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Theme: Differential calculus of functions of many variables. Application of the gradient vector in the linear model of international trade. Integral calculus of functions of one variable

Part 3. The definite integral and its application

Economic problems (Example 1)

The mean value of a function on the interval $[a, b]$**:**

$$f(c) = \frac{1}{b-a} \cdot \int_a^b f(x)dx.$$

Example 1. Let us assume that the value of the UAH expressed in the US dollars from day 1 to day 30 of a certain month is represented by the function of time: $f(t) = t^2 + \frac{900}{t^3}$. Find the mean value of the UAH in this month.

Economic problems (Example 1)

Example 1. Let us assume that the value of the UAH expressed in the US dollars from day 1 to day 30 of a certain month is represented by the function of time: $f(t) = t^2 + \dfrac{900}{t^3}$. Find the mean value of the UAH in this month.

Solution. First of all, let's find the definite integral:

$$\int_1^{30}\left(x^2 + \frac{900}{x^3}\right)dx = \int_1^{30}\left(x^2 + 900\cdot x^{-3}\right)dx = \left(\frac{1}{3}x^3 - \frac{900}{2}x^{-2}\right)\Bigg|_1^{30} = \left(\frac{x^3}{3} - \frac{450}{x^2}\right)\Bigg|_{x=30} -$$

$$-\left(\frac{x^3}{3} - \frac{450}{x^2}\right)\Bigg|_{x=1} = \frac{30^3}{3} - \frac{450}{30^2} - \frac{1^3}{3} + \frac{450}{1^2} = 9000 - \frac{1}{2} - \frac{1}{3} + 450 \approx 9449.$$

Economic problems (Example 1)

Solution. First of all, let's find the definite integral:

$$\int_1^{30}\left(x^2+\frac{900}{x^3}\right)dx=\int_1^{30}\left(x^2+900\cdot x^{-3}\right)dx=\left(\frac{1}{3}x^3-\frac{900}{2}x^{-2}\right)\Bigg|_1^{30}=\left(\frac{x^3}{3}-\frac{450}{x^2}\right)\Bigg|_{x=30}-$$

$$-\left(\frac{x^3}{3}-\frac{450}{x^2}\right)\Bigg|_{x=1}=\frac{30^3}{3}-\frac{450}{30^2}-\frac{1^3}{3}+\frac{450}{1^2}=9000-\frac{1}{2}-\frac{1}{3}+450\approx 9449.$$

According to the formula of the mean value of a function on the given interval $[1, 30]$, we have:

$$f(c)=\frac{1}{30-1}\int_1^{30}\left(x^2+\frac{900}{x^3}\right)dx\approx\frac{1}{29}\cdot 9449\approx 325.8.$$

Thus, the mean value of the UAH in this month is approximately 325.8.

Economic problems (Example 2)

Example 2. Let us assume that the production costs $f(x)=10+20x+0.03x^2$ The volume of manufacture changes from 100 UAH to 300 UAH per year. Find the mean value of the cost of production.

Solution. According to the formula of the mean value of a function on the given interval $[100, 300]$, we have:

Economic problems (Example 2)

Example 2. Let us assume that the production costs $f(x)=10+20x+0.03x^2$ The volume of manufacture changes from 100 UAH to 300 UAH per year. Find the mean value of the cost of production.

Solution. According to the formula of the mean value of a function on the given interval $[100, 300]$, we have:

$$f(c)=\frac{1}{300-100}\cdot\int_{100}^{300}\left(10+20x+0,03x^2\right)dx=\frac{1}{200}\cdot\left(10x+20\cdot\frac{x^2}{2}+0,03\cdot\frac{x^3}{3}\right)\Bigg|_{100}^{300}=$$

$$=\frac{1}{200}\cdot\left[\left(10x+10x^2+0,01x^3\right)\Big|_{x=300}-\left(10x+10x^2+0,01x^3\right)\Big|_{x=100}\right]=$$

$$=\frac{1}{200}\cdot(3000+900000+270000-1000-100000-10000)=$$

$$=\frac{1}{200}\cdot(1173000-120000)=5865-600=5265.$$

Thus, the mean value of the cost of production is 5 265 UAH.

Economic problems (Example 3)

Example 3. The value of productivity of equipment at certain factory from the day of start (the first day) to the next day (the second day) is represented by the function of time $f(t) = 2^t$. Find the average value of productivity of this equipment.

Solution. a) In order to find the average value of productivity of equipment we will calculate the average value of the given function $f(t) = 2^t$ on the interval $[0, 2]$. Due to the formula of the mean value of a function on the interval:

$$f(c) = \frac{1}{2-1}\int_1^2 2^x dx = \left.\frac{2^x}{\ln 2}\right|_1^2 = \frac{2^2}{\ln 2} - \frac{2^1}{\ln 2} = \frac{2}{\ln 2} \approx \frac{2}{0.6931472} \approx 2.88539.$$

Thus, the mean value of the cost of productivity of the given equipment is approximately 2.88539.

Economic problems (Example 4)

Example 4. The Lorenz curve of the income distribution within a certain group is given by the formula: $L(x)=0.8x^2+0.28x$. Determine the degree of equality of the income distribution.

Solution. Let's find the Gini coefficient

$$G=2\int_0^1\left(x-(0.8x^2+0.28x)\right)dx=2\int_0^1\left(-0.8x^2+0.72x\right)dx=$$

$$=2\left(-0.8\cdot\frac{x^{2+1}}{2+1}+0.72\cdot\frac{x^{1+1}}{1+1}\right)\Bigg|_0^1=2\left(-\frac{4}{15}\cdot x+0.36\cdot x^2\right)\Bigg|_0^1=$$

$$=2\left(-\frac{4}{15}\cdot x+0.36\cdot x^2\right)\Bigg|_{x=1}-2\left(-\frac{4}{15}\cdot x+0.36\cdot x^2\right)\Bigg|_{x=0}=2\left(-\frac{4}{15}+0.36\right)\approx 0.2.$$

Economic problems (Example 6)

Example 6. A firm's monthly marginal cost of a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ of the given month if:

a) $MC(x) = x + 30$ and the level of production is raised from 4 units to 6 units;

b) $MC(x) = 3(x-5)^2$ and the level of production is raised from 6 units to 10 units.

Economic problems (Example 7)

Example 7. The laws of supply and demand have the form:

$$p = 186 - x^2 \qquad p = 20 + \frac{11}{6}x$$

a) Find the point of market equilibrium.

b) Find consumers' benefit under the condition of establishment of market equilibrium.

c) Find suppliers' benefit under the condition of establishment of market equilibrium.

Economic problems (Example 7)

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$$\begin{cases} p = 186 - x^2 \\ p = 20 + \dfrac{11}{6}x \end{cases}$$

Economic problems (Example 7)

a) Find the point of market equilibrium.

$$\begin{cases} p = 186 - x^2 \\ p = 20 + \dfrac{11}{6}x \end{cases} \Rightarrow x_0 = 12,\ p_0 = 42$$

Economic problems (Example 7)

b) Find consumers' benefit under the condition of establishment of market equilibrium.

$$C = \int_0^{x_0} p_c(x)dx - p_0x_0 = \int_0^{12}\left(186 - x^2\right)dx - 12\cdot 42 = \left(186x - \frac{x^3}{3}\right)\Bigg|_0^{12} - 504 = 1152$$

(units of money)

Economic problems (Example 7)

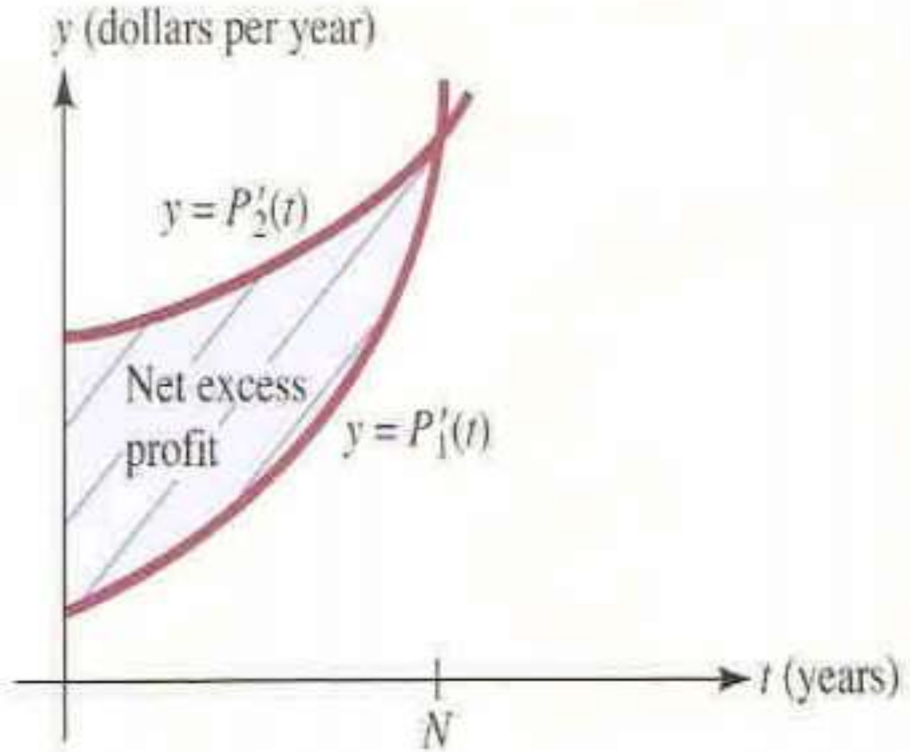
c) Find suppliers' benefit under the condition of establishment of market equilibrium.

$$P = p_0 x_0 - \int_0^{x_0} p_s(x)dx = 12 \cdot 42 - \int_0^{12} \left(20 + \frac{11}{6} x \right) dx = 504 - \left(20x - \frac{11}{6} \frac{x^2}{2} \right) \Bigg|_0^{12} = 132$$

(units of money)

Economic problems

Net Excess Profit



Net excess profit as the area between rate of profitability curves.

