Theme 1. The limit of a function and continuity

Lecture plan

1. A definition of a function. A domain of the function. A range of the function values. A limit of a numerical sequence. Some properties of limits.
2. Infinitesimals and their main properties. Comparison of two infinitesimal values.
3. Infinitude values (infinitely large variable). Comparison of two infinitude values.
4. A limit of a function at a point. One-sided limits.
5. Basic "rules-theorems" of calculation of limits. Some properties of limits.
6. The 1-st and the 2-nd remarkable limits and their consequences.
7. Techniques of calculations of limits.
8. Calculation of a limit of a rational function.
9. Calculation of limits of functions with irrational expressions.
10. Calculating limits of functions using the 1-st and the 2-nd remarkable limits and their consequences

1. A definition of the limit of a function at a point. A domain of the function definition. A range of the function values. A limit of a numerical sequence. Some properties of limits

Definition*.* If for every value of a variable *x* belonging to some set *X* ($x \in X$) there corresponds by the rule *f* a definite number $y \in Y$ then it is said that on the set *X* the *function* $y = f(x)$ is given.

Definitions*.* The set *X* is called the *domain of the function definition,* the set YY is *a range of the function values.*

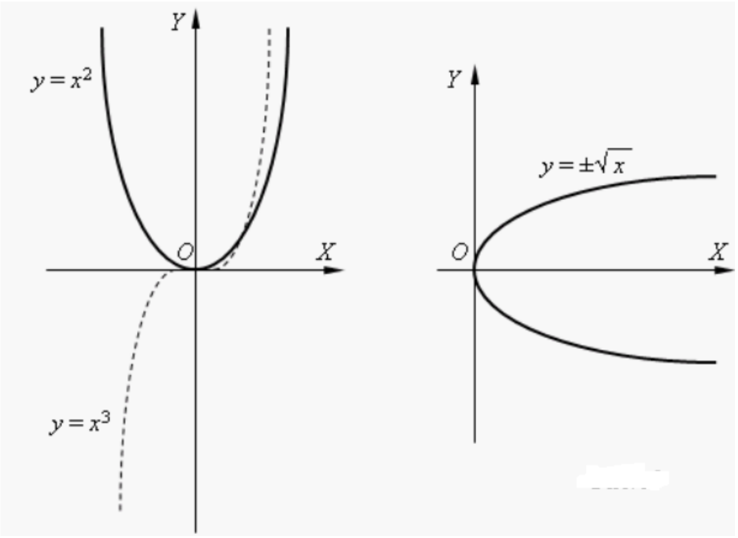
Definitions*.* Thus *x* is called an *independent variable* or an *argument, y* is a *function.*

Definition*.* If for each value of variable the $x \in X$ there is more than one correspondent value $y \in Y$ (and even an indefinite set of them), then this function is called *multivalued* as against *single valued function* defined above.

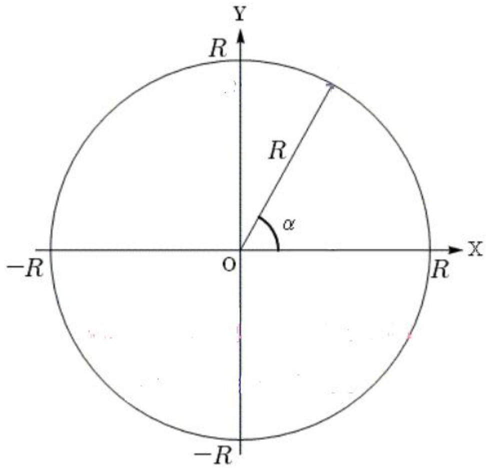
Example. $x^2 + y^2 = R^2$ is an equation of a circle, R is a radius.

Further we will consider single valued functions.

One of the main calculus operations is an operation of limiting transition. Let us consider the simplest form of this operation based on the concept of the limit of the so-called numerical sequence.



Definition. If for each number *n* belonging to the set of natural numbers 1, 2, ..., n, ... we put in correspondence some real number x_n by *a certain law*, then the set of enumerated real numbers $x_1, x_2, \ldots, x_n$ is called a *numerical sequence* or just a *sequence.* Briefly the sequence is designated by the symbol $\{x_n\}$.



For example, the symbol $\left\{\frac{1}{n}\right\}$ presents the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$

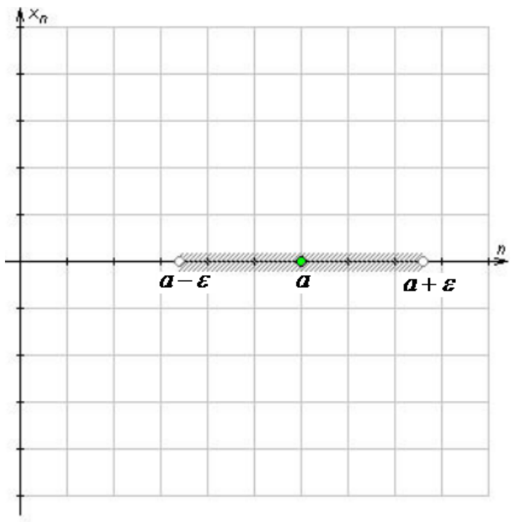
Definitions. The numerical sequence $\{x_n\}$ is called *bounded from above (below)* if there exists such a real number *M (m)* that $\forall\, n$ the following inequality $x_n \le M$ $(x_n \ge M)$ is true, thus the number *M* is called the upper bound, *m* is the lower bound..

