$

Example 22. Find $\lim\limits_{x\to\infty}\left(\dfrac{x^2-2}{2x^2+3}\right)^{x^2}$.

Solution. Here $\lim\limits_{x\to\infty}\left(\dfrac{x^2-2}{2x^2+3}\right)=\dfrac{1}{2}$ consequently

$$\lim_{x\to\infty}\left(\frac{x^2-2}{2x^2+3}\right)^{x^2}=\left(\frac{1}{2}\right)^{\lim\limits_{x\to\infty}x^2}=\frac{1}{2^{\lim\limits_{x\to\infty}x^2}}=0,$$

because $2^{\lim\limits_{x\to\infty} x^2}$ is the infinitely large magnitude.

Example 23. Find $\lim\limits_{x\to\infty}\left(1+\dfrac{k}{x}\right)^{mx}$.

Solution. Let us substitute $t=\dfrac{k}{x}$ then as $x\to\infty, t\to 0$ and $x=\dfrac{k}{t}$. Writing this substitution under the symbol of the limit we obtain

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{mx}=\lim_{t\to 0}(1+t)^{mk/t}=\lim_{t\to 0}\left[(1+t)^{1/t}\right]^{mk}=e^{mk}.$$

Since $\lim\limits_{t\to 0}(1+t)^{1/t}=e$ then the limit equals to e^{km}.

Example 24. Find $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^{x}$.

Solution. In order to find this limit we add and subtract unity in the numerator and use the transformation of the Remark ("c") then we have

$$\lim_{x\to\infty}\left(\frac{x+1-1}{1+x}\right)^{x}=\lim_{x\to\infty}\left(1+\frac{-1}{1+x}\right)^{x}=\lim_{x\to\infty}\left[\left(1+\frac{-1}{1+x}\right)^{-(x+1)}\right]^{-\frac{x}{x+1}}=$$

Applying the solution of the preceding example we get

$$\lim_{x\to\infty}\left(1+\frac{-1}{1+x}\right)^{-(x+1)}=\lim_{t\to 0}(1-t)^{-1/t}=e \quad \text{where } t=\frac{1}{x+1} \quad \text{then}$$

$$e^{\lim\limits_{x\to\infty}\left(-\frac{x}{1+x}\right)}=e^{-1}.$$

Example 25. Find $\lim\limits_{x\to\infty}\left(\dfrac{x-2}{x+2}\right)^{x+2}$.

Solution. Dividing the numerator and the denominator of this fraction by x and using the solution of the example 18 we obtain

$$\lim_{x\to\infty}\left(\frac{1-2/x}{1+5/x}\right)^x\left(\frac{1-2/x}{1+5/x}\right)^2=\frac{\lim\limits_{x\to\infty}(1-2/x)^x\lim\limits_{x\to\infty}(1-2/x)^2}{\lim\limits_{x\to\infty}\left(1+5/x\right)^x\lim\limits_{x\to\infty}\left(1+5/x\right)^2}=\frac{e^{-2}}{e^5}=e^{-7}.$$

18. Continuity of a Function

It had been said above that the notion of the continuity of functions is fundamental in mathematics that is why it needs particular attention.

Let a function $y=f(x)$ be defined at a point $x=x_0$ and some neighborhood with the centre at the point x_0. Let us denote $y_0=f(x_0)$ and let x_0 have an increment Δx. Then this function will have the new value equal to $y_0+\Delta y=f(x_0+\Delta x)$ and the increment of the function will be equal to $\Delta y=f(x_0+\Delta x)-f(x_0)$.

Definition 20. A function $y=f(x)$ is called *continuous at a point* $x=x_0$ if it is defined in some neighborhood x_0 and

$$\lim_{\Delta x\to 0}\Delta y=0.$$

It means that an infinitesimal increment of a function corresponds to an infinitesimal increment of argument. Sometimes it is convenient to use another definition.

Definition 21. A function $y=f(x)$ is said to be *continuous at a point* $x=x_0$ if it is defined in some neighborhood of the point x_0 and if the limit of the function as the independent variable x tending to x_0 exists and it is equal to the particular value of the function at the point

$$\lim_{x\to x_0} f(x)=f(x_0).$$

It follows from this definition that the limit of the continuous function to be found as $x\to x_0$ it is sufficiently to substitute the x_0 instead of x in the expression of the function. It means that the symbol of the function and the symbol of the limit may be rearranged.

We can perform the arithmetic operations on continuous functions. It is established by the following theorems:

Theorem 1. The algebraic sum of a finite number of functions continuous at the point x_0 is a continuous function at that point;

Theorem 2. The product of a finite number of functions continuous at the point x_0 is a continuous function at that point;

Theorem 3. The quotient of two functions continuous at the point x_0 is a continuous function at the point x_0 provided that the denominator does not turn into zero at that point;

Theorem 4. A function of a function composed of a finite number of continuous functions is a continuous function.

Remark. If the limit of a function $\varphi(x)$ as $x \to x_0$ or as $x \to \infty$ exists and it is equal to a finite number and the function is continuous at the point $x = x_0$ then

$$\lim_{x \to x_0} f(\varphi(x)) = f(\lim_{x \to x_0} \varphi(x)).$$

These theorems can be read in all mathematical text-books therefore we don't introduce their proofs.

All elementary functions are continuous in the domain of their definition.

Let us show this assertion by examples

$$y = \sin x; \; y + \Delta y = \sin(x + \Delta x);$$

$$\Delta y = \sin(x + \Delta x) - y \quad \text{or} \quad \Delta y = \sin(x + \Delta x) - \sin x;$$

$$\Delta y = 2\sin\frac{x + \Delta x - x}{2}\cos\frac{x + \Delta x + x}{2};$$

$$\Delta y = 2\sin\frac{\Delta x}{2}\cos\left(x + \frac{\Delta x}{2}\right);$$

but $\lim\limits_{\Delta x \to 0} \Delta y = \lim\limits_{\Delta x \to 0} 2\sin(\Delta x / 2)\cos(x + \Delta x / 2) = 0.$

Using the indicated above notion of the left-hand sided limit we can write

$$\lim_{x \to x_0 - 0} f(x) = f(x_0 - 0).$$

Analogously for the right-hand sided limit

$$\lim_{x \to x_0 + 0} f(x) = f(x_0 + 0).$$

Now we introduce the third definition of the continuity of a function which means the necessary and sufficient condition of the continuity at the point $x = x_0$, namely

$$f(x_0 - 0) = f(x_0) = f(x_0 + 0).$$

In the case if there is no the double equality it is said that the point x_0 is the point of the discontinuity of a function $y = f(x)$.

Let us present the classification of points of the discontinuity of a function by following definition.

Definition 22. A function $y = f(x)$ has a discontinuity as $x \to x_0$ if it is defined both from the left and from the right of x_0 but at the point x_0, at least one of the continuity conditions is not fulfilled.