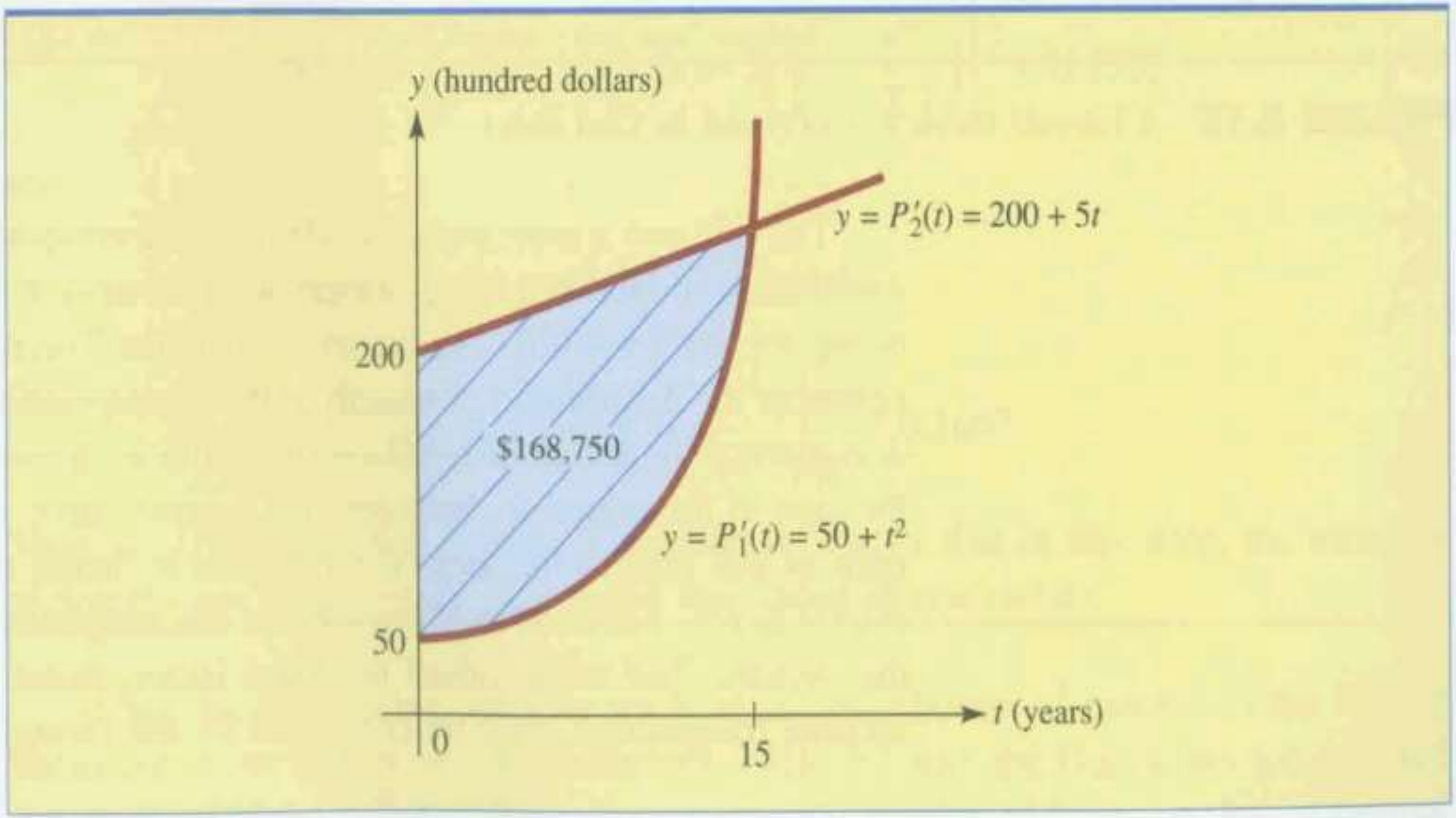
Economic problems

Example

- Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=50+t^2$ hundred dollars per year, while a second investment will be generating profit at the rate of $P_2'(t)=200+5t$ hundred dollars per year.

a. For how many years does the rate of profitability of the second investment exceed that of the first?
b. Compute the net excess profit for the time period determined in part (a). Interpret the net excess profit as an area.

Economic problems



Net excess profit for one investment plan over another.

Economic problems

A manufacturer has found that marginal cost is $3q^2 - 60q + 400$ dollars per unit when q units have been produced.

The total cost of producing the first 2 units is $900.

Q: What is the total cost of producing the first 5 units?

1. Definite integral.
Formula of Newton-Leibnitz

Let the function $f(x)$ be determined on the interval $[a,b]$

Let us divide the interval $[a,b]$ into *n* subintervals by *n* points

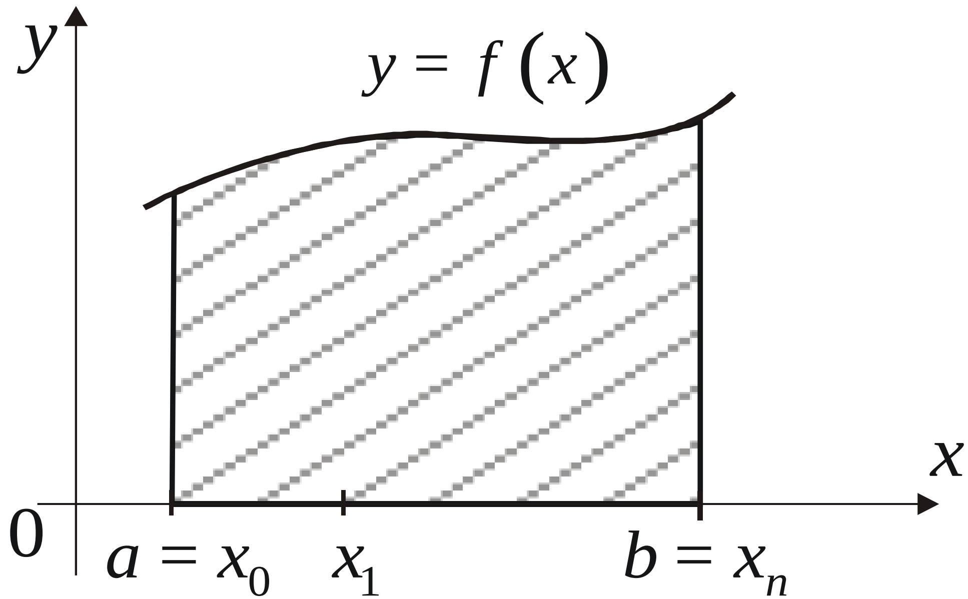
$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$

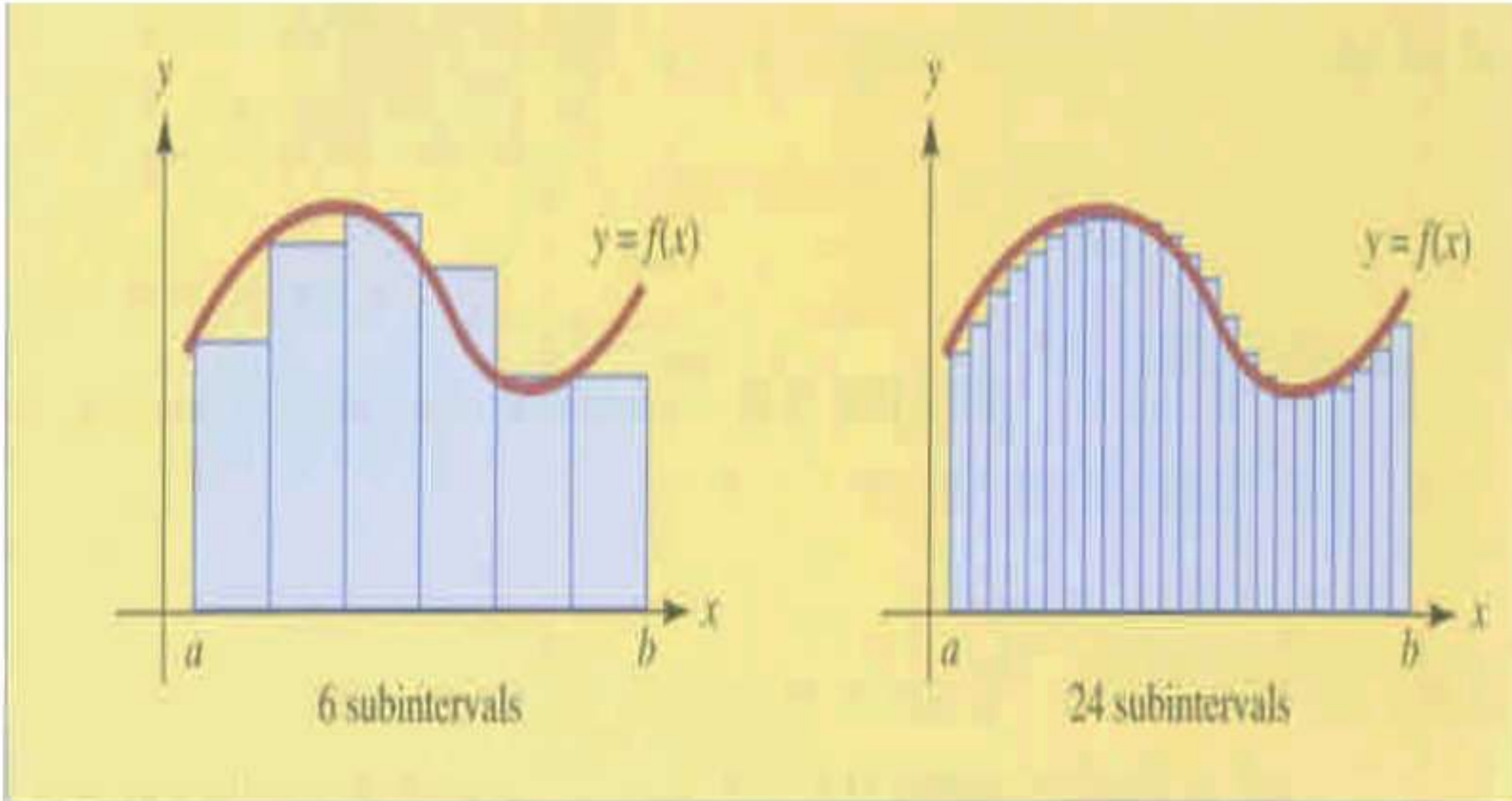
On each subinterval $[x_{i-1}, x_i]$ we choose some point and compound the sum

$$S_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + ... + f(\xi_n)\Delta x_n$$

where

$\Delta x_i = x_i - x_{i-1}$ is the length of each interval.

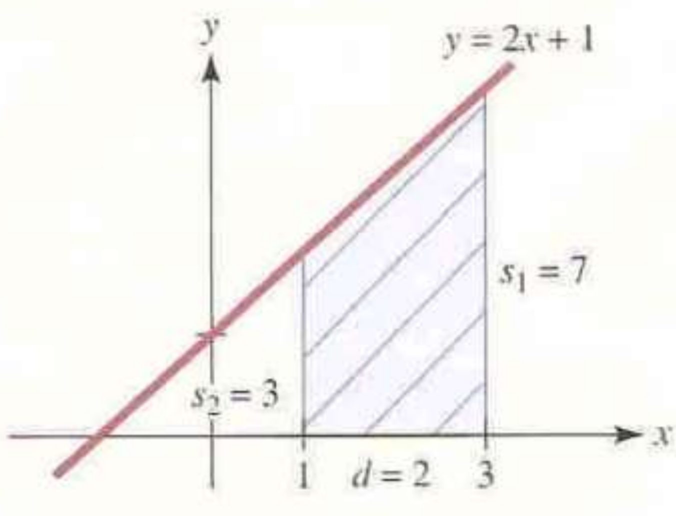

y
y = f (x)
x
0
a = x0
x1
b = xn



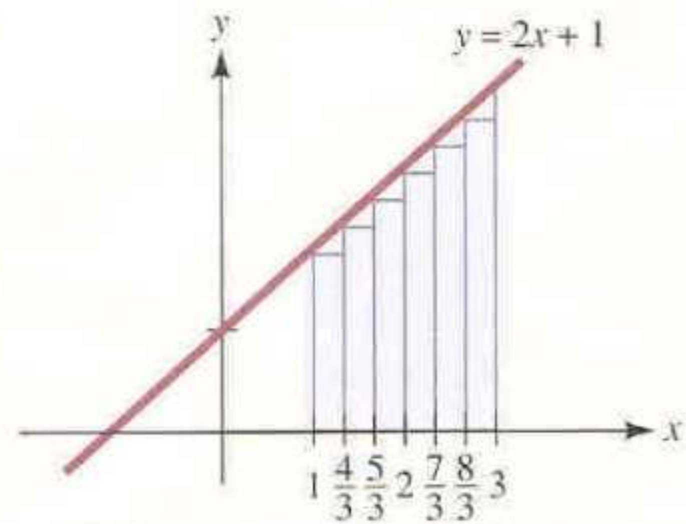
The approximation improves as the number of subintervals increases

Q: Find any difference between these two figures?