Definition*.* The least of all upper bounds of a numerical sequence is called the *exact upper bound* and is designated by the latin word *"supremum"* ("the highest"), or briefly $\sup x_n$.

Definition*.* The greatest of all lower bounds of a numerical sequence is called the *exact lower bound* and is designated by the latin word *"infimum"* ("the lowest"), or briefly $\inf x_n$.

Definition*.* The numerical sequence $\{x_n\}$ is called *bounded* if it is bounded from above and below, i.e. if $\forall n$ the following inequality $m \le x_n \le M$.

Definition. The number *a* is called the *limit* of the numerical sequence $\{x_n\}$ if for any given positive number ε we can point such a number $N(\varepsilon)$ that all values x_n $\forall n > N$ will satisfy the inequality $|x_n - a| < \varepsilon$.



If the number *a* is the limit of the numerical sequence $\{x_n\}$, then one says that x_n approaches the limit *a* and writes $x_n \to a$ or $\lim\limits_{n\to\infty} x_n = a$.

Let each value of the variable x_n has a corresponding point on the numerical axis. Let us write the inequality $|x_n - a| < \varepsilon$ as a double inequality $-\varepsilon < x_n - a < \varepsilon$ or $-\varepsilon + a < x_n < \varepsilon + a$.

Thus, the set of points belonging to the interval $(-\varepsilon + a, \varepsilon + a)$ and satisfying the inequality $|x_n - a| < \varepsilon$ is called the ε-neighbourhood of the point *a*. Then the geometrical interpretation of the limit can be the following:

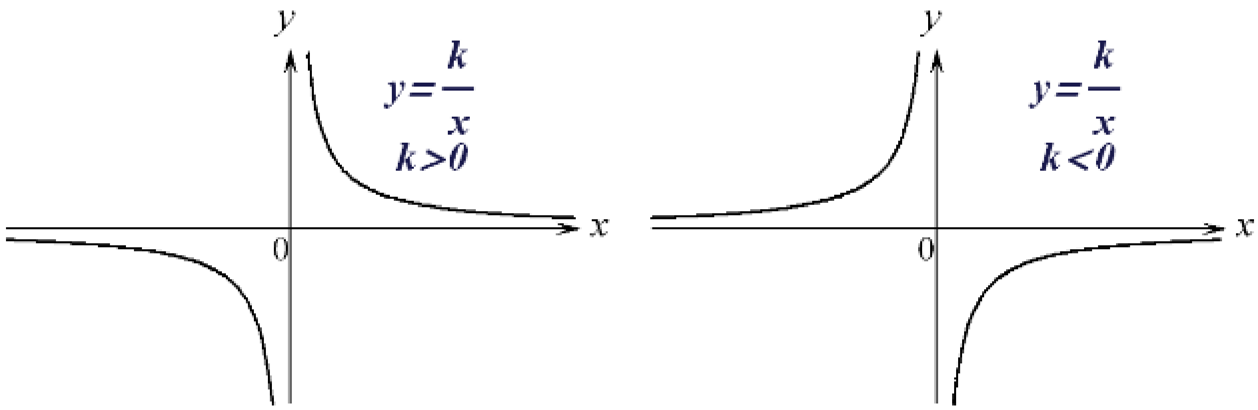
The number *a* is the limit of the variable x_n if for any indefinitely small positive number ε all values of the variable x_n, starting with some number N will be located in the ε-neighbourhood of the point *a.* Outside of this interval there can be only a finite number of elements.

Definitions. A numerical sequence which has a finite limit is called a *convergent sequence,* otherwise - a *divergent sequence.*

For example, the sequence $\left\{2^n\right\}$ is a *divergent sequence*, the sequence $\left\{\frac{1}{n}\right\}$ is a *convergent sequence.*

Some properties of limits.

1. A convergent numerical sequence is bounded.
2. If a numerical sequence has a limit then it is the only one.



2. Infinitesimals and their main properties. Comparison of two infinitesimal values

Definition. The numerical sequence $\{x_n\}$ is called *infinitesimal* if $\lim\limits_{n\to\infty} x_n = 0$.

Example: $x_n = \frac{1}{n}$.

We can show that for infinitesimals the following statements are true:

1. The sum of a *finite* number of infinitesimal values is also an infinitesimal value.
2. The product of a *finite* number of infinitesimal values is also an infinitesimal value.
3. The product of bounded and infinitesimal values is an infinitesimal value.

Let the variables α_n and β_n be infinitesimal values, i.e. $\lim\limits_{n\to\infty} \alpha_n = 0$, $\lim\limits_{n\to\infty} \beta_n = 0$.

Then:

1) If $\lim\limits_{n\to\infty} \frac{\alpha_n}{\beta_n} = \left\|\frac{0}{0}\right\| = C = const$, then α_n and β_n are called *infinitesimal* values *of the same* order: $\alpha_n \sim C\beta_n$.

2) If $\lim\limits_{n\to\infty} \frac{\alpha_n}{\beta_n} = \left\|\frac{0}{0}\right\| = 1$, then α_n and β_n are called *equivalent infinitesimals:* $\alpha_n \sim \beta_n$

3) If $\lim\limits_{n\to\infty} \frac{\alpha_n}{\beta_n} = \left\|\frac{0}{0}\right\| = 0$, then α_n is an infinitesimal *of the higher order* than β_n, i.e. α_n approaches zero faster than β_n.