We usually distinguish between two basic kinds of discontinuity.

a) discontinuity of the first kind.

Such a case occurs when there exist the limits on the right and of the left and they are finite, i.e. when the second condition of continuity is fulfilled and the rest of the condition for at least one of them is not fulfilled.

Example 26. It is required to determine the character of the point of discontinuity:

$$y = f(x) = 2 + 1 / \left(1 + 2^{\frac{1}{1-x}}\right).$$

Solution. The point $x = 1$ is the point of discontinuity since $1 - x = 0$.

Let us evaluate the left-hand sided limit of the function at that point

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0}\left(2+1/\left(1+2^{\frac{1}{1-x}}\right)\right).$$

Since $x \to 1-0$, i.e. remaining less than unity, then $1-x \geq 0$, hence

$$\lim_{x\to 1-0}(1-x)=0,\ \alpha(x)=1-x$$

is the infinitesimal, as $x \to 1$ ($\alpha(x) \geq 0$). The inverse value to the infinitesimal $\alpha(x)$ is the infinitely large magnitude, i.e. $\lim_{x\to 1-0} 1/\alpha(x) = +\infty$ consequently $\lim_{x\to 1-0} 2^{1/\alpha(x)} = +\infty$ and finally $\lim_{x\to 1-0} f(x) = 2$.

Now we research the right-hand sided limit

$$\lim_{x\to 1+0} f(x) = \lim_{x\to 1+0}\left(2+\frac{1}{1+2^{\frac{1}{1-x}}}\right).$$

Denoting $1-x=\beta(x)$, where $\beta(x) \leq 0$ we have $\lim_{x\to 1+0}\beta(x)=0$ since $\beta(x)$ is the infinitesimal as $x \to 1+0$, then $\lim_{x\to 1+0} 1/\beta(x) = -\infty$ and

$$\lim_{x\to 1+0} 2^{1/\beta(x)} = \lim_{x\to 1+0}\left(\frac{1}{2}\right)^{-1/\beta(x)} = 0,$$

it follows that $\lim_{x\to 1+0}\left(-1/\beta(x)\right) = +\infty$, and $\frac{1}{2} < 1$ then

$$\lim_{x\to 1+0} f(x) = \lim_{x\to 1+0}\left(2+\frac{1}{1+2^{\frac{1}{1-x}}}\right) = 2+1=3.$$

So the point $x=1$ is the point of discontinuity of the first kind. The graph of this function in the neighborhood of the point $x=1$ is shown in Fig. 10.

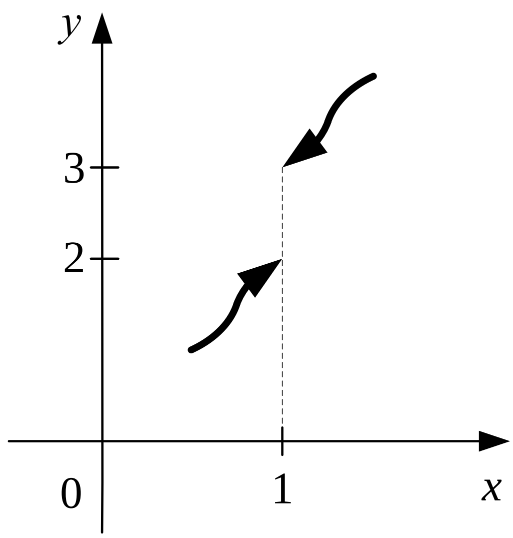


Fig. 10. The graph of the function $y = f(x) = 2 + 1/\left(1 + 2^{\frac{1}{1-x}}\right)$

in the neighborhood of the point $x = 1$

Remark. But if $f(x)$ is not defined for $x = x_0$ we say that the function $f(x)$ has a removable singularity at the point $x = x_0$. The meaning of this term is explained as following: if the point x_0 is added to the domain of definition of the function $f(x)$ and if at that new point the value of function is put equal to the common value of the left-hand and right-hand limits the (new) function $f(x)$ thus obtained is continuous at the point x_0. For example, the function $y = \dfrac{\sin x}{x}$ is not defined at the point $x = 0$. But since $\lim\limits_{x \to 0} \sin x / x = 1$ we can introduce a new function, defined for all the values of x and coinciding with the old one $x \neq 0$ which is everywhere continuous:

$$y = \begin{cases} \sin x/x, & \text{for} \quad x \neq 0 \\ 1, & \text{for} \quad x = 0 \end{cases}.$$

The term "removable singularity" is also applied when a function $f(x)$ is defined at the point x_0 and possesses coinciding left-hand and right-hand limits which are not equal to $f(x)$. For instance, such is the point $x = 0$ for the function

$$y = f(x) = \begin{cases} x, & \text{for} \quad x \neq 0 \\ 2, & \text{for} \quad x = 0 \end{cases}.$$

Its graph can be obtained from of the continuous function $y = x$ if its point $(0;0)$ is ‘torn out’ and moved 2 units of length upward along Oy.

b) discontinuity of the second kind.

This is the case, when $\lim\limits_{x \to x_0} f(x)$ either on the left or on the right is equal to $\pm\infty$.

Example 27. It is required to research the point of discontinuity for the function $y = f(x) = 3^{\frac{1}{x}}$.

Solution. This function has a discontinuity at the point $x = 0$. Let us consider the left-hand limit $\lim\limits_{x \to 0-0} 3^{\frac{1}{x}} = 0,$ since $\lim\limits_{x \to 0-0} \dfrac{1}{x} = -\infty.$ Now we proceed to the right-hand limit $\lim\limits_{x \to 0+0} 3^{\frac{1}{x}} = +\infty,$ it follows from $\lim\limits_{x \to 0+0} \dfrac{1}{x} = +\infty.$

So $x = 0$ is the point of discontinuity of the second kind. The graph of that function in the neighborhood of the point $x = 0$ is shown in Fig. 11.

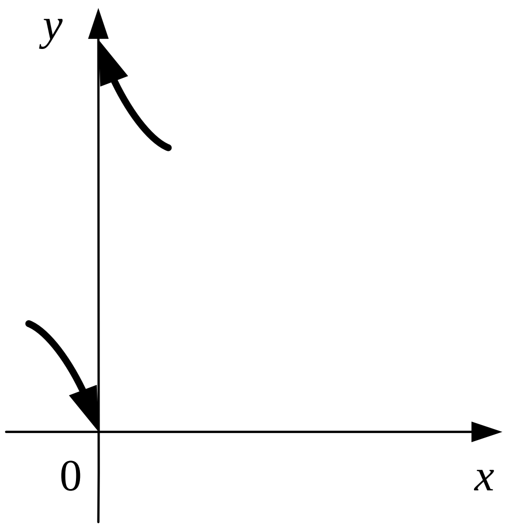


Fig. 11. The function $y = f(x) = 3^{\frac{1}{x}}$

19. Properties of Continuous Functions

Definition 23. A function $f(x)$ is called continuous on a closed interval $[a;b]$ if it is continuous at every point of the interval, and at its end-points

$$\lim_{x \to a+0} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b-0} f(x) = f(b).$$

These properties are formulated in the following theorems.