Let R be the region under the graph of $f(x) = 2x + 1$ over the interval $1 \leqq x \leqq 3$, as shown if Figure Compute the area of R as the limit of a sum.



(a) The region R under the line $y = 2x + 1$ over $1 \leq x \leq 3$ (a trapezoid)



(b) The region R subdivided by six approximating rectangles

Approximating the area under a line with rectangles

The definite integral of the function $f(x)$ on the interval $[a,b]$ is called the limit

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)\,dx$$

where ***a*** is the lower limit, ***b*** is the upper limit of integration.

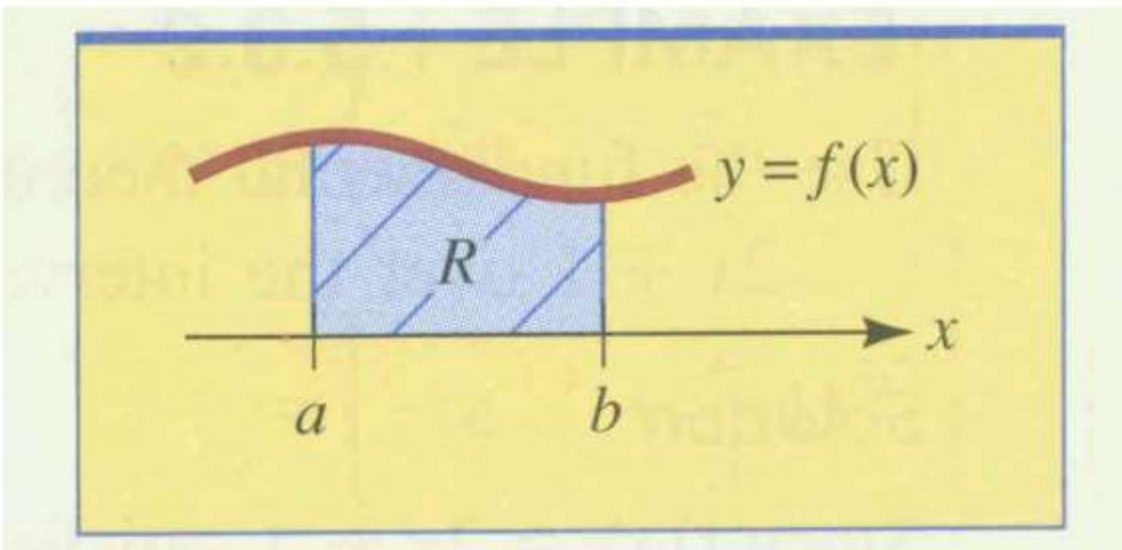
If $F(x)$ is an antiderivative for a continuous function $f(x)$ on the interval $[a,b]$ then *the formula of Newton-Leibnitz* is true

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Area as a Definite Integral

If $f(x)$ is continuous and $f(x) \geq 0$ on the interval $a \leq x \leq b$, Then the region R under the curve $y = f(x)$ over the interval $a \leq x \leq b$ has area given by the definite integral

$$\int_a^b f(x)dx$$



Fundamental Theorem of Calculus

If the function $f(x)$ is continuous on the interval $a \leq x \leq b$, then

$$\int_a^b f(x)dx = \mathrm{F}(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$ on $a \leq x \leq b$.

Example 1. Calculate the definite integral: $\int_1^2 \left(x^2 - 3x + 1\right) dx.$

Solution.

$$\int_1^2 \left(x^2 - 3x + 1\right) dx = \int_1^2 x^2 dx - 3\int_1^2 x dx + \int_1^2 dx = \frac{x^3}{3}\Bigg|_1^2 - 3\cdot\frac{x^2}{2}\Bigg|_1^2 +$$

$$+x\Big|_1^2 = \frac{8}{3} - \frac{1}{3} - \frac{3}{2}\left(4-1\right) + 2 - 1 = \frac{7}{3} - \frac{9}{2} + 1 = -\frac{7}{6}.$$

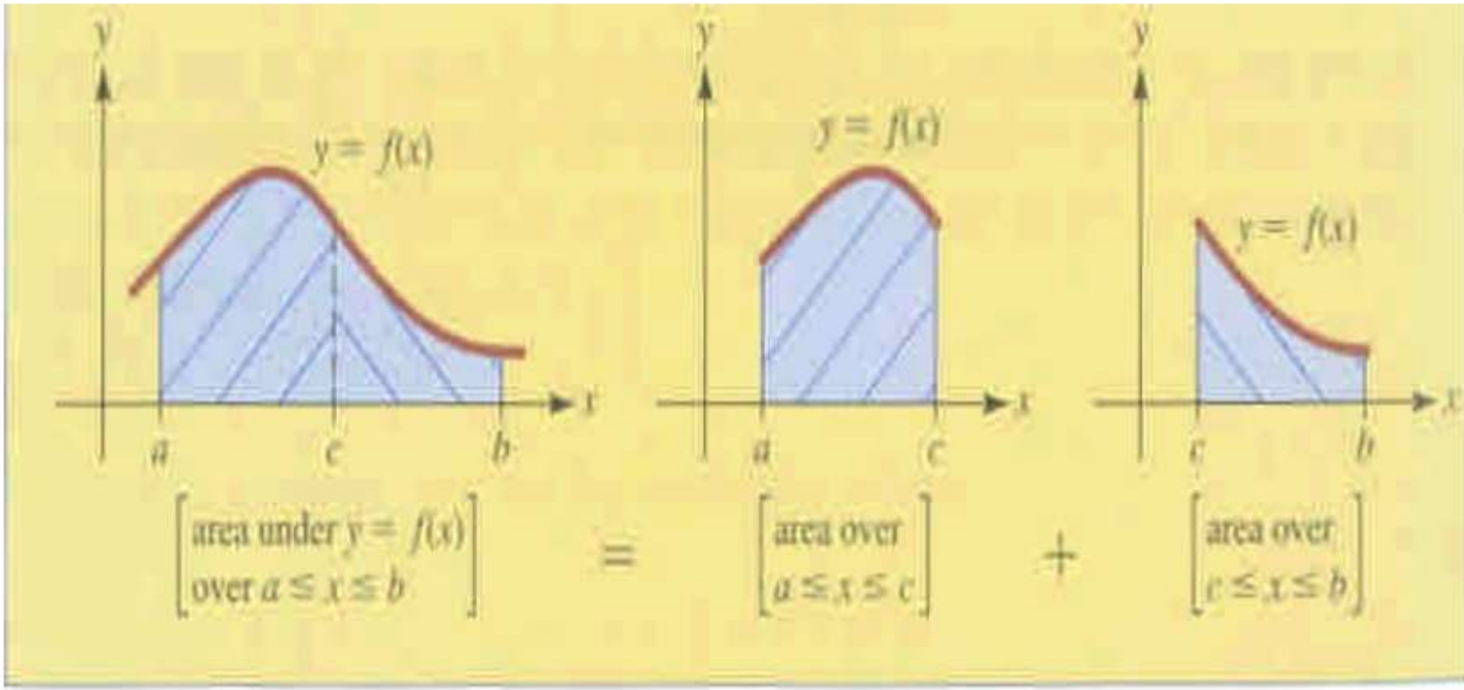
The basic properties of *a definite integral:*

1) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2) $\int_a^a f(x)\,dx = 0.$

The basic properties of *a definite integral:*

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad c \in [a, b]$



The subdivision rule for definite integrals (case where $f(x) \geq 0$).

The basic properties of *a definite integral:*

1) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2) $\int_a^a f(x)\,dx = 0.$

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad c \in [a, b]$

4) $\int_a^b \left(f_1(x) \pm f_2(x) \right) dx = \int_a^b f_1(x)\,dx \pm \int_a^b f_2(x)\,dx$

5) $\int_a^b C \cdot f(x)\,dx = C \cdot \int_a^b f(x)\,dx$ where **C** is a constant

6) If functions $f(x)$ and $\varphi(x)$ on the interval $[a,b]$, where $a<b$ satisfy the condition $f(x)\le\varphi(x)$, then

$$\int_a^b f(x)dx \le \int_a^b \varphi(x)dx$$

7) Estimation of the definite integral. If *m* and *M* are the least and the greatest values of the function $f(x)$ on the interval $[a,b]$ i.e. $m\le f(x)\le M$, then

$$m(b-a)\le\int_a^b f(x)dx \le M(b-a)$$

8) Theorem of the mean. If the function $y = f(x)$ is continuous on the interval $[a,b]$, where $a < b$, then there exists such

a value that $\xi \in [a,b]$

$$\int_a^b f(x)dx = f(\xi)\cdot(b-a)$$

$$f_{mean}(x) = f(\xi) = \frac{1}{b-a}\cdot\int_a^b f(x)dx$$

9) $\int_{-a}^{a} f(x)dx = 0$, where $y = f(x)$ is an odd function.

10) $\int_{-a}^{a} f(x)dx = 2\cdot\int_0^a f(x)dx$, where $y = f(x)$ is an even function.

2. Rules of definite integral calculation

1) INTEGRATION BY PARTS

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

where $u = u(x)$ and $v = v(x)$ are continuous differential functions on the interval $[a,b]$ and

$$uv\Big|_a^b = u(b)v(b) - u(a)v(a)$$

Example 2. Calculate $\int_0^1 xe^{-x}dx$

Solution. Let's use the method of integration by parts:

$$u = x \qquad dv = e^{-x}dx$$

then

$$du = dx \qquad v = \int e^{-x}dx = -e^{-x}$$

We obtain

$$\int_0^1 xe^{-x}dx = x\left(-e^{-x}\right)\Big|_0^1 - \int_0^1 \left(-e^{-x}\right)dx = -e^{-1} + \int_0^1 e^{-x}dx =$$

$$= -\frac{1}{e} - e^{-x}\Big|_0^1 = -\frac{1}{e} - \left(e^{-1} - e^{0}\right) = -\frac{1}{e} - \frac{1}{e} + 1 = \frac{e-2}{e}.$$

2) THE SUBSTITUTION OF A VARIABLE

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt$$

where $x = \varphi(t)$ and $\varphi'(t)$ are continuous functions on the interval $[\alpha, \beta]$, $\varphi(\alpha) = a$, $\varphi(\beta) = b$.

Example 3. Calculate $\int_0^1 \frac{dx}{x^2 + 4x + 5}$.