4) If $\lim\limits_{n\to\infty} \frac{\alpha_n}{\beta_n} = \left\|\frac{0}{0}\right\| = \infty$, then β_n is an infinitesimal of the higher order than α_n, i.e. β_n approaches zero faster than α_n.

3. Infinitude values (infinitely large variable). Comparison of two infinitude values

In a certain sense infinitesimal values are opposed to *infinitude values* (infinitely large variables or just infinitudes).

Definition. The numerical sequence $\{x_n\}$ is called *infinitude* (infinitely large variables) if $\lim\limits_{n\to\infty} x_n = \infty$.

Example: $x_n = n$.

There is a simple connection between infinitesimal and infinitude values, and namely the following statements are true:

1. If the variable x_n is infinitude then its inverse value $\alpha_n = \frac{1}{x_n}$ is infinitesimal.

2. If the variable α_n (not vanishing) is infinitesimal then its inverse value $x_n = \dfrac{1}{\alpha_n}$ is infinitude.

Further we will designate *an infinitude value* by the *symbol "∞".*

Similarly to the comparison of two infinitesimal values we can carry out the comparison of two infinitude values.

Let the variables u_n and v_n be infinitude, i.e. $\lim\limits_{n\to\infty} u_n = \infty$, $\lim\limits_{n\to\infty} v_n = \infty$. Then:

1) If $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \left\|\dfrac{\infty}{\infty}\right\| = C = const$, then u_n and v_n are called infinitude values of the same order.

2) If $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \left\|\dfrac{\infty}{\infty}\right\| = 1$, then $u_n \sim v_n$, i.e. u_n and v_n are called equivalent infinitude values.

3) If $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \left\|\dfrac{\infty}{\infty}\right\| = 0$, then v_n is an infinitude value of the higher order then u_n, i.e. v_n grows (растет) faster than u_n.

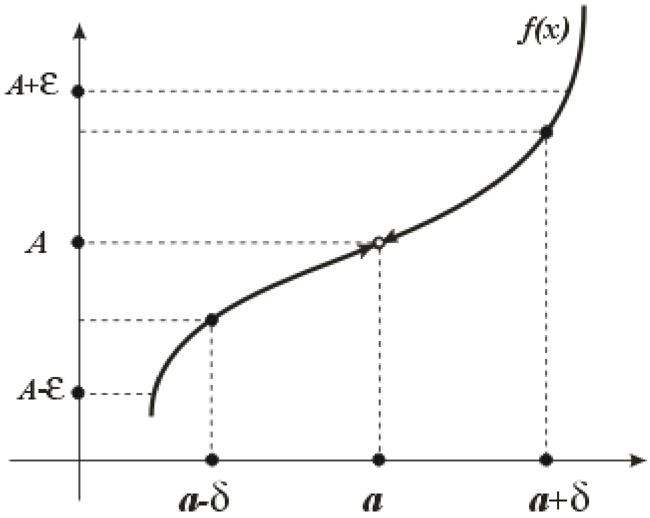
4) If $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \left\|\dfrac{\infty}{\infty}\right\| = \infty$, then u_n is an infinitude value of the higher order then v_n, i.e. u_n grows faster than v_n.

4. A limit of a function at a point. One-sided limits.

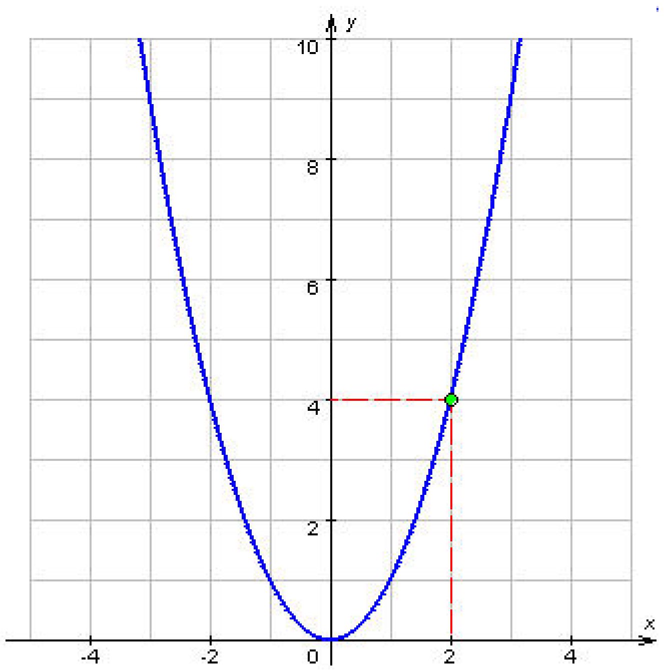
Let the function $f(x)$ be defined in some *neighborhood* of the point $x = a$, except, maybe, the point a itself.

Definition. The number *A* is called the *limit of the function* $y = f(x)$ at $x \to a$ (*x* approaches *a*) if for any *convergent* to a sequence $x_1, x_2, \ldots, x_n, \ldots$ of values of the arguments *x* $(x_n \neq a)$ the corresponding sequence $f(x_1), f(x_2), \ldots, f(x_n), \ldots$ of values of the function *converges* to *A.* This limit is designated as: $\lim\limits_{x\to a} f(x) = A$.