Theorem 1. If a function is continuous in a closed interval $[a;b]$ there exist at least one point in which the function assumes the greatest value M and at least one point in which it assumes the least value m on that interval (see Fig.12).

Theorem 2. If a function is continuous in a closed interval $[a;b]$ then it means that $f(x)$ assumes any value at least one time concluded between its least and its greatest values. It means that for arbitrary $c \in (a;b)$ there exists $f(c)$ where $f(c) \in (m;M)$ (see Fig. 13).

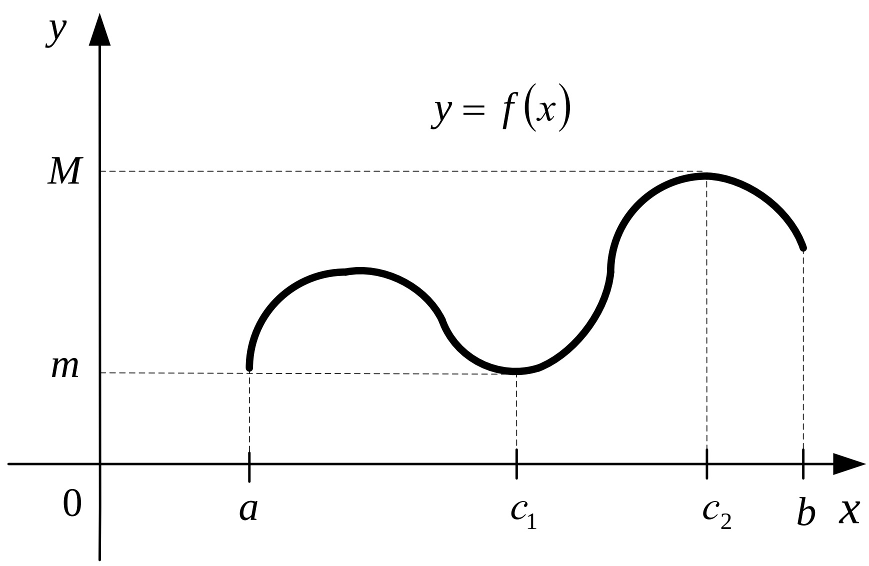


Fig. 12. A continuous function in a closed interval $[a;b]$

Theorem 3. If a function is continuous in a closed interval $[a;b]$, it assumes values of different signs at its ($f(a) \cdot f(b) < 0$) and points, there exists at least one point, lying inside the interval in which the function turns into zero (see Fig. 14).

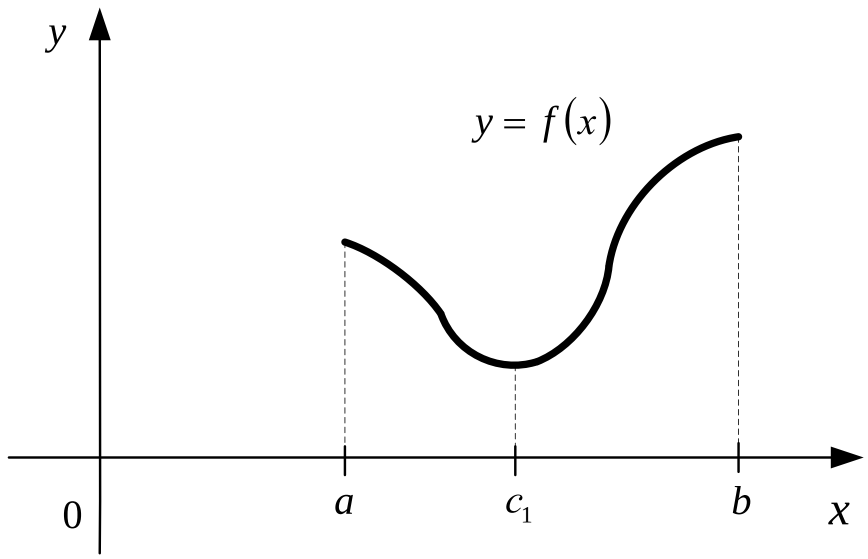


Fig. 13. A continuous function in a closed interval $[a;b]$

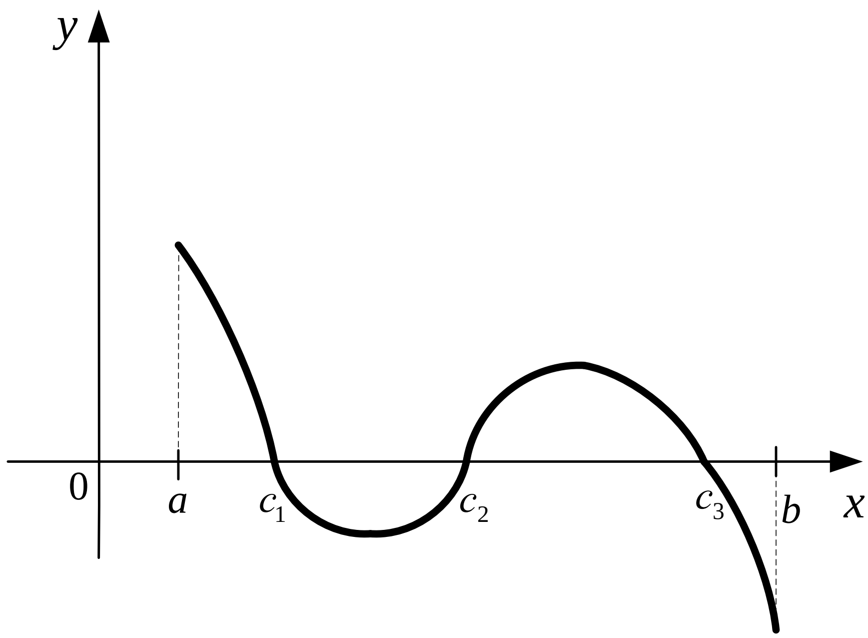


Fig. 14. A continuous function in a closed interval $[a;b]$

Individual tasks. Finding limits.
Finding and researching the point of discontinuity for the function $f(x)$

Variant 1

1) $\lim\limits_{x\to 1}\dfrac{(2x+3)(4x-3)}{5-x}$;

2) $\lim\limits_{x\to\infty}\dfrac{3x^3-5x^2+2}{2x^3+5x-1}$;

3) $\lim\limits_{x\to 2}\dfrac{3x^2+x-14}{x^2+3x-10}$;

4) $\lim\limits_{x\to 0}\dfrac{\sin 3x-\sin x}{\mathrm{tg}x}$;

5) $\lim\limits_{x\to 0}\left(\dfrac{1}{x\sin x}-\dfrac{1}{x^2}\right)$;

6) $\lim\limits_{x\to\infty}\dfrac{7x^2-14x+17}{12x^3+7x-3}$;

7) $\lim\limits_{x\to-\infty}\dfrac{x^5-2x+4}{2x^4+3x^2+1}$;

8) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+3x}-1}{x}$;

9) $\lim\limits_{x\to\infty}\left(\dfrac{x-4}{x+3}\right)^{x+1}$;

10) $f(x)=\dfrac{1}{2-2^{\frac{1}{x}}}$.