Solution. Let’s allocate the perfect square

$$x^2 + 4x + 5 = x^2 + 2\cdot 2x + 4 + 1 = (x+2)^2 + 1.$$

$$x^2+4x+5=x^2+2\cdot 2x+4+1=\left(x+2\right)^2+1.$$

Let's substitute $x+2=t, \quad dx=dt.$

Let's change limits of integration:

if $x=0,$ then $t=2$;

if $x=1$ then $t=3.$

We have

$$\int_0^1 \frac{dx}{x^2+4x+5}=\int_0^1 \frac{dx}{\left(x+2\right)^2+1}=\int_2^3 \frac{dt}{t^2+1}=arctgt\Big|_2^3=$$

$$=arctg3-arctg2.$$

HOMEWORK

Let *f(x)* and *g(x)* be functions that are continuous on the interval $-2 \le x \le 5$ and that satisfy

$$\int_{-2}^{5} f(x)dx = 3 \quad \int_{-2}^{5} g(x)dx = -4 \quad \int_{3}^{5} g(x)dx = 7$$

Use this information to evaluate each of these definite integrals:

a. $\int_{-2}^{5} [2f(x) - 3g(x)]dx$ b. $\int_{-2}^{3} f(x)dx$

$$\int_{0}^{1} 8x(x^2 + 1)^3 dx$$

$$\int_{1/4}^{2} \left(\frac{\ln x}{x} \right) dx$$

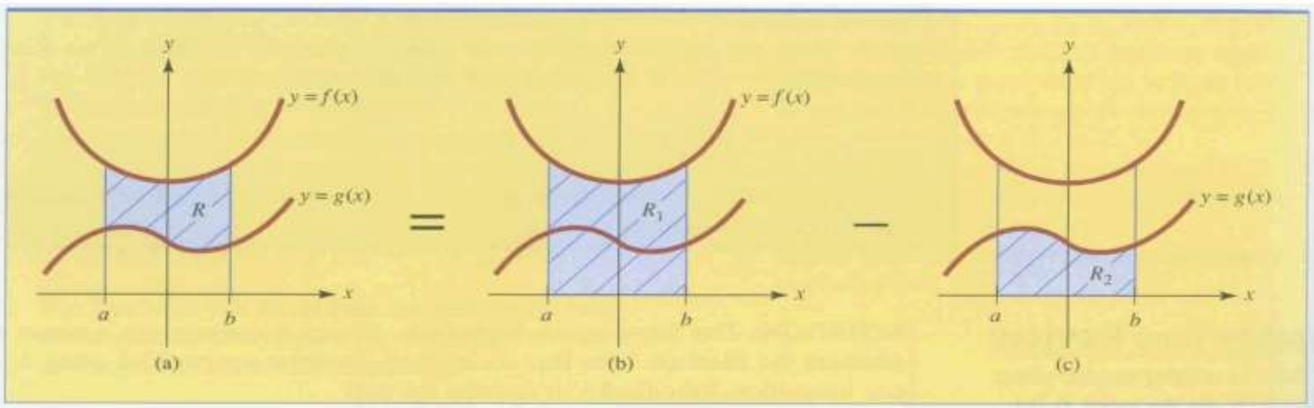
3. Application of definite integral for geometry problems.

Calculation of the area of a plane figure

The Area Between Two Curves

If $f(x)$ and $g(x)$ are continuous with $f(x) \geq g(x)$ on the interval $a \leq x \leq b$, then the area A between the curves $y = f(x)$ and $y = g(x)$ over the interval is given by

$$A = \int_a^b [f(x) - g(x)]dx$$



Area of R = area of R_1 − area of R_2.

The Area Between Two Curves

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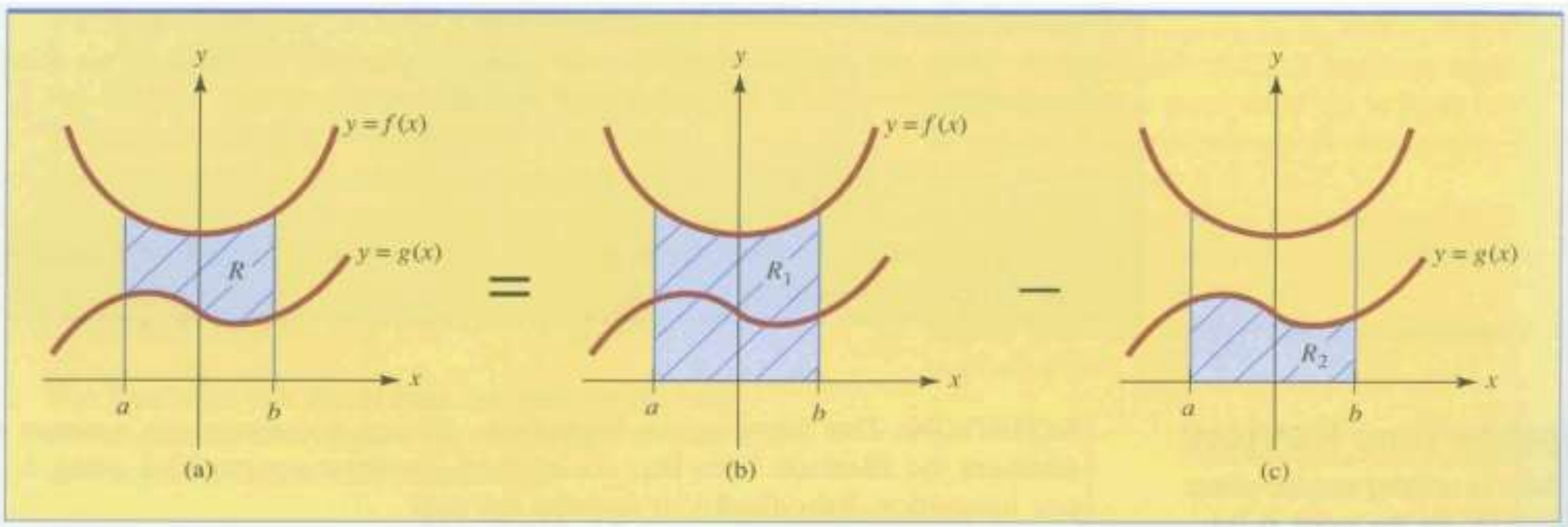


Area of R = area of R_1 − area of R_2.

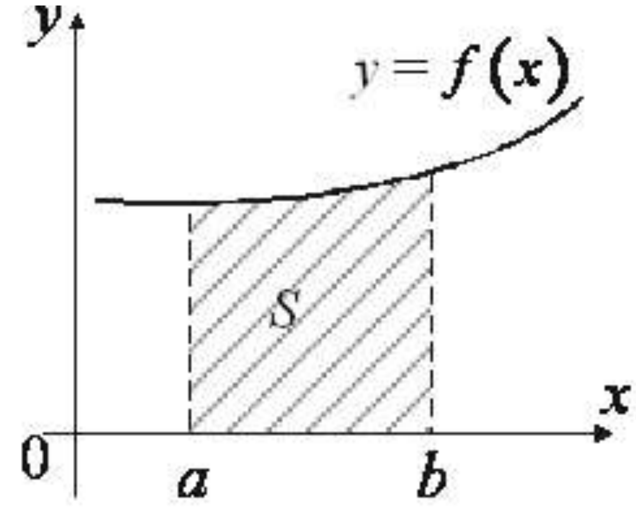
3. Application of definite integral for geometry problems.

Calculation of the area of a plane figure

If $f(x) \geq 0 \quad \forall\, x \in [a,b]$ then $\int\limits_a^b f(x)dx$ is numerically equal to

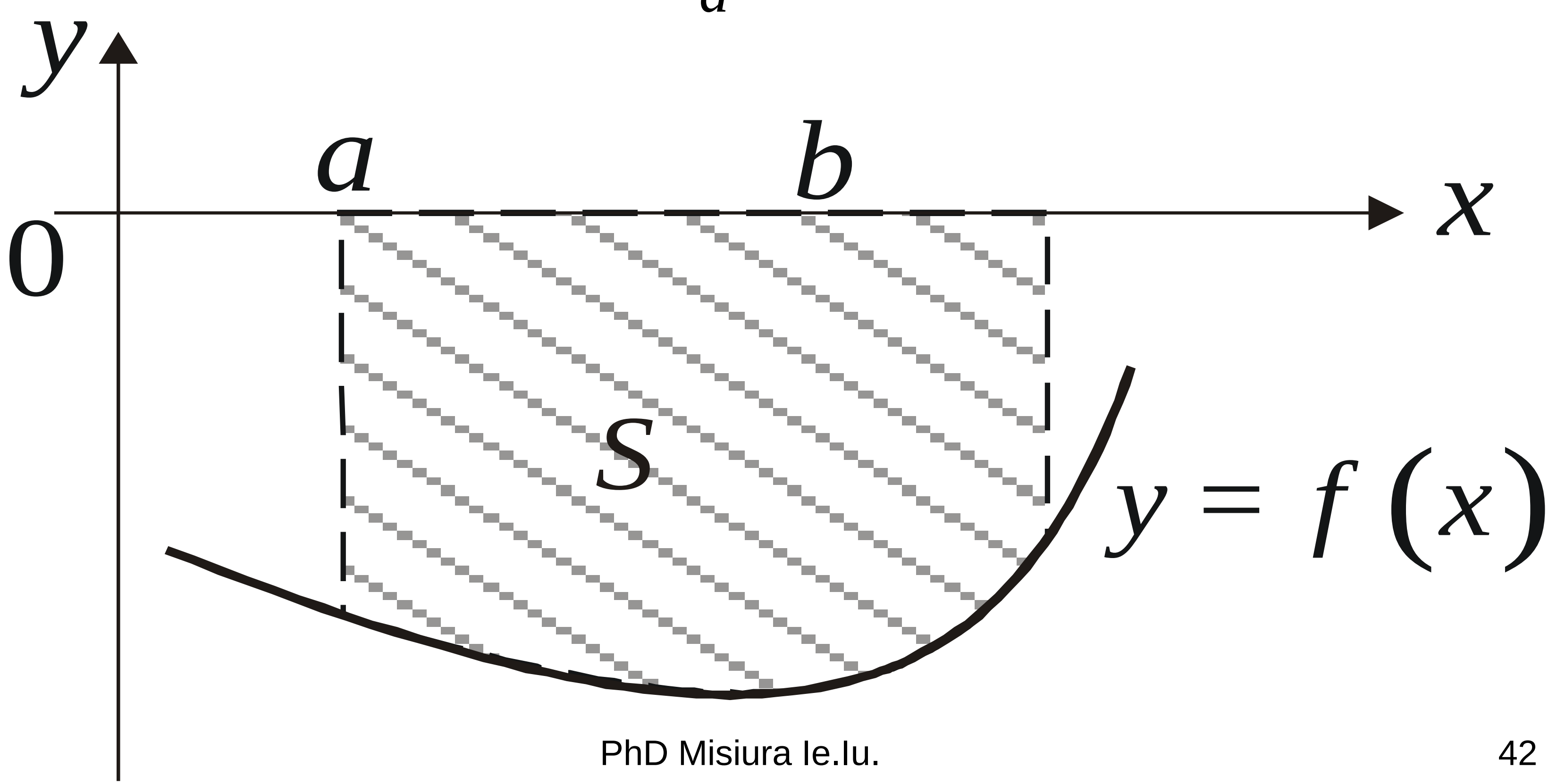
the area of the curvilinear trapezoid with the base $[a,b]$
bounded by the straight lines $x = a$, $x = b$, $y = 0$
and the curve $y = f(x)$:

$$S = \int\limits_a^b f(x)dx$$

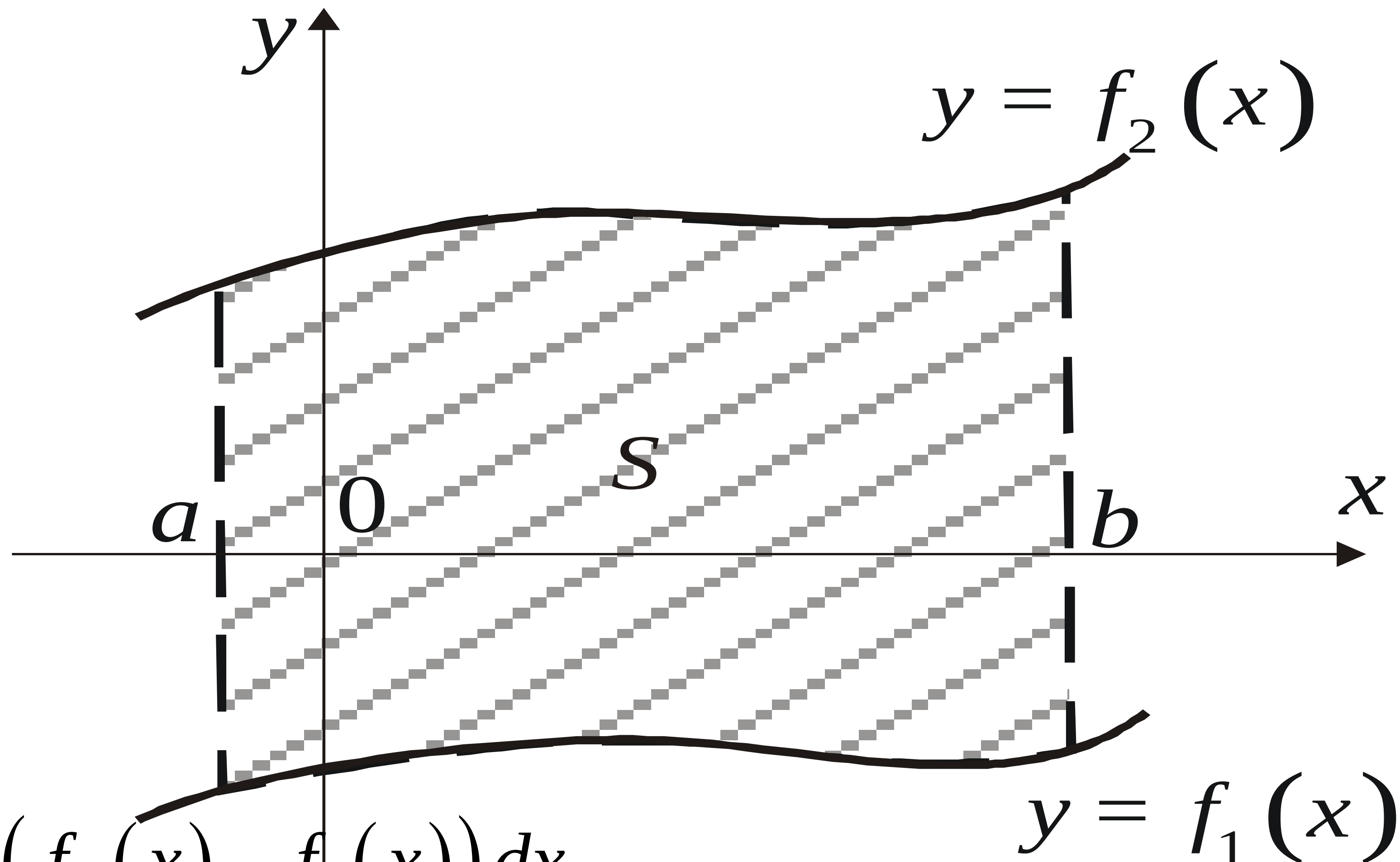


If the curvilinear trapezoid is bounded by the straight lines $x = a$, $x = b$, $y = 0$ and the curve $y = f(x)$, $f(x) \leq 0$ then its area is

$$S = -\int_a^b f(x)dx$$



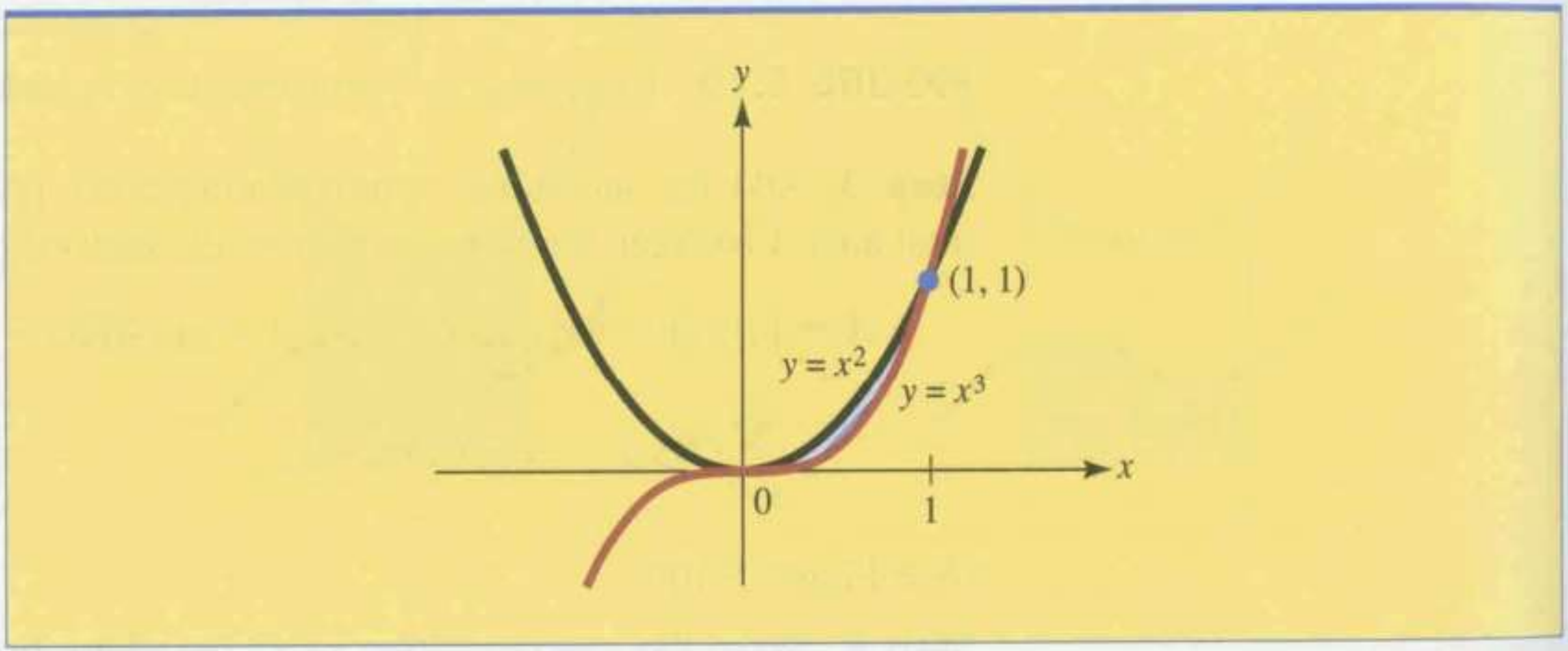
If $f_2(x) \geq f_1(x)$ then the area, bounded by these curves and the straight lines $x = a$ and $x = b$ equals



$$S = \int_a^b \left(f_2(x) - f_1(x) \right) dx$$

Example 1

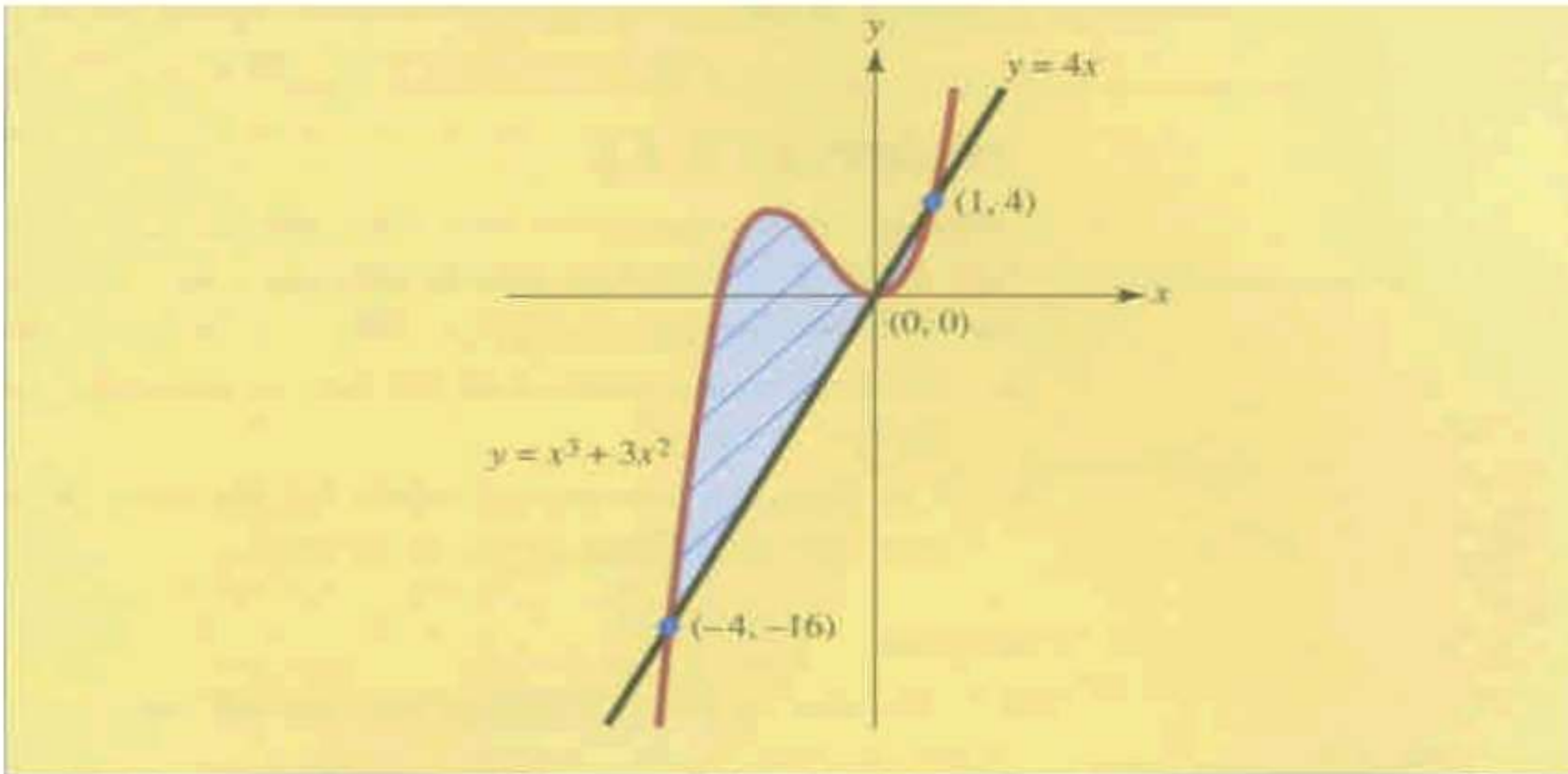
Find the area of the region R enclosed by the curves $y = x^3$ and $y = x^2$



The region enclosed by the curves $y = x^2$ and $y = x^3$.