Cauchy's definition. The number $A \in R$ is called **the *limit of the function*** $y = f(x)$ at $x \to a$ (*x* approaches *a*) and designated as $\lim\limits_{x \to a} f(x) = A$ if for any ε there exists δ such that any $0 < |x - a| < \delta$ will satisfy the inequality $|f(x) - A| < \varepsilon$.



Example. Find $\lim\limits_{x \to 2} x^2$.



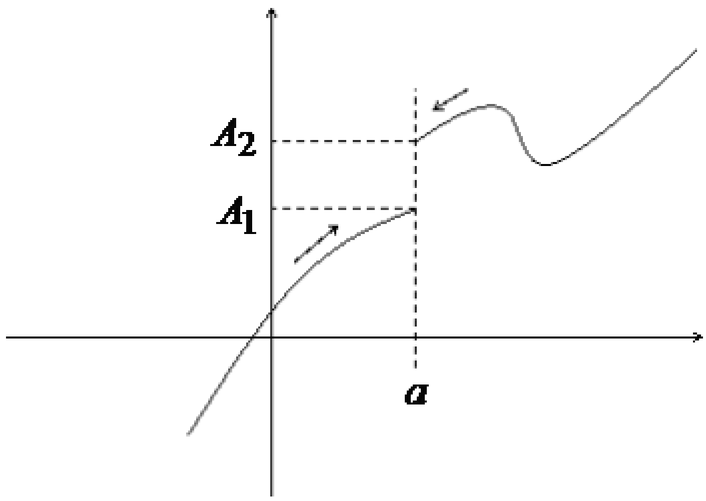
Definition. The number A is called ***the right (left) limit of the function*** $y = f(x)$ at the point $x = a$ if for any converging to a sequence $x_1, x_2, \ldots, x_n, \ldots$ of elements of which are larger (smaller) than a, the corresponding sequence $f(x_1), f(x_2), \ldots, f(x_n), \ldots$ of values of the function converges to *A.*

For the right limit of the function the following notation is used:

$$\lim_{x \to a-0} f(x) = A_1 \text{ or } f(a-0) = A_1.$$

Accordingly, the left limit of the function is denoted as follows:

$$\lim_{x \to a+0} f(x) = A_2 \text{ or } f(a+0) = A_2.$$



5. Basic "rules-theorems" of calculation of limits. Some properties of limits.

If we know that the functions $f(x)$ and $g(x)$ have limits and these limits are finite, then:

1. If a function is constant, i.e. $f(x) = C = const$, then $\lim\limits_{x \to a} f(x) = C$.
2. A limit of the sum is equal to the sum of limits, i.e. $\lim\limits_{x \to a} \left(f(x) + g(x) \right) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$.
3. A limit of the product is equal to the product of limits, i.e. $\lim\limits_{x \to a} \left(f(x) \cdot g(x) \right) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$
4. A limit of the quotient is equal to the quotient of limits, i.e. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, if $\lim\limits_{x \to a} g(x) \neq 0$.

Some properties of limits:

1) $\lim\limits_{x\to\infty} C\cdot x=\infty$; 2) $\lim\limits_{x\to\infty} \frac{C}{x}=0$;

3) $\lim\limits_{x\to\infty} \frac{x}{C}=\infty$; 4) $\lim\limits_{x\to 0} \frac{C}{x}=\infty$;

5) $\lim\limits_{x\to 0} C\cdot x=0$; 6) $\lim\limits_{x\to 0} \frac{x}{C}=0$.

6. The 1-st and the 2-nd remarkable limits and their consequences:

1) the 1-st remarkable limit:

$$\lim_{x\to 0}\frac{\sin x}{x}=\left\|\frac{0}{0}\right\|=1,$$

2) the 2-nd remarkable limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\left\|1^\infty\right\|=e,$$

where $e\approx 2{,}718$,

$$\lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}}=\left\|1^\infty\right\|=e.$$

Consequences from the 1-st remarkable limit (infinitesimals)	*Consequences from the 2-nd remarkable limit (infinitesimals)*
1. $\arcsin x \underset{x\to 0}{\sim} x$	1. $e^x-1 \underset{x\to 0}{\sim} x$
2. $tgx \underset{x\to 0}{\sim} x$	2. $a^x-1 \underset{x\to 0}{\sim} x\ln a$
3. $arctgx \underset{x\to 0}{\sim} x$	3. $\ln(1+x) \underset{x\to 0}{\sim} x$

7. Techniques of calculations of limits.

At calculating limits of a function we use the rule of limiting transition under the sign of a continuous function. This rule is formulated as follows:

$$\lim_{x\to a} f(x)=f\left(\lim_{x\to a} x\right).$$

All elementary functions are *continuous* in their domains of definition.

While calculating limits, first of all, we have to replace an argument of a function by its limiting value and find out whether there is indetermination.

Indeterminate forms expressions are the following:

$$\left\|\frac{0}{0}\right\|, \left\|\frac{\infty}{\infty}\right\|, \left\|\infty \cdot 0\right\|, \left\|\infty - \infty\right\|, \left\|\infty^{0}\right\|, \left\|0^{0}\right\|, \left\|1^{\infty}\right\|.$$

If after the substitution of a limiting value of an argument we obtain an *indeterminate form,* then we should carry out some identical transformations that will *eliminate the indetermination* and then the sought limit is calculated.