Variant 2

1) $\lim\limits_{x\to 3}(x+4)(2x-7)$;

2) $\lim\limits_{x\to\infty}\dfrac{4x^3+4x+1}{2x^3-4x^2+5}$;

3) $\lim\limits_{x\to-7}\dfrac{x^2+10x+21}{x^2+5x-14}$;

4) $\lim\limits_{x\to-2}\dfrac{\arcsin(x+2)}{x^2+2x}$;

5) $\lim\limits_{x\to 0}\left(\dfrac{1}{e^x-1}-\dfrac{1}{x}\right)$;

6) $\lim\limits_{x\to-\infty}\dfrac{3x^4+5x+10}{2x^3+15x+2}$;

7) $\lim\limits_{x\to\infty}\dfrac{2x^2+3x-5}{7x^3-2x^2+1}$;

8) $\lim\limits_{x\to 7}\dfrac{\sqrt{2+x}-3}{x-7}$;

9) $\lim\limits_{x\to\infty}\left(\dfrac{2x-4}{2x+3}\right)^{2x+1}$;

10) $f(x)=\dfrac{x^2-16}{x-4}$.

Variant 3

1) $\lim\limits_{x\to 11}\left(\frac{1}{11}x^2+x+3\right);$

2) $\lim\limits_{x\to -\infty}\frac{5x^4-3x^2+7}{x^4+2x^3+1};$

3) $\lim\limits_{x\to 4}\frac{2x^2-4x-16}{x^2-5x+4};$

4) $\lim\limits_{x\to 1}\frac{\sin 3(x-1)}{x-x^2};$

5) $\lim\limits_{x\to 0}\left(\frac{1}{\ln(x+1)}-\frac{1}{x}\right);$

6) $\lim\limits_{x\to \infty}\frac{4x^4+7x+6}{2x^3+10x^2-3};$

7) $\lim\limits_{x\to -\infty}\frac{3x^2+7x-4}{x^5+2x-1};$

8) $\lim\limits_{x\to -4}\frac{3-\sqrt{x^2-7}}{\sqrt{x+8}-2};$

9) $\lim\limits_{x\to \infty}\left(\frac{5x^2+2}{5x^2}\right)^x;$

10) $f(x)=3+\frac{1}{1+2^{\frac{1}{x-3}}}.$

Variant 4

1) $\lim\limits_{x\to -1}(2x-1)(x+2);$

2) $\lim\limits_{x\to \infty}\frac{3x-x^6}{x^2-2x+5};$

3) $\lim\limits_{x\to -4}\frac{x^2+8x+16}{2x^2+9x+4};$

4) $\lim\limits_{x\to 0}\frac{\sin 5x}{\operatorname{arctg} 2x};$

5) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{\ln(x+1)}\right);$

6) $\lim\limits_{x\to \infty}\frac{4x^2+3x-2}{5x^2+3x-1};$

7) $\lim\limits_{x\to -\infty}\frac{2x^2-x+7}{3x^4-5x^2+10};$

8) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x};$

9) $\lim\limits_{x\to \infty}\left(\frac{2x+1}{2x-4}\right)^{x-1};$

10) $f(x)=1+\frac{1}{2-2^{\frac{x}{x-1}}}.$

Variant 5

1) $\lim\limits_{x\to 2}\left(\frac{1}{2}x^2-5x-6\right)$;

2) $\lim\limits_{x\to\infty}\frac{x^3-4x^2+28x}{5x^3+3x^2+x-1}$;

3) $\lim\limits_{x\to 6}\frac{x^2-5x-6}{x^2-3x-18}$;

4) $\lim\limits_{x\to 0}\frac{\operatorname{arctg} 5x}{\sin x-\sin 3x}$;

5) $\lim\limits_{x\to 0}\left(\frac{1}{2x}-\frac{1}{\ln(2x+1)}\right)$;

6) $\lim\limits_{x\to\infty}\frac{3x^3-2x+5}{2x^2+4x-3}$;

7) $\lim\limits_{x\to-\infty}\frac{2x^3+7x-1}{3x^4+2x+5}$;

8) $\lim\limits_{x\to 1}\frac{\sqrt{3+x}-2}{x-1}$;

9) $\lim\limits_{x\to\infty}\left(\frac{x+1}{x-3}\right)^{x-4}$;

10) $f(x)=\ln(\sin x)$.

Variant 6

1) $\lim\limits_{x\to 5}\left(x^2-2x+3\right)$;

2) $\lim\limits_{x\to-\infty}\frac{3x^2+10x+3}{2x^2+5x-3}$;

3) $\lim\limits_{x\to 7}\frac{x^2-10x+21}{x^2-14x+49}$;

4) $\lim\limits_{x\to 0}\frac{2x\cdot\sin x}{\operatorname{arctg}^2(-3x)}$;

5) $\lim\limits_{x\to 0}\left(\frac{1}{\ln(2x+1)}-\frac{1}{2x}\right)$;

6) $\lim\limits_{x\to\infty}\frac{5x^3+7x^2+5}{10x^4-11x+8}$;

7) $\lim\limits_{x\to\infty}\frac{3x^4-2x+1}{3x^2+2x-5}$;

8) $\lim\limits_{x\to 0}\frac{\sqrt{x+1}-\sqrt{4-x}}{x}$;

9) $\lim\limits_{x\to\infty}\left(\frac{x-1}{x+2}\right)^{x+2}$;

10) $f(x)=\frac{1}{5+3^{\frac{1}{x-3}}}$.