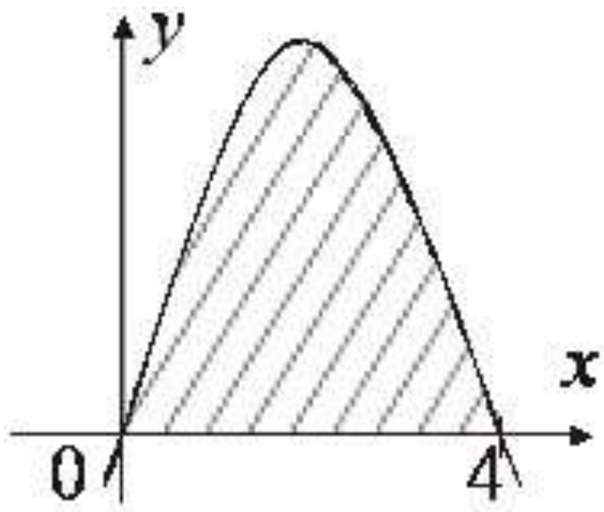
Example 2

Find the area of the region enclosed b the line $y = 4x$ and the curve $y = x^3 + 3x^2$



The region enclosed by the … d the curve $y = x^3 + 3x^2$

Example 4. Calculate the area of the figure bounded by the parabola $y = 4x - x^2$ and the X-axis.

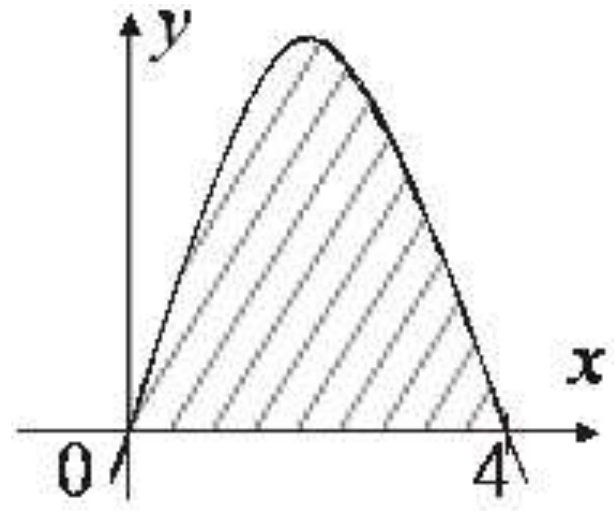


Solution. Find the limits of integration, i.e. the points of intersection of the given curve and X-axis ($y=0$):

$$4x-x^2=0$$

$$x(4-x)=0$$

$$x=0 \qquad x=4$$

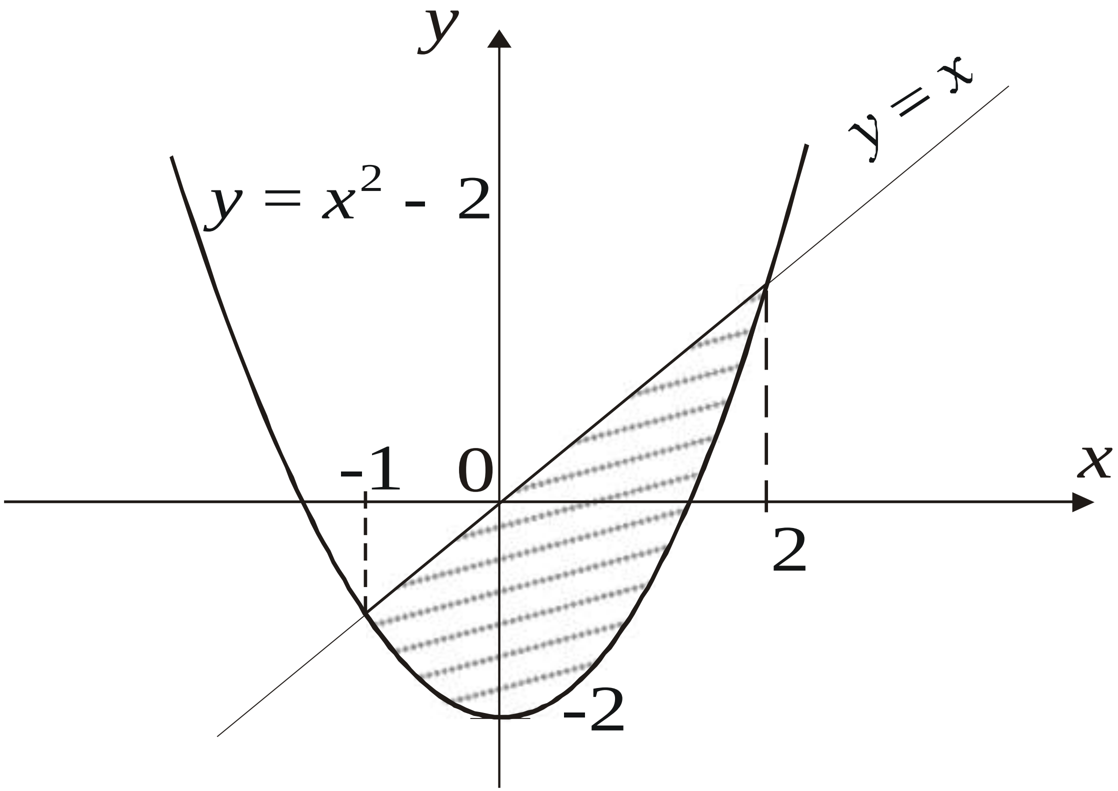


Let's calculate the area of the figure by the formula:

$$S=\int_0^4\left(4x-x^2\right)dx=4\int_0^4 xdx-\int_0^4 x^2dx=2x^2\Big|_0^4-\frac{x^3}{3}\Big|_0^4=$$

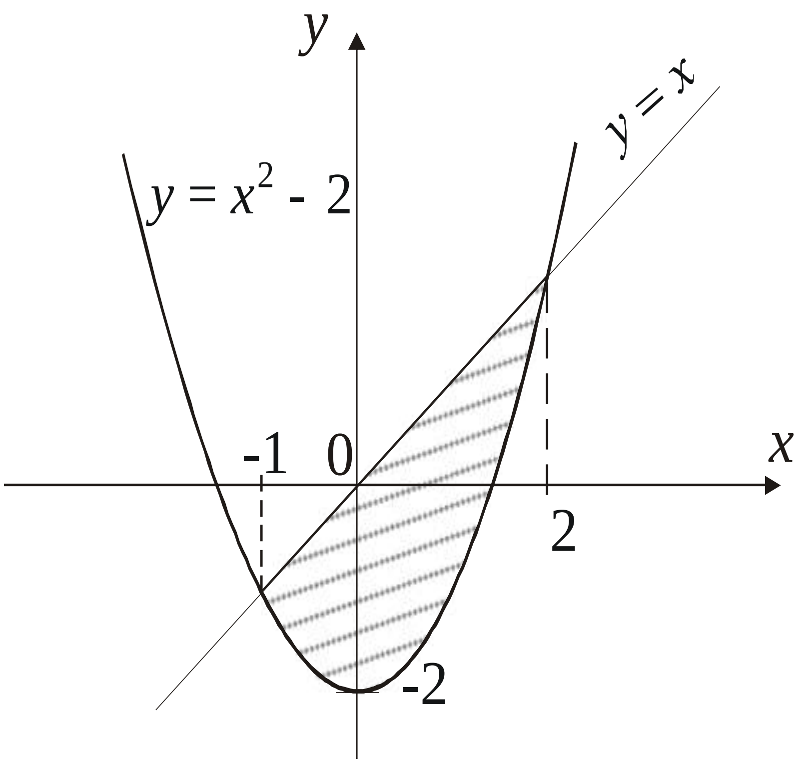
$$=32-\frac{64}{3}=\frac{32}{3}$$ (sq. units)

Example 5. Calculate the area of the figure bounded by curves $y = x^2 - 2$ and $y = x$



Solution. Find the limits of integration, i.e. the points of intersection of given curves:

$$\begin{cases} y = x^2 - 2, \\ y = x \end{cases}$$



$$x^2 - 2 = x$$

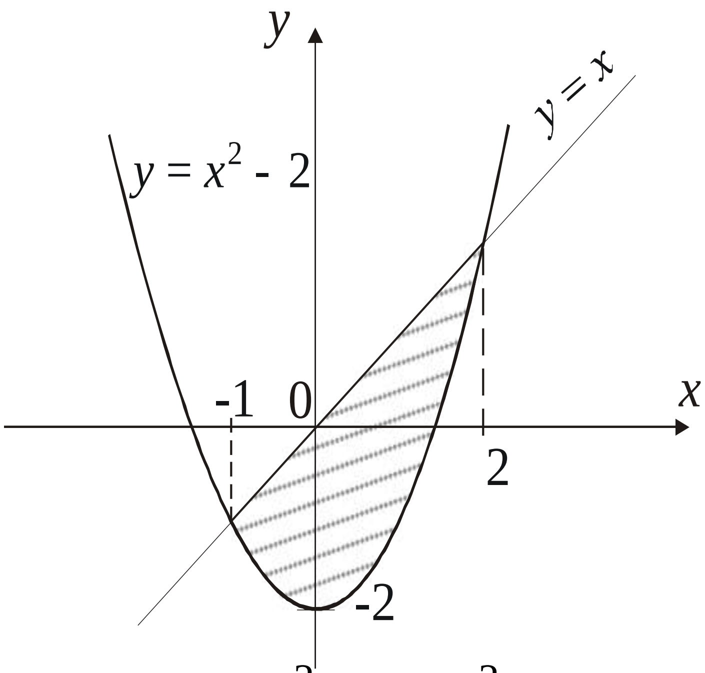
$$x^2 - x - 2 = 0$$

Let's find roots:

$$x = \frac{1 \pm \sqrt{1+8}}{2}$$

$$x_1 = -1 \qquad x_2 = 2$$

Since $f_2(x) \geq f_1(x)$ then the area, bounded by these curves is defined by the formula:



$$S = \int_a^b \left(f_2(x) - f_1(x) \right) dx$$

$$S = \int_{-1}^{2} \left(x - \left(x^2 - 2 \right) \right) dx = \int_{-1}^{2} \left(x - x^2 + 2 \right) dx = \frac{x^2}{2} \Big|_{-1}^{2} - \frac{x^3}{3} \Big|_{-1}^{2} +$$

$$+ 2x \Big|_{-1}^{2} = \frac{1}{2}\left(4 - (-1)^2 \right) - \frac{1}{3}\left(2^3 - (-1)^3 \right) + 2\left(2 - (-1) \right) =$$