Let us sequentially consider the standard cases of evaluation of the indefinite expressions.

8. Calculation of a limit of a rational function.

$\lim\limits_{x \to a} \dfrac{P(x)}{Q(x)}$, where $P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$ and $Q(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_{m-1} x + b_m$ are polynomials of the orders *n* and *m*, $n, m \in N$.

1. $\lim\limits_{x \to a} \dfrac{P(x)}{Q(x)} = \left\|\dfrac{0}{0}\right\|$

To eliminate the indetermination let us carry out identical transformations, i.e. to pick out in the numerator and the denominator the factor approaching *0,* namely $(x - a)$.

In this case we should remember the following:

1) The consequence of the *Bezout's theorem:* if a is the root of the polynomial $P_n(x)$, i.e. $P_n(a) = 0$, then $P_n(x)$ is divided by the binomial $(x - a)$ without a remainder:

$$P_n(x) = (x - a) \cdot R_{n-1}(x)$$

2) The square trinomial $P_2(x) = ax^2 + bx + c$, where $D \geq 0$ ($D = b^2 - 4ac$ is the discriminant of $ax^2 + bx + c = 0$), can be presented as a product of linear factors:

$$ax^2 + bx + c = a(x - x_1)(x - x_2),$$

where x_1 and x_2 are roots of the square trinomial.

3) Vieta theorem

$$x_1 + x_2 = -\frac{b}{a}, \qquad\qquad x_1 \cdot x_2 = \frac{c}{a}.$$

Example 1. Let's calculate $\lim\limits_{x\to 1} \dfrac{x^4-1}{2x^3-x^2-1} = \left\| \dfrac{1-1}{2-1-1} = \dfrac{0}{0} \right\|$

Solution. In this case we have to eliminate the indetermination, i.e. to pick out in the numerator and the denominator the factor tending to zero. It is obvious that $x=1$ is the root of the polynomials in the numerator and the denominator.

To do this let us present the numerator as $x^4 - 1 = \left(x^2\right)^2 - 1^2 = \left(x^2+1\right)\left(x^2-1\right) = \left(x^2+1\right)(x-1)(x+1)$. By the consequence of Bezout's theorem the denominator is divided by $(x-1)$ without the remainder:

$$\begin{array}{r|l} -\;\; 2x^3 - x^2 - 1 & x-1 \\ \underline{2x^3 - 2x^2} \quad\;\; & \overline{2x^2 + x + 1} \\ x^2 - 1 & \\ -\;\; \underline{x^2 - x} & \\ x - 1 & \\ -\;\; \underline{x - 1} & \\ 0 & \end{array}$$

Now the denominator can be presented as

$$2x^3 - x^2 - 1 = \left(2x^2 + x + 1\right)(x-1).$$

Thus in the numerator and the denominator we have picked out the factor $(x-1)$, tending to 0 as $x \to 1$. So we can finish calculation of the given limit, i.e.

$$\lim_{x\to 1} \frac{x^4-1}{2x^3-x^2-1} = \left\| \frac{1-1}{2-1-1} = \frac{0}{0} \right\| = \lim_{x\to 1} \frac{\left(x^2+1\right)(x-1)(x+1)}{\left(2x^2+x+1\right)(x-1)} =$$

$$= \lim_{x\to 1} \frac{\left(x^2+1\right)(x+1)}{2x^2+x+1} = \left\| \frac{(1+1)(1+1)}{2+1+1} = \frac{2\cdot 2}{4} \right\| = 1.$$

Example 2. Let's calculate $A = \lim\limits_{x\to 2} \dfrac{x^2-6x+8}{x^2-8x+12} = \left\| \dfrac{2^2-6\cdot 2+8}{2^2-8\cdot 2+12} = \dfrac{0}{0} \right\|$

Solution. In this case we have to eliminate the indetermination, i.e. to pick out in the numerator and the denominator the factor tending to zero. It is obvious that $x = 2$ is the root of the polynomials in the numerator and the denominator.

Let's find roots of the square trinomial $ax^2 + bx + c = 0$ and present it as

$$ax^2 + bx + c = a(x - x_1)(x - x_2),$$

where

$$D = b^2 - 4ac,$$

$$x_1 = \frac{-b + \sqrt{D}}{2a}, \quad x_2 = \frac{-b - \sqrt{D}}{2a}.$$

Let's find roots of the polynomials in the numerator and the denominator:

$$x^2 - 6x + 8 = 0$$

$$D = b^2 - 4ac = (-6)^2 - 4 \cdot 1 \cdot 8 = 36 - 32 = 4$$

$$x_1 = \frac{-b + \sqrt{D}}{2a} = \frac{-(-6) + \sqrt{4}}{2 \cdot 1} = \frac{6 + 2}{2} = \frac{8}{2} = 4$$

$$x_2 = \frac{-b - \sqrt{D}}{2a} = \frac{-(-6) - \sqrt{4}}{2 \cdot 1} = \frac{6 - 2}{2} = \frac{4}{2} = 2$$

$$a(x - x_1)(x - x_2) = (x - 4)(x - 2)$$

$$x^2 - 8x + 12 = 0$$

$$D = b^2 - 4ac = (-8)^2 - 4 \cdot 1 \cdot 12 = 64 - 48 = 16$$

$$x_1 = \frac{-b + \sqrt{D}}{2a} = \frac{-(-8) + \sqrt{16}}{2 \cdot 1} = \frac{8 + 4}{2} = \frac{12}{2} = 6$$

$$x_2 = \frac{-b - \sqrt{D}}{2a} = \frac{-(-8) - \sqrt{16}}{2 \cdot 1} = \frac{8 - 4}{2} = \frac{4}{2} = 2$$

$$a(x - x_1)(x - x_2) = (x - 6)(x - 2)$$

Let's substitute into the sought limit:

$$A = \lim_{x \to 2} \frac{(x-2)(x-4)}{(x-2)(x-6)} = \lim_{x \to 2} \frac{x-4}{x-6} = \frac{2-4}{2-6} = \frac{-2}{-4} = \frac{1}{2}$$