Variant 7

1) $\lim\limits_{x\to 2}\left(7x^2-8x+9\right);$

2) $\lim\limits_{x\to -\infty}\dfrac{-3x^4+x^2+x}{x^4+3x-2};$

3) $\lim\limits_{x\to 3}\dfrac{x^2+2x-15}{x^2-8x+15};$

4) $\lim\limits_{x\to \frac{1}{2}}\dfrac{\arcsin(1-2x)}{4x^2-1};$

5) $\lim\limits_{x\to 0} x\cdot\ln x;$

6) $\lim\limits_{x\to\infty}\dfrac{7x^2-x^4-3x}{4x^4+3x-2};$

7) $\lim\limits_{x\to -\infty}\dfrac{3x^6-5x^2+2}{2x^3+4x-5};$

8) $\lim\limits_{x\to 4}\dfrac{3-\sqrt{5+x}}{1-\sqrt{5-x}};$

9) $\lim\limits_{x\to\infty}\left(\dfrac{5x-1}{5x+1}\right)^{x+3};$

10) $f(x)=\dfrac{|x|}{x}.$

Variant 8

1) $\lim\limits_{x\to 1}\left(3x^2-2x+4\right);$

2) $\lim\limits_{x\to\infty}\dfrac{2x^2+7x+3}{5x^2-3x+4};$

3) $\lim\limits_{x\to 4}\dfrac{2x^2-5x-12}{x^2+x-20};$

4) $\lim\limits_{x\to 0}\dfrac{1-e^{3x}}{\operatorname{tg} 2x};$

5) $\lim\limits_{x\to 0}\sin x\cdot\ln x;$

6) $\lim\limits_{x\to\infty}\dfrac{2x^5-4x+7}{6x^4+2x-10};$

7) $\lim\limits_{x\to -\infty}\dfrac{5x^2-4x+2}{4x^3+2x-5};$

8) $\lim\limits_{x\to 2}\dfrac{\sqrt{3x-2}-2}{\sqrt{x+2}-2};$

9) $\lim\limits_{x\to\infty}\left(\dfrac{2x-1}{2x+3}\right)^{x+3};$

10) $f(x)=2+\dfrac{1}{1+2^{\frac{1}{1-x}}}.$

Variant 9

1) $\lim\limits_{x\to 9}\dfrac{(x+1)(x-2)}{6-3x^2}$;

2) $\lim\limits_{x\to\infty}\dfrac{-x^2+3x+1}{3x^2+x-5}$;

3) $\lim\limits_{x\to 1}\dfrac{2x^2-4x+2}{x^2-5x+4}$;

4) $\lim\limits_{x\to 1}\dfrac{\arcsin 5(x-1)}{x-1}$;

5) $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}-\dfrac{1}{\ln x}\right)$;

6) $\lim\limits_{x\to\infty}\dfrac{8x^5+4x^4-3}{5x^4-3x^2+6}$;

7) $\lim\limits_{x\to-\infty}\dfrac{7x^2+5x+9}{1+4x-x^3}$;

8) $\lim\limits_{x\to 1}\dfrac{\sqrt{5x-1}-2}{x-1}$;

9) $\lim\limits_{x\to\infty}\left(\dfrac{3x+1}{3x-1}\right)^{4x}$;

10) $f(x)=2^{-2^{\frac{1}{1-x}}}$.

Variant 10

1) $\lim\limits_{x\to -2}(x-3)(2x+5)$;

2) $\lim\limits_{x\to\infty}\dfrac{3x^4+x^2-6}{2x^2+3x+1}$;

3) $\lim\limits_{x\to -4}\dfrac{x^2+8x+16}{2x^2+9x+4}$;

4) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\operatorname{arctg} 2x}$;

5) $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$;

6) $\lim\limits_{x\to\infty}\dfrac{7x^3+8x^2+1}{12x^3-9x+5}$;

7) $\lim\limits_{x\to-\infty}\dfrac{5x^2-4x+2}{4x^3+2x-5}$;

8) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$;

9) $\lim\limits_{x\to\infty}\left(\dfrac{4x-1}{4x+5}\right)^{3x-1}$;

10) $f(x)=\dfrac{1}{1-e^{\frac{x}{x-2}}}$.

Variant 11

1) $\lim\limits_{x\to 2}\left(5x^2-2x+4\right);$

2) $\lim\limits_{x\to\infty}\dfrac{4x^2+5x-7}{2x^2-x+10};$

3) $\lim\limits_{x\to 3}\dfrac{3x^2-7x-6}{2x^2+x-21};$

4) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\arcsin 2x};$

5) $\lim\limits_{x\to 0}\operatorname{tg}x\cdot\ln x;$

6) $\lim\limits_{x\to\infty}\dfrac{3x^2-x+4}{2x^3-4x^2+3};$

7) $\lim\limits_{x\to-\infty}\dfrac{7x^5+6x^4-x^3}{2x^2+6x+1};$

8) $\lim\limits_{x\to 1}\dfrac{\sqrt{3x+1}-2}{x-1};$

9) $\lim\limits_{x\to\infty}\left(\dfrac{2x-1}{2x+1}\right)^x;$

10) $f(x)=\dfrac{2}{3+5^{\frac{1}{x-2}}}.$

Variant 12

1) $\lim\limits_{x\to 2}\dfrac{(2x-5)(3x-4)}{x+2};$

2) $\lim\limits_{x\to\infty}\dfrac{3x^4+2x+1}{x^4-x^3+3x};$

3) $\lim\limits_{x\to-2}\dfrac{x^2-2x-8}{x^2+4x+4};$

4) $\lim\limits_{x\to 0}\dfrac{\operatorname{tg}x}{1-e^{-x}};$

5) $\lim\limits_{x\to 0}x^{\sin x};$

6) $\lim\limits_{x\to\infty}\dfrac{6x^5+4x+3}{10x^3+5x^2-1};$

7) $\lim\limits_{x\to-\infty}\dfrac{4-3x-2x^2}{3x^4+5x};$

8) $\lim\limits_{x\to 3}\dfrac{\sqrt{5x+1}-4}{x^2+2x-15};$

9) $\lim\limits_{x\to\infty}\left(\dfrac{2x-1}{2x+3}\right)^{x-4};$

10) $f(x)=\dfrac{1}{3+5^{\frac{1}{x}}}.$

Variant 13

1) $\lim\limits_{x\to -2}\dfrac{(9-4x)(x+5)}{x-2}$;

2) $\lim\limits_{x\to -\infty}\dfrac{5x^3-3x^2+7}{2x^4+3x^2+1}$;

3) $\lim\limits_{x\to 8}\dfrac{x^2-3x-40}{x^2-10x+16}$;

4) $\lim\limits_{x\to 0}\dfrac{2\arcsin 3x}{\mathrm{tg}3x}$;

5) $\lim\limits_{x\to 0}\dfrac{1-e^{3x}}{x^2}$;

6) $\lim\limits_{x\to \infty}\dfrac{8x^3+5x-1}{2x^3+4x^2+3}$;

7) $\lim\limits_{x\to -\infty}\dfrac{7-3x^4}{2x^3+3x^2-5}$;

8) $\lim\limits_{x\to -1}\dfrac{\sqrt{5+x}+2}{-1-x}$;

9) $\lim\limits_{x\to \infty}\left(\dfrac{3x+4}{3x-1}\right)^{4x-3}$;

10) $f(x)=\dfrac{2^{\frac{1}{x}}-1}{2^{\frac{1}{x}}+1}$.