$$= \frac{3}{2} - 3 + 6 = \frac{9}{2}.$$

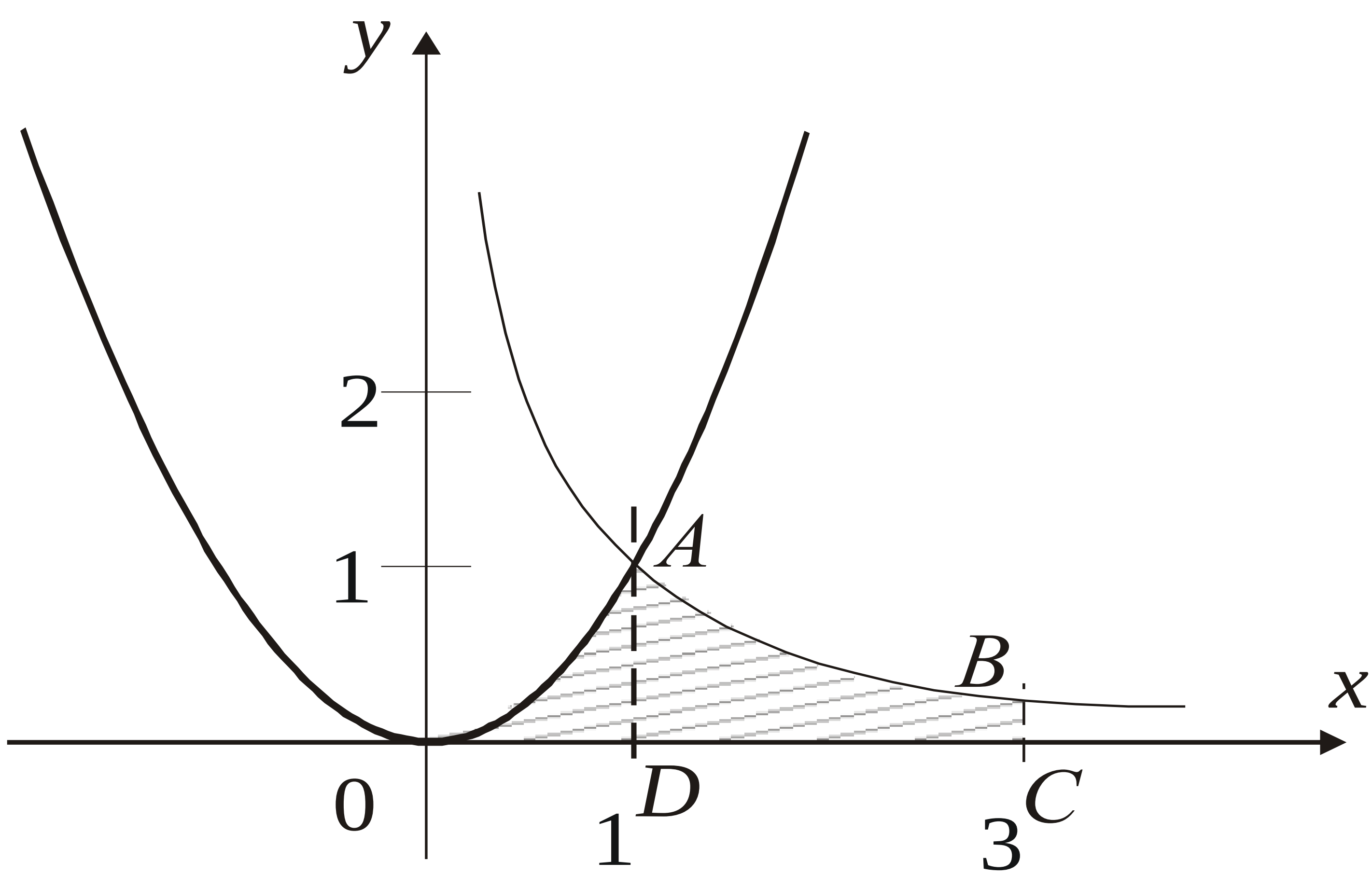
Tasks for the test

Task 1. Calculate the area of the figure bounded by curves:

$$y = x^2 \qquad y = \frac{1}{x} \qquad x = 3 \qquad y = 0$$

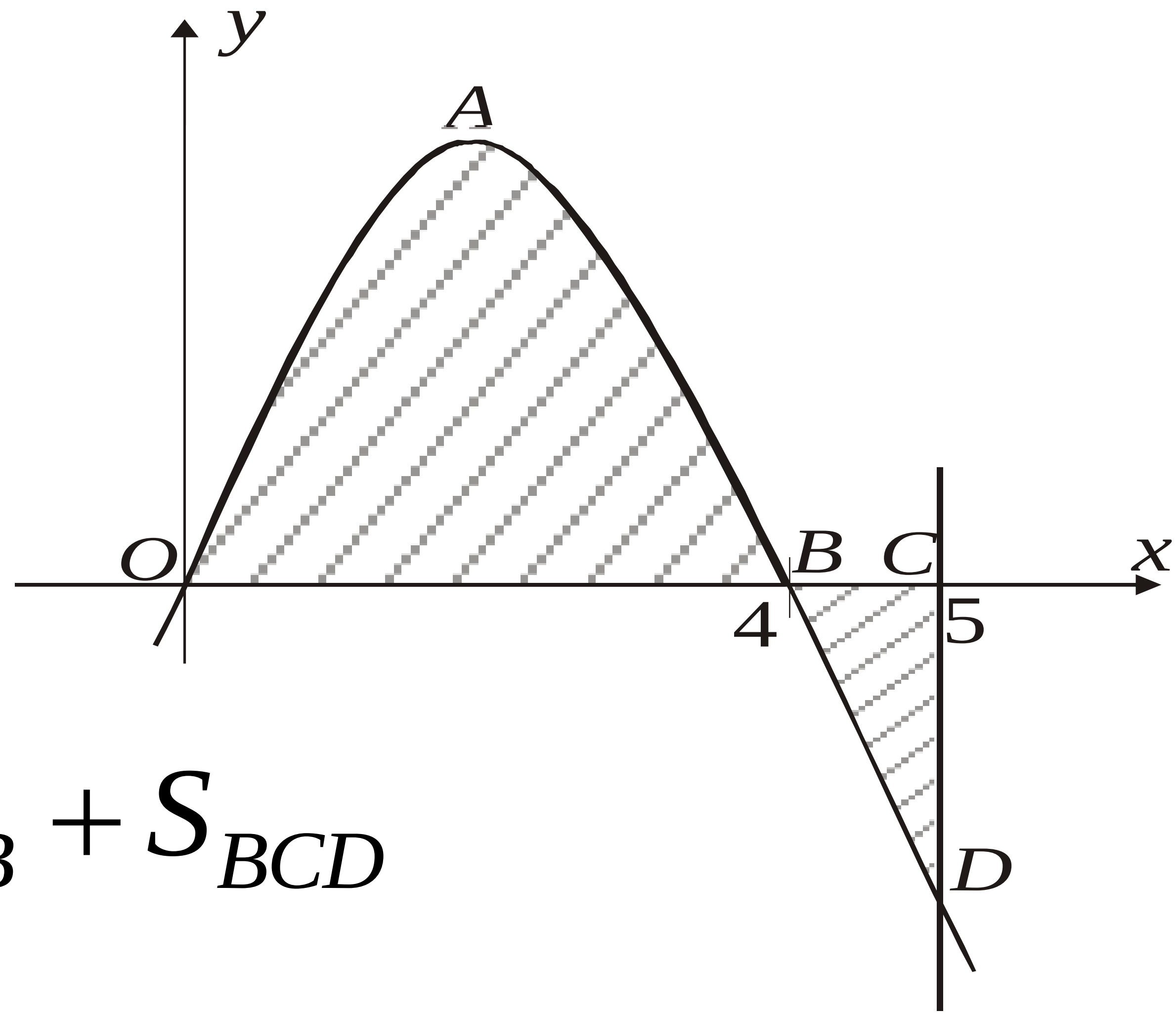
Task 2. Calculate the area of the figure bounded by curves:

$$y = 4x - x^2 \qquad y = 0 \qquad x = 5$$



$$S = S_{OAD} + S_{DABC}$$

$$S = \frac{1}{3} + ln\,3$$



$$S = S_{OAB} + S_{BCD}$$

$$S = \frac{32}{3} + \frac{7}{3} = 13$$

4. Application of definite integral for geometry problems.

Calculation of the rotation solid volume

If the curvilinear trapezoid bounded by the curve $y = f(x)$ and the straight lines $x = a$, $x = b$, $y = 0$, it is rotated round the X-axis then the rotation solid volume can be obtained by the formula:

$$V_x = \pi \int_a^b y^2 dx$$

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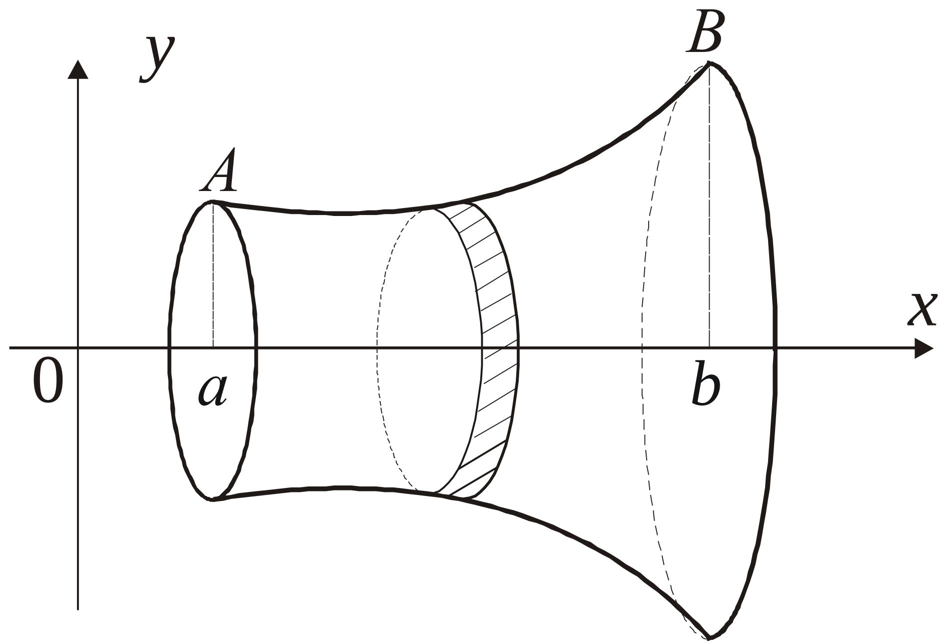
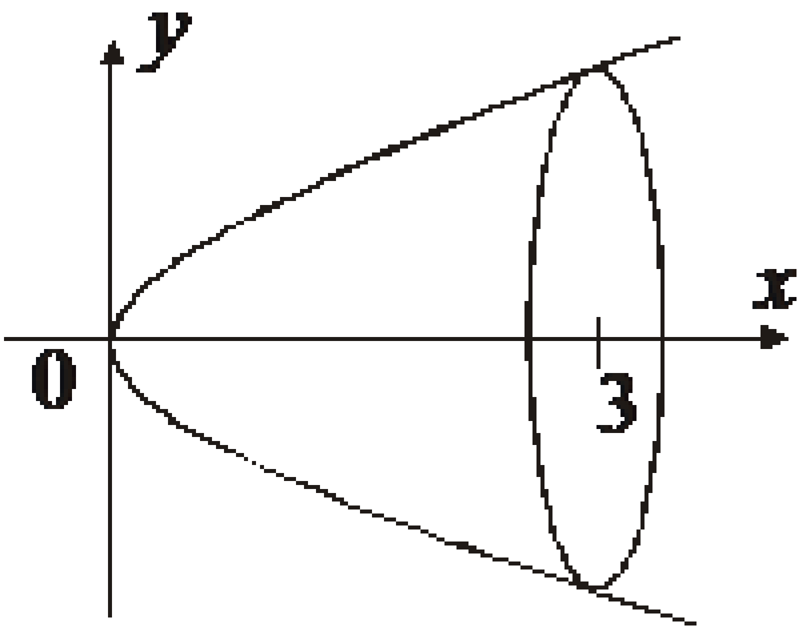


