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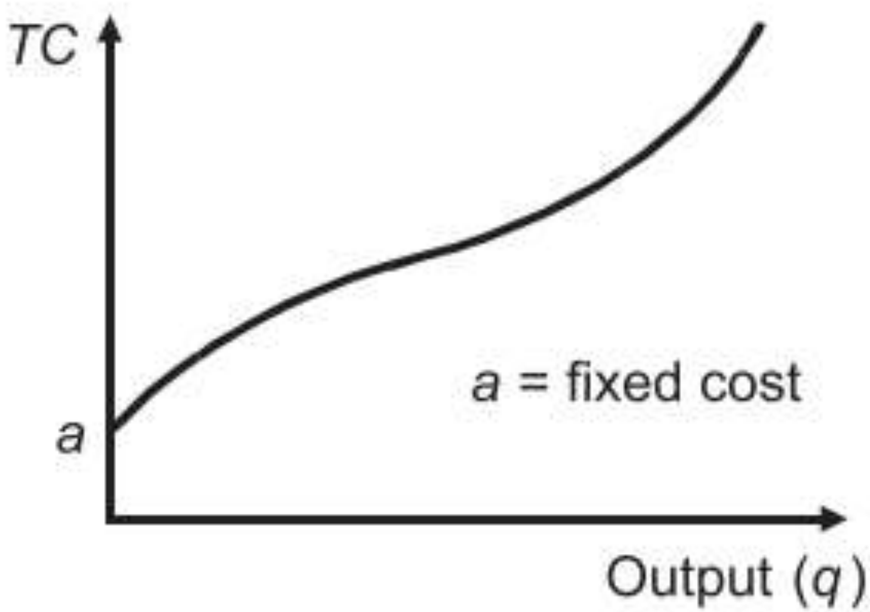
TOPIC:
Functions and their graphs. Graphs in economic modeling. Simple and compound interest in economic studies

ECONOMIC PROBLEMS