Variant 14

1) $\lim\limits_{x\to 10}(4x-20)(2x-10)$;

2) $\lim\limits_{x\to \infty}\dfrac{5x^2-3x+1}{1+2x-x^4}$;

3) $\lim\limits_{x\to \frac{1}{3}}\dfrac{9x^2-1}{9x^2-6x+1}$;

4) $\lim\limits_{x\to 4}\dfrac{\arcsin(4-x)}{x^2-3x-4}$;

5) $\lim\limits_{x\to 0}\dfrac{x^2}{1-e^{3x}}$;

6) $\lim\limits_{x\to \infty}\dfrac{6x^2+7x+2}{2x^2-2x+5}$;

7) $\lim\limits_{x\to -\infty}\dfrac{8x^4+7x^3-3}{3x^3-5x+1}$;

8) $\lim\limits_{x\to 5}\dfrac{\sqrt{3x+1}-4}{x^2-5x}$;

9) $\lim\limits_{x\to \infty}\left(\dfrac{x-1}{x+3}\right)^{x+2}$;

10) $f(x)=\dfrac{1}{3+3^{\frac{1}{x}}}$.

Variant 15

1) $\lim\limits_{x\to 4}\left(\frac{1}{2}x+1\right)(x-5)$;

6) $\lim\limits_{x\to\infty}\frac{8x^3+5x-4}{3x^4+6x+11}$;

2) $\lim\limits_{x\to\infty}\frac{2x^3+7x-2}{3x^3-x-4}$;

7) $\lim\limits_{x\to-\infty}\frac{2x^3+3x^2+5}{3x^2-4x+1}$;

3) $\lim\limits_{x\to\frac{1}{2}}\frac{8x^3-1}{4x^2-4x+1}$;

8) $\lim\limits_{x\to 8}\frac{\sqrt{x+1}-3}{4-\sqrt{x+8}}$;

4) $\lim\limits_{x\to 0}\frac{\cos x-\cos^3 x}{x^2}$;

9) $\lim\limits_{x\to\infty}\left(\frac{5x+1}{5x-1}\right)^{2x}$;

5) $\lim\limits_{x\to 0}\frac{2x}{3^{2x}-1}$;

10) $f(x)=e^{-\frac{1}{x^2-4}}$.

Variant 16

1) $\lim\limits_{x\to 5}(9x-5)(x+5)$;

6) $\lim\limits_{x\to\infty}\frac{25x^4+7x^3-3}{8x^5+5x+4}$;

2) $\lim\limits_{x\to\infty}\frac{18x^2+5x}{8-9x^2-3x}$;

7) $\lim\limits_{x\to-\infty}\frac{2x^3-3x+1}{7x+5}$;

3) $\lim\limits_{x\to\frac{1}{3}}\frac{9x^2-6x+1}{3x^2+2x-1}$;

8) $\lim\limits_{x\to 0}\frac{\sqrt{3x+1}-\sqrt{1-2x}}{x^2}$;

4) $\lim\limits_{x\to 0}\frac{\cos 3x-\cos 5x}{x^2}$;

9) $\lim\limits_{x\to\infty}\left(\frac{2x+4}{2x-1}\right)^{2x-3}$;

5) $\lim\limits_{x\to 0}\frac{3^{2x}-1}{2x}$;

10) $f(x)=\frac{x^3+8}{x+2}$.

Variant 17

1) $\lim\limits_{x\to 2}\left(\frac{1}{2}x^2+5x-1\right)$;

2) $\lim\limits_{x\to -\infty}\frac{11x^3+3x}{2x^2-2x+1}$;

3) $\lim\limits_{x\to 0.1}\frac{10x^2-21x+2}{x^2+0.9x-0.1}$;

4) $\lim\limits_{x\to 0}\frac{\text{arctg}\,5x}{e^{10x}-1}$;

5) $\lim\limits_{x\to 0} x\cdot\text{ctg}\,x$;

6) $\lim\limits_{x\to\infty}\frac{x^5+7x^4-12}{3x^5+6x^3-3x}$;

7) $\lim\limits_{x\to\infty}\frac{10x-7}{3x^4+2x^3+1}$;

8) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{3x}$;

9) $\lim\limits_{x\to\infty}\left(\frac{x+3}{x-2}\right)^{x-3}$;

10) $f(x)=\ln(\cos x)$.

Variant 18

1) $\lim\limits_{x\to 7}\frac{x^2-5x-4}{x+7}$;

2) $\lim\limits_{x\to\infty}\frac{8x^2+4x-5}{4x^2-3x+2}$;

3) $\lim\limits_{x\to -3}\frac{x^2-x-12}{x^2+x-6}$;

4) $\lim\limits_{x\to 0}\frac{\cos 3x-\cos x}{\sin 5x}$;

5) $\lim\limits_{x\to\pi/2}\text{tg}\,x\cdot\left(x-\frac{\pi}{2}\right)$;

6) $\lim\limits_{x\to\infty}\frac{3x^5+4x^2-2}{7x^4-2x+5}$;

7) $\lim\limits_{x\to -\infty}\frac{8x^2+3x+5}{4x^3-2x^2+1}$;

8) $\lim\limits_{x\to 0}\frac{\sqrt{x^2+1}-1}{x}$;

9) $\lim\limits_{x\to\infty}\left(\frac{3x+1}{3x-1}\right)^{x+2}$;

10) $f(x)=(1+x)\,\text{arctg}\frac{1}{1-x^2}$.

Variant 19

1) $\lim\limits_{x\to -6}(2x-8x+9)$;

2) $\lim\limits_{x\to\infty}\dfrac{8x^4-4x^2+3}{2x^4+1}$;

3) $\lim\limits_{x\to -\frac{1}{2}}\dfrac{4x^2+4x+1}{2x^2+3x+1}$;

4) $\lim\limits_{x\to 1}\dfrac{\arcsin(1-x)}{x^2-1}$;

5) $\lim\limits_{x\to 0}\arcsin x\cdot \operatorname{tg} x$;

6) $\lim\limits_{x\to\infty}\dfrac{6x^3+4x^2-3}{7x^4+3x+6}$;

7) $\lim\limits_{x\to -\infty}\dfrac{6x^3+5x^2-3}{2x^2-x+7}$;

8) $\lim\limits_{x\to 4}\dfrac{\sqrt{2x+1}-3}{2-\sqrt{2x-4}}$;

9) $\lim\limits_{x\to\infty}\left(\dfrac{4x-1}{4x+1}\right)^{2x}$;

10) $f(x)=e^{-\frac{1}{x^2}}$.

Variant 20

1) $\lim\limits_{x\to -4}(3x^2+2x+4)$;

2) $\lim\limits_{x\to\infty}\dfrac{3x^2+4x-7}{x^4-2x^3+1}$;

3) $\lim\limits_{x\to 5}\dfrac{x^2-7x+10}{x^2-10x+25}$;

4) $\lim\limits_{x\to 0}\dfrac{\sin x-\sin 5x}{x\operatorname{tg} 2x}$;

5) $\lim\limits_{x\to 1}\ln(1-x)(1-x)$;

6) $\lim\limits_{x\to\infty}\dfrac{4x^3+8x-3}{3x^3+5x^2+5}$;

7) $\lim\limits_{x\to -\infty}\dfrac{3x^4+5x}{2x^2-3x-7}$;

8) $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{\sqrt{x+4}-2}$;

9) $\lim\limits_{x\to\infty}\left(\dfrac{3x-2}{3x+3}\right)^{2x}$;

10) $f(x)=\dfrac{1}{1+e^{\frac{1}{1-x}}}$.