2) $\lim\limits_{x\to\infty}\dfrac{P(x)}{Q(x)}=\left\|\dfrac{\infty}{\infty}\right\|$

We pick out x to the greatest power in the numerator and the denominator and reduce the obtained fraction. After this we substitute $x=\infty$ into the limit and use the following limit

$$\lim_{x\to\infty}\frac{C}{x^{\alpha}}=0.$$

In this case it is useful to note that if $x\to\infty$, then

$a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n\sim a_0x^n$ (*an infinitude value*)

$b_0x^m+b_1x^{m-1}+\ldots+b_{m-1}x+b_m\sim b_0x^m$ (*an infinitude value*)

Thus, $\lim\limits_{x\to\infty}\dfrac{P_n(x)}{Q_m(x)}=\begin{cases}0, & if \quad n<m\\ \dfrac{a_0}{b_0}, & if \quad n=m\\ \infty, & if \quad n>m\end{cases}.$

Example 3. Let’s calculate $\lim\limits_{x\to\infty}\dfrac{3x^3+4x^2+7}{2x^3-3x^2+11}$.

Solution. Let’s replace x by $x=\infty$.

$$\lim_{x\to\infty}\frac{3x^3+4x^2+7}{2x^3-3x^2+11}=\left\|\frac{\infty}{\infty}\right\|=$$

Let’s eliminate the indetermination $\left\|\dfrac{\infty}{\infty}\right\|$. For this we pick out the factor x to the greatest power in the numerator and the denominator, i.e. x^3.

$$=\lim_{x\to\infty}\frac{x^3\left(3+\frac{4}{x}+\frac{7}{x^3}\right)}{x^3\left(2-\frac{3}{x}+\frac{11}{x^3}\right)}=\lim_{x\to\infty}\frac{3+\frac{4}{x}+\frac{7}{x^3}}{2-\frac{3}{x}+\frac{11}{x^3}}=$$

Let’s use the following limit

$$\lim_{x\to\infty}\frac{C}{x^{\alpha}}=0.$$

We have

$$= \lim_{x\to\infty} \frac{3+\frac{4}{x}+\frac{7}{x^3}}{2-\frac{3}{x}+\frac{11}{x^3}} = \frac{3+0+0}{2-0+0} = \frac{3}{2}$$

Let’s calculate this limit using equivalencies:

$$\lim_{x\to\infty} \frac{3x^3+4x^2+7}{2x^3-3x^2+11} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to\infty} \frac{3x^3}{2x^3} = \frac{3}{2}.$$

In this case we have the indetermination $\frac{\infty}{\infty}$. To eliminate of this indetermination we pick out in the numerator and the denominator *the factor x to the greatest power*, i.e. x^3. Then we divide the numerator and the denominator by x^3 and calculate the limit.

Here $n = m = 3$.

Example 4. Let’s calculate $\lim\limits_{x\to\infty} \frac{x^{10}+7x^3+5x+1}{x^8+2x^4-8x}$.

Solution. Let’s replace x by $x = \infty$.

$$\lim_{x\to\infty} \frac{x^{10}+7x^3+5x+1}{x^8+2x^4-8x} = \left\|\frac{\infty}{\infty}\right\| =$$

Let’s eliminate the indetermination $\left\|\frac{\infty}{\infty}\right\|$. For this we pick out the factor *x* to the greatest power in the numerator and the denominator, i.e. x^{10}.

$$= \lim_{x\to\infty} \frac{x^{10}\cdot\left(1+\frac{7}{x^7}+\frac{5}{x^9}+\frac{1}{x^{10}}\right)}{x^{10}\cdot\left(\frac{1}{x^2}+\frac{2}{x^6}-\frac{8}{x^9}\right)} =$$

$$= \lim_{x\to\infty} \frac{1+\frac{7}{x^7}+\frac{5}{x^9}+\frac{1}{x^{10}}}{\frac{1}{x^2}+\frac{2}{x^6}-\frac{8}{x^9}} = \frac{1+0+0+0}{0+0-0} = \frac{1}{0} = \infty$$

Here $n = 10, m = 8$ and $n > m$.

Homework. Calculate this limit: $\lim\limits_{x\to\infty}\dfrac{x^5-7x^3+11}{x^9-x^4+25}$.

Here $n=5, m=9$ and $n<m$.