If the curvilinear trapezoid bounded by the curve $y = f(x)$ and the straight lines $x = a$, $x = b$, $y = 0$, it is rotated round the Y-axis then the rotation solid volume can be obtained by the formula:

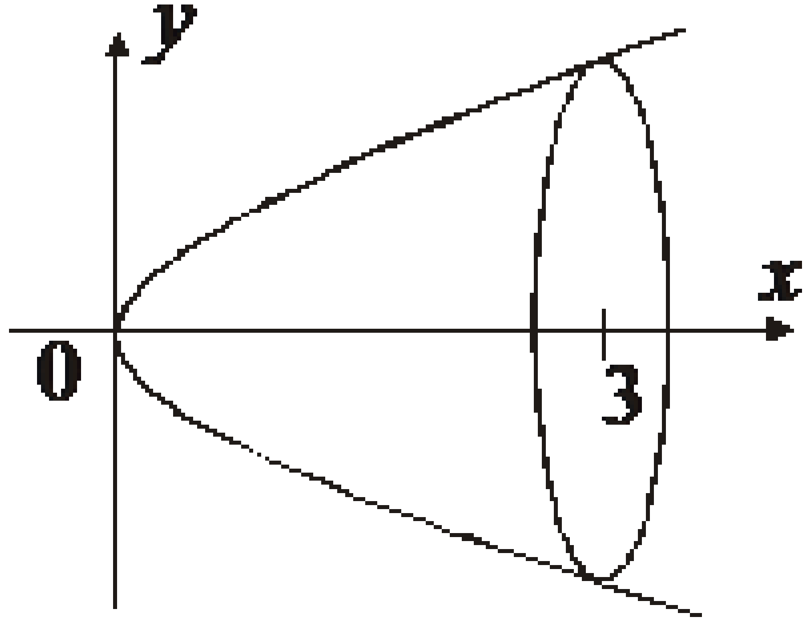
$$V_y = 2\pi \int_a^b xy dx$$

Example. Calculate a solid volume obtained by the figure rotation round the X-axis bounded by the lines:

$$y^2 = 2x \qquad x = 3$$



$$a = 0 \qquad b = 3$$



$$V_x = \pi \int_0^3 y^2 dx = \pi \int_0^3 2x dx = \pi \cdot 2 \frac{x^2}{2} \Big|_0^3 = 9\pi$$

If a figure is bounded by the curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$

$\left(0 \le f_1(x) \le f_2(x)\right)$ and the straight lines $x = a$, $x = b$,

it is rotated round the X-axis then the rotation solid volume is

$$V_x = \pi \int_a^b \left(y_2^2 - y_1^2 \right) dx = \pi \int_a^b \left(f_2^{\,2}(x) - f_1^{\,2}(x) \right) dx$$

If a figure is bounded by the curves $x_1 = \varphi_1(y)$ and $x_2 = \varphi_2(y)$

$\left(0 \le \varphi_1(y) \le \varphi_2(y)\right)$ and the straight lines $y = c$, $y = d$,

it is rotated round the Y-axis then the rotation solid volume is

$$V_y = \pi \int_c^d \left(x_2^2 - x_1^2 \right) dy = \pi \int_c^d \left(\varphi_2^2(y) - \varphi_1^2(y) \right) dy$$

Application of a definite integral for economic problems

Problem 1. Let the function $z = f(t)$ describe the changing of the productivity of an enterprise under the time ***t***. Then the volume of production ***V*** produced by the time $[t_1, t_2]$ is defined by the formula:

$$V = \int_{t_1}^{t_2} f(t)dt$$

Problem 1. Determine the volume of production V produced by an employee over the second working hour if the productivity is defined by the function:

$$f(t) = \frac{2}{3t + 4} + 3$$

Problem 1. Determine the volume of production V produced by an employee over **the second working hour** if the productivity is defined by the function:

$$f(t) = \frac{2}{3t + 4} + 3$$

The second working hour is the time from the first to the second hour.

Solution. The volume of production V is defined by the formula:

$$V = \int_{t_1}^{t_2} f(t)\,dt$$

In this case

$$V = \int_1^2 \left(\frac{2}{3t+4} + 3 \right) dt = \left(\frac{2}{3} \ln|3t+4| + 3t \right) \Bigg|_1^2 = \frac{2}{3}\ln 10 + 6 - \frac{2}{3}\ln 7 - 3 = \frac{2}{3}\ln\frac{10}{7} + 3$$

Tasks

Task 1.1. Let the function of the productivity have the form:

(details per an hour),

$$f(t) = -0{,}0033t^2 - 0{,}089t + 20{,}96$$

where is a time segment from the beginning of a day. Determine the quantity of the details which are produced by an employee over the first working hour and the second working hour and. Find the mean value of the quantity of the details which are produced by an employee during a working day (8 hours). Make an analysis of the obtained values in the problem.

Tasks

Task 1.2. Let the function of the productivity have the form:

$$f(t) = 100 + 10t$$ (details per an hour),

where is a time segment from the beginning of a day. Determine the quantity of the details which are produced by an employee over the first working hour and the second working hour and. Find the mean value of the quantity of the details which are produced by an employee during a working day (8 hours). Make an analysis of the obtained values in the problem.

Problem 2. Find the mean value of the inputs

$$K(x) = 3x^2 + 4x + 1$$

expressed in units of money, if the volume of production changes from 0 to 3 units. Indicate the volume of production, under which the inputs take on the mean value.

Solution. Let's apply the mean value theorem:

$$K(c)=\frac{1}{x_2-x_1}\int_{x_1}^{x_2} K(x)dx$$

In this case

$$K(c)=\frac{1}{3-0}\int_0^3\left(3x^2+4x+1\right)dx=\frac{1}{3}\left(3\frac{x^3}{3}+4\frac{x^2}{2}+x\right)\Bigg|_0^3=\frac{1}{3}\left(x^3+2x^2+x\right)\Big|_0^3=$$

$=\frac{1}{3}\left(27+18+3\right)=16$ (units of money).

The mean value of the inputs equals 16, i.e. $K(c) = 16$

Let's define the volume of production, under which the inputs take on the mean value, i.e. solve the equation:

$$K(c) = 16$$

$$3c^2 + 4c + 1 = 16$$

$$3c^2 + 4c - 15 = 0$$

$$3c^2 + 4c - 15 = 0$$

Let's find the discriminant and roots of this quadratic equation:

$$D = 196 \qquad c_1 = -3 \qquad c_2 = \frac{5}{3}$$

$$3c^2 + 4c - 15 = 0$$

Let's find the discriminant and roots of this quadratic equation:

$$D = 196 \qquad c_1 = -3 \qquad c_2 = \frac{5}{3}$$

This root $c_1 = -3$ is not the value of the volume of production, because it is less than 0. Therefore ***c*** equals $\frac{5}{3}$ i.e. $c = \frac{5}{3}$ units of production.

TASKS

Task 2.1. Find the mean value of the inputs

$$K(x) = 3x + 2 - \frac{1}{x}$$

expressed in units of money, if the volume of production changes from 0 to 3 units. Indicate the volume of production, under which the inputs take on the mean value.

Problem 3. The laws of supply and demand have the form:

$$p = 186 - x^2 \qquad p = 20 + \frac{11}{6}x$$

a) Find the point of market equilibrium.

b) Find consumers' benefit under the condition of establishment of market equilibrium.

c) Find suppliers' benefit under the condition of establishment of market equilibrium.

a) Find the point of market equilibrium.

$$\begin{cases} p = 186 - x^2 \\ p = 20 + \dfrac{11}{6}x \end{cases}$$

a) Find the point of market equilibrium.

$$\begin{cases} p = 186 - x^2 \\ p = 20 + \dfrac{11}{6}x \end{cases} \Rightarrow x_0 = 12,\ p_0 = 42$$

b) Find consumers' benefit under the condition of establishment of market equilibrium.

$$C=\int_{0}^{x_0} p_c(x)dx-p_0x_0=\int_{0}^{12}\left(186-x^2\right)dx-12\cdot 42=\left(186x-\frac{x^3}{3}\right)\Bigg|_{0}^{12}-504=1152$$

(units of money)

c) Find suppliers' benefit under the condition of establishment of market equilibrium.

$$P = p_0 x_0 - \int_0^{x_0} p_s(x)dx = 12 \cdot 42 - \int_0^{12} \left(20 + \frac{11}{6}x\right)dx = 504 - \left(20x - \frac{11}{6}\frac{x^2}{2}\right)\Bigg|_0^{12} = 132$$

(units of money)

TASKS

Task 3.1. The laws of supply and demand have the form:

$$p = \frac{120}{x+2} \qquad p = 10 + \frac{5}{2}x$$

a) Find the point of market equilibrium.
b) Find consumers' benefit under the condition of establishment of market equilibrium.
c) Find suppliers' benefit under the condition of establishment of market equilibrium.

Logistic differential equations are used, amongst other applications, to model population growth in biology an capital yield in economics. The following differential equation is an example of a logistic equation.

$$\frac{dy}{dt} = y\left(1 - \frac{y}{4}\right) \qquad (\dagger)$$

(a) Find the *constant* (or equilibrium) solutions of (†).

(b) Use your answer to (a) to write down the solution with initial value $y = 4$ when $t = 0$.

(c) What does the answer to (a) tell us about the *range* of solution with initial value $y = 1$ when $t = 0$?

(d) Verify that for any $C \geq 0$

$$y = \frac{4}{1 + Ce^{-t}}$$

is a solution of (†).

Economic interpretation

Slope as marginal rate of change

A very clear way to see how calculus helps us interpret economic information and relationships is to compare total, average, and marginal functions.

Take, for example, a total cost function, TC:

$$TC = Q^2 + 9Q$$

What about the change in marginal cost? That way, we can not only evaluate costs at a particular level,

but we can see how our marginal costs are changing as we increase or decrease our level of production.

$TC = Q^2 + 9Q$

take the first derivative to get MC:

$$\text{MC} = \frac{dTC}{dQ} = 2Q + 9$$

take the second derivative to get change in MC:

$$\frac{d}{dQ}(MC) = \frac{d}{dQ}\left(\frac{dTC}{dQ}\right) = \frac{d^2TC}{dQ^2} = 2$$