Variant 21

1) $\lim\limits_{x\to -3} \frac{(x+1)(x+4)}{3-3x^2}$;

2) $\lim\limits_{x\to \infty} \frac{7x^3+4x}{x^3-3x+2}$;

3) $\lim\limits_{x\to -4} \frac{x^2-x-20}{x^2+x-12}$;

4) $\lim\limits_{x\to 1} \frac{\operatorname{arctg} 2(x-1)}{x^2-x}$;

5) $\lim\limits_{x\to 0} \left(\frac{1}{\sin x}-\frac{1}{x}\right)$;

6) $\lim\limits_{x\to \infty} \frac{x^3+2x^2-3}{5x^4-x^2+4}$;

7) $\lim\limits_{x\to -\infty} \frac{8x^5-4x^3+3}{2x^3+x-7}$;

8) $\lim\limits_{x\to 1} \frac{\sqrt{2x-1}-1}{x-1}$;

9) $\lim\limits_{x\to \infty} \left(\frac{4x+1}{4x-1}\right)^{x-5}$;

10) $f(x)=e^{\frac{1}{x+1}}$.

Variant 22

1) $\lim\limits_{x\to 3} (2x^2-3x)(7x-15)$;

2) $\lim\limits_{x\to \infty} \frac{1+4x-x^4}{x+3x^2+2x^4}$;

3) $\lim\limits_{x\to -2} \frac{x^2+4x+4}{x^3+8}$;

4) $\lim\limits_{x\to 0} \frac{\arcsin^2 x}{x\cdot \sin x}$;

5) $\lim\limits_{x\to 0} \left(\frac{1}{x}-\frac{1}{\sin x}\right)$;

6) $\lim\limits_{x\to \infty} \frac{2x^3-4x^2+3}{4x^2+7x+5}$;

7) $\lim\limits_{x\to -\infty} \frac{2x^2-7x+1}{x^3+4x^2-3}$;

8) $\lim\limits_{x\to 5} \frac{\sqrt{x+4}-3}{\sqrt{x-1}-2}$;

9) $\lim\limits_{x\to \infty} \left(\frac{x+2}{x-1}\right)^{3x}$;

10) $f(x)=\frac{|x-3|}{x-3}$.

Variant 23

1) $\lim\limits_{x\to\frac{1}{2}}\left(4x^2 - x - 10\right);$

2) $\lim\limits_{x\to\infty}\dfrac{2x^3+7x^2-2}{6x^3-4x+3};$

3) $\lim\limits_{x\to 7}\dfrac{2x^2-5x-63}{x^2-6x-7};$

4) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\operatorname{arctg} 2x};$

5) $\lim\limits_{x\to 0}(\sin x)^x;$

6) $\lim\limits_{x\to\infty}\dfrac{3x^2-5x+14}{7x^3+2x^2-3};$

7) $\lim\limits_{x\to-\infty}\dfrac{5x^4-2x^3+3}{2x^2+3x-7};$

8) $\lim\limits_{x\to 5}\dfrac{\sqrt{x-1}-2}{x-5};$

9) $\lim\limits_{x\to\infty}\left(\dfrac{2x+1}{2x-3}\right)^{x-5};$

10) $f(x)=\dfrac{1}{1-e^{\frac{x}{1-x}}}.$

Variant 24

1) $\lim\limits_{x\to 1}\dfrac{(x-4)(3x+4)}{x+1};$

2) $\lim\limits_{x\to\infty}\dfrac{8x^3+x^2-7}{2x^2-5x+3};$

3) $\lim\limits_{x\to -3}\dfrac{x^2-2x-15}{x^2+4x+3};$

4) $\lim\limits_{x\to 1}\dfrac{\operatorname{tg}2(x-1)}{1-e^{4(x-1)}};$

5) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{\sin x};$

6) $\lim\limits_{x\to\infty}\dfrac{6x^3+x+1}{x^3+5x^2-1};$

7) $\lim\limits_{x\to-\infty}\dfrac{3x+1}{x^3-5x^2+4x};$

8) $\lim\limits_{x\to 2}\dfrac{\sqrt{4x+1}-3}{x^3-8};$

9) $\lim\limits_{x\to\infty}\left(\dfrac{x-4}{x+3}\right)^{5x};$

10) $f(x)=3+\dfrac{1}{1+7^{\frac{1}{1-x}}}.$

Variant 25

1) $\lim\limits_{x\to 1}\dfrac{(x-4)(4x-3)}{4-x}$;

2) $\lim\limits_{x\to -\infty}\dfrac{x-2x^2+5x^4}{2+3x^2+x^4}$;

3) $\lim\limits_{x\to 2}\dfrac{2x^2+x-10}{x^2-3x+2}$;

4) $\lim\limits_{x\to 2}\dfrac{\sin 3(x-2)}{\operatorname{tg}(x-2)}$;

5) $\lim\limits_{x\to 0}(\operatorname{tg} x)^{2x}$;

6) $\lim\limits_{x\to \infty}\dfrac{4x^2-5x+7}{2x^3+3x-1}$;

7) $\lim\limits_{x\to -\infty}\dfrac{3x^4+2x^2-8}{8x^3-4x+5}$;

8) $\lim\limits_{x\to 0}\dfrac{\sqrt{4+3x}-2}{x}$;

9) $\lim\limits_{x\to \infty}\left(\dfrac{4x-4}{4x-3}\right)^{x-2}$;

10) $f(x)=\dfrac{x^2-4}{x-2}$.

Variant 26

1) $\lim\limits_{x\to 11}\left(\dfrac{1}{22}x-1\right)(5x+6)$;

2) $\lim\limits_{x\to -\infty}\dfrac{3x^4-2x^2-7}{3x^4+3x+5}$;

3) $\lim\limits_{x\to \frac{1}{5}}\dfrac{5x^2+4x-1}{5x^2-6x+1}$;

4) $\lim\limits_{x\to -1}\dfrac{x+1}{\arcsin(2x+2)}$;

5) $\lim\limits_{x\to 0}\left(\dfrac{1}{\operatorname{tg} x}\right)^{x}$;

6) $\lim\limits_{x\to \infty}\dfrac{7x^2-5x+4}{3x^4+2x-2}$;

7) $\lim\limits_{x\to \infty}\dfrac{3x^4+2x-4}{3x^2-4x+1}$;

8) $\lim\limits_{x\to 5}\dfrac{\sqrt{x-4}-1}{x-5}$;

9) $\lim\limits_{x\to \infty}\left(\dfrac{4x-1}{4x+2}\right)^{x-3}$;

10) $f(x)=e^{\frac{1}{x}}$.