9. Calculation of limits of functions with irrational expressions

While calculating the limits of functions which have an irrational expression, which vanishes as $x\to a$ we should pick out the factor $x-a\to 0$. We can do it by getting rid of irrationality in the numerator and the denominator by multiplying the given fraction by the correspondent conjugate factor. At that the following formula is often used:

$$a^2-b^2=(a-b)(a+b), \qquad a^3\pm b^3=(a\pm b)\left(a^2\mp ab+b^2\right).$$

Example Let's calculate $\lim\limits_{x\to 8}\dfrac{\sqrt{9+2x}-5}{x^2-6x-16}$

Solution. We have

$$\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{x^2-6x-16}=\left\|\frac{0}{0}\right\|$$

As $x\to 8$ then $x-8\to 0$. Let us pick out the factor $(x-8)$ in the numerator and the denominator.

The numerator has the irrational function $\sqrt{9+2x}-5$ or $\sqrt{9+2x}-5=a-b$.

To get rid of irrationality in the numerator and the denominator we multiply the given fraction by the correspondent conjugate factor $a+b=\sqrt{9+2x}+5$.

We obtain the product $(a-b)(a+b)$. It's $(a-b)(a+b)=a^2-b^2$

Thus in the numerator we get the following expression:

$$\left(\sqrt{9+2x}+5\right)\left(\sqrt{9+2x}-5\right).$$

Let us present the denominator as the following product: $x^2-6x-16=(x-8)(x+2)$. Thus calculation of the given limit is as follows:

$$\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{x^2-6x-16}=\left\|\frac{0}{0}\right\|=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}-5\right)\left(\sqrt{9+2x}+5\right)}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$=\lim_{x\to 8}\frac{\left(\sqrt{9+2x}\right)^2-5^2}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=\lim_{x\to 8}\frac{9+2x-25}{\left(x^2-6x-16\right)\left(\sqrt{9+2x}+5\right)}=$$

$$= \lim_{x\to 8} \frac{2x-16}{(x-8)(x+2)\left(\sqrt{9+2x}+5\right)} = \lim_{x\to 8} \frac{2(x-8)}{(x-8)(x+2)\left(\sqrt{9+2x}+5\right)} =$$

$$= \frac{2}{(8+2)\cdot\left(\sqrt{9+2\cdot 8}+5\right)} = \frac{2}{10\cdot 10} = \frac{1}{50}.$$

3) While disclosing indeterminations like $\left\|\infty-\infty\right\|$ we have to carry out identical transformations allowing to reduce this indetermination to $\left\|\frac{0}{0}\right\|$ or $\left\|\frac{\infty}{\infty}\right\|$. For example, in the case of a low-level index of root we can do it by multiplying and dividing the given expression by the «conjugate» expression.

Example 6. Let's calculate $\lim\limits_{x\to\infty}\left(\sqrt{x^2+3x-2}-x\right)$.

Solution. Solution. Let's replace x by $x=\infty$.

$$\lim_{x\to\infty}\left(\sqrt{x^2+3x-2}-x\right) = \left\|\infty-\infty\right\| =$$

We multiply and divide the given expression by the «conjugate» expression.

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+3x-2}-x\right)\left(\sqrt{x^2+3x-2}+x\right)}{\left(\sqrt{x^2+3x-2}+x\right)} =$$

$$= \lim_{x\to\infty} \frac{x^2+3x-2-x^2}{\sqrt{x^2+3x-2}+x} = \lim_{x\to\infty} \frac{3x-2}{\sqrt{x^2+3x-2}+x} = \left\|\frac{\infty}{\infty}\right\| =$$

$$= \lim_{x\to\infty} \frac{x\cdot\left(3-\frac{2}{x}\right)}{\sqrt{x^2\cdot\left(1+\frac{3}{x}-\frac{2}{x^2}\right)}+x} = \lim_{x\to\infty} \frac{x\cdot\left(3-\frac{2}{x}\right)}{x\cdot\left(\sqrt{1+\frac{3}{x}-\frac{2}{x^2}}+1\right)} =$$

$$= \lim_{x\to\infty} \frac{3-\frac{2}{x}}{\sqrt{1+\frac{3}{x}-\frac{2}{x^2}}+1} = \frac{3-0}{\sqrt{1+0-0}+1} = \frac{3}{1+1} = \frac{3}{2}.$$

10. Calculating limits of functions using the 1-st and the 2-nd remarkable limits and their consequences

1) The 1-st remarkable limit:

$$\lim_{x\to 0}\frac{\sin x}{x}=\left\|\frac{0}{0}\right\|=1$$

To disclose indeterminations like $\left\|\frac{0}{0}\right\|$ we should use the 1-st remarkable limit, carry out the elementary transformations with the numerator and the denominator and apply the trigonometrical formulas.

Consequences from the first remarkable limit:

1. $\lim\limits_{x\to 0}\frac{x}{\sin x}=1;$

2. $\lim\limits_{x\to 0}\frac{\arcsin x}{x}=1;$

3. $\lim\limits_{x\to 0}\frac{\operatorname{tg} x}{x}=1;$

4. $\lim\limits_{x\to 0}\frac{\operatorname{arctg} x}{x}=1.$

$$f(x)\to 0$$

$\lim\limits_{x\to 0}\frac{\sin f(x)}{f(x)}=\left\|\frac{0}{0}\right\|=1,$

6. $\lim\limits_{x\to 0}\frac{\arcsin f(x)}{f(x)}=\left\|\frac{0}{0}\right\|=1$

7. $\lim\limits_{x\to 0}\frac{tg\ f(x)}{f(x)}=\left\|\frac{0}{0}\right\|=1,$

8. $\lim\limits_{x\to 0}\frac{arctg\ f(x)}{f(x)}=\left\|\frac{0}{0}\right\|=1$

Example 7. Let’s calculate $\lim\limits_{x\to 0}\frac{\sin 5x}{x}$

Solution. $\lim\limits_{x\to 0}\frac{\sin 5x}{x}=\left\|\frac{0}{0}\right\|=$

We apply the consequence $\lim\limits_{x\to 0}\frac{\sin f(x)}{f(x)}=\left\|\frac{0}{0}\right\|=1$. Here $f(x)=5x$.

We obtain

$$=\lim_{x\to 0}\underbrace{\frac{\sin 5x}{5x}}_{\to 1}\cdot 5=5$$

2) The 2-nd remarkable limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\left\|1^{\infty}\right\|=e,$$

where $e\approx 2{,}718$,

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \left\|1^{\infty}\right\| = e.$$

To disclose indeterminations like $\left\|1^{\infty}\right\|$ we should use the 2-nd remarkable limit and carry out transformations with the base and the exponent.

Example 8. Let's calculate $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x-4}\right)^{x}$.

Solution. Let's find the limit of the base:

$$\lim_{x\to\infty}\frac{x+2}{x-4} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to\infty}\frac{x\cdot\left(1+\frac{2}{x}\right)}{x\cdot\left(1-\frac{4}{x}\right)} = \lim_{x\to\infty}\frac{1+\frac{2}{x}}{1-\frac{4}{x}} = \frac{1+0}{1-0} = 1.$$

We obtain

$$\lim_{x\to\infty}\left(\frac{x+2}{x-4}\right)^{x} = \left\|1^{\infty}\right\| =$$

If in the given example we add and subtract from the base of the power, then the expressions will remain unchanged.

$$= \lim_{x\to\infty}\left(1+\frac{x+2}{x-4}-1\right)^{x} = \lim_{x\to\infty}\left(1+\frac{6}{x-4}\right)^{x} =$$

Let us carry out the identical transformation:

$$= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-4}{6}}\right)^{x} = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-4}{6}}\right)^{\frac{x-4}{6}\frac{6}{x-4}x} =$$

The limit of the base in the obtained expression is the value *e*:

$$= \lim_{x\to\infty}\left(\underbrace{\left(1+\frac{1}{\frac{x-4}{6}}\right)^{\frac{x-4}{6}}}_{\to e}\right)^{\frac{6}{x-4}\cdot x} = \lim_{x\to\infty} e^{\frac{6}{x-4}\cdot x} =$$

The limit of the power is calculated as follows:

$$= e^{\lim\limits_{x\to\infty}\frac{6x}{x-4}} = \left\|\frac{\infty}{\infty}\right\| = e^{\lim\limits_{x\to\infty}\frac{6x}{x\cdot\left(1-\frac{4}{x}\right)}} =$$

$$= e^{\lim\limits_{x\to\infty}\frac{6}{1-\frac{4}{x}}} = e^{\frac{6}{1-0}} = e^{6}.$$

Consequences of the second remarkable limit:

1. $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab};$

2. $\lim\limits_{x\to 0}\left(1+ax\right)^{1/bx} = e^{a/b};$

3. $\lim\limits_{x\to 0}\frac{a^x-1}{x} = \ln a;$

4. $\lim\limits_{x\to\infty}\left(1+\frac{a}{bx}\right)^{x} = e^{a/b};$

$\lim\limits_{x\to 0}\left(1+ax\right)^{bx} = e^{ab};$

6. $\lim\limits_{x\to 0}\frac{e^x-1}{x} = 1.$

7. $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x} = 1;$

8. $\lim\limits_{x\to 0}\frac{a^x-1}{x} = \ln a;$

9. $\lim\limits_{x\to 0}\frac{(1+x)^m-1}{x} = m$

Theoretical questions

1. Give a definition of a function.
2. What do you call a domain of the function?
3. Give a definition of a range of the function values.
4. What variable is called independent?
5. What variable is called argument?
6. What function is called single valued?
7. What function is called multivalued?
8. What do you call a limit of a numerical sequence?
9. What do you call the infinitesimal function?
10. Write down main properties of infinitesimals.
11. What do you call the infinitude function?
12. Write down main properties of Infinitude values.
13. What is a limit of a function at $x \to x_0$?
14. Give a definition of one-sided limits.
15. Write down basic "rules-theorems" of calculation of limits.
16. Call all properties of limits.
17. What remarkable limits do you know?
18. Write down the 1-st remarkable limit and its consequences.
19. Write down the 2-nd remarkable limit and its consequences.

20. What do you call an indeterminate form?
21. What indeterminate forms do you know? Call all the types of indeterminate forms.
22. Give a definition a rational function.
23. Call the consequence of Bezout's theorem*:*
24. Describe the calculation of a limit of a rational function with the indeterminate form $\left\|\frac{0}{0}\right\|$.
25. Describe the calculation of a limit of a rational function with the indeterminate form $\left\|\frac{\infty}{\infty}\right\|$.
26. What do you call a conjugate factor?
27. Describe the calculation of limits of functions with irrational expressions.
28. Describe the calculation of limits using the 1-st remarkable limit and its consequences.

29. Describe the calculation of limits using the 2-nd remarkable limit and its consequences.