Variant 27

1) $\lim\limits_{x\to 2}\left(5-2x-x^2\right)$;

2) $\lim\limits_{x\to -\infty}\dfrac{7x^3-2x+4}{2x^2+x-5}$;

3) $\lim\limits_{x\to 2}\dfrac{2x^2-2x-4}{x^2-4x+4}$;

4) $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 3x}{x^2}$;

5) $\lim\limits_{x\to 0}(\cos x)^{1/x}$;

6) $\lim\limits_{x\to \infty}\dfrac{12x^3+4x-7}{4x^3+5x^2-3}$;

7) $\lim\limits_{x\to \infty}\dfrac{2x-13}{x^7-3x^5-4x}$;

8) $\lim\limits_{x\to 0}\dfrac{1-\sqrt{1-x^2}}{x^2}$;

9) $\lim\limits_{x\to \infty}\left(\dfrac{3x-1}{3x+3}\right)^{x-4}$;

10) $f(x)=\dfrac{1}{2+2^{\frac{1}{x}}}$.

Variant 28

1) $\lim\limits_{x\to 2}\left(\dfrac{1}{2}x+5\right)(x-3)$;

2) $\lim\limits_{x\to \infty}\dfrac{4x^3+5x^2-3x}{3x^2+x-10}$;

3) $\lim\limits_{x\to \frac{1}{4}}\dfrac{4x^2+3x-1}{4x^2-17x+4}$;

4) $\lim\limits_{x\to 0}\dfrac{1-e^{-2x}}{\operatorname{arctg} 2x}$;

5) $\lim\limits_{x\to 0}(\operatorname{ctg} x)^{\sin x}$;

6) $\lim\limits_{x\to \infty}\dfrac{12x^3+7x-2}{6x^3+5x+10}$;

7) $\lim\limits_{x\to -\infty}\dfrac{2x^2-3x+1}{x^3+2x^2+5}$;

8) $\lim\limits_{x\to 3}\dfrac{\sqrt{2x-1}-\sqrt{5}}{x-3}$;

9) $\lim\limits_{x\to \infty}\left(\dfrac{x-1}{x+4}\right)^{3x}$;

10) $f(x)=\operatorname{arctg}\dfrac{1}{x-5}$.

Variant 29

1) $\lim\limits_{x\to 1}(2x-6)(20x-10)$;

2) $\lim\limits_{x\to -\infty}\dfrac{2x^2+10x-11}{3x^4-2x+5}$;

3) $\lim\limits_{x\to \frac{1}{3}}\dfrac{3x^2+2x-1}{9x^2-6x+1}$;

4) $\lim\limits_{x\to 0}\dfrac{\arcsin^2 3x}{x\cdot \operatorname{arctg}(-x)}$;

5) $\lim\limits_{x\to \infty}(\ln x)^{1/x}$;

6) $\lim\limits_{x\to \infty}\dfrac{3x^5+2x^4-13}{6x^5+x^3+10}$;

7) $\lim\limits_{x\to \infty}\dfrac{x^3-81}{3x^2+4x+2}$;

8) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+3x}-1}{2x}$;

9) $\lim\limits_{x\to \infty}\left(\dfrac{2x-4}{2x+3}\right)^{3x-1}$;

10) $f(x)=\dfrac{x^3+1}{x+1}$.

Variant 30

1) $\lim\limits_{x\to 3}(x^2-x+3)$;

2) $\lim\limits_{x\to \infty}\dfrac{5x^2-3x+1}{3x^2+x-5}$;

3) $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2-1}$;

4) $\lim\limits_{x\to 0}\dfrac{2x\cdot(e^{2x}-1)}{\operatorname{arctg}^2(-2x)}$;

5) $\lim\limits_{x\to 0}(\operatorname{tg} x)^{\sin x}$;

6) $\lim\limits_{x\to \infty}\dfrac{10x^3+x^2+5}{5x^4-3x+8}$;

7) $\lim\limits_{x\to -\infty}\dfrac{7x^3+3x-4}{2x^2-5x+1}$;

8) $\lim\limits_{x\to 0}\dfrac{2-\sqrt{4-x}}{x}$;

9) $\lim\limits_{x\to \infty}\left(\dfrac{4x-1}{4x+2}\right)^{3x+2}$;

10) $f(x)=\operatorname{arctg}\dfrac{1}{x}$.

Theoretical questions

1. Set. Element of set.
2. Union. Intersection. Difference.
3. Closed interval, open interval, half-open interval.
4. Neighborhood of a point.
5. Closed set. Bounded above (below) set.
6. Notion of a function.
7. Domain of definition of a function.
8. Range of a function.
9. Independent variable.
10. Dependent variable.
11. Ways to define a function (analytic, tabular and graphic methods).
12. Rational function.
13. Composite function.
14. Implicit function.
15. Inverse function.
16. Properties of a function: boundedness and unboundedness, increasing and decreasing of a function, oddness and evenness, periodicity.
17. Numerical sequence.
18. Definition of a limit of a sequence.
19. Infinitesimals. Infinitely large values.
20. Definition of a limit of a function.
21. Arithmetical operations above functions which have finite limits.
22. First remarkable limit.
23. Second remarkable limit.
24. Comparison of infinitesimals.
25. Basic theorems about limits.
26. Left (right)-side limit.
27. Continuous function at a point.
28. Properties of continuous functions.
29. Classification of breaks.
30. First kind discontinuity.
31. Second kind discontinuity.
32. Removable discontinuity.

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Educational Edition

Guidelines to practical tasks in introduction to mathematical analysis of the academic discipline "Higher and Applied Mathematics" for foreign and English-learning full-time students of training direction 6.030601 "Management" of specialization "Business Administration"

Compiled by **Ie. Misiura**
V. Titarev

Editorial director **L. Malyarets**

Editor **N. Ovcharenko**
Proof-reader

Навчальне видання

Методичні рекомендації до виконання практичних завдань зі вступу до математичного аналізу з навчальної дисципліни «Вища та прикладна математика» для іноземних студентів та студентів, що навчаються англійською мовою, напряму підготовки 6.030601 «Менеджмент» спеціалізації «Бізнес-адміністрування» денної форми навчання

Укладачі **Місюра Євгенія Юріївна**
Титарев Віктор Георгієвич

Відповідальний за випуск **Малярець Л. М.**

Редактор **Овчаренко Н.М.**
Коректор

План 2014 р. Поз. № 38.
Підп. до друку. Формат 60×90 1/16. Папір MultiCopy. Друк на Riso.
Ум.-друк. арк. 4. Обл.-вид. арк. Тираж 50 прим. Зам. №

Свідоцтво про внесення до Державного реєстру суб'єктів видавничої справи ***Дк №481 від 13.06.2001р.***

Видавець і виготівник – видавництво ХНЕУ ім. С. Кузнеця, 61001, м. Харків, пр. Леніна, 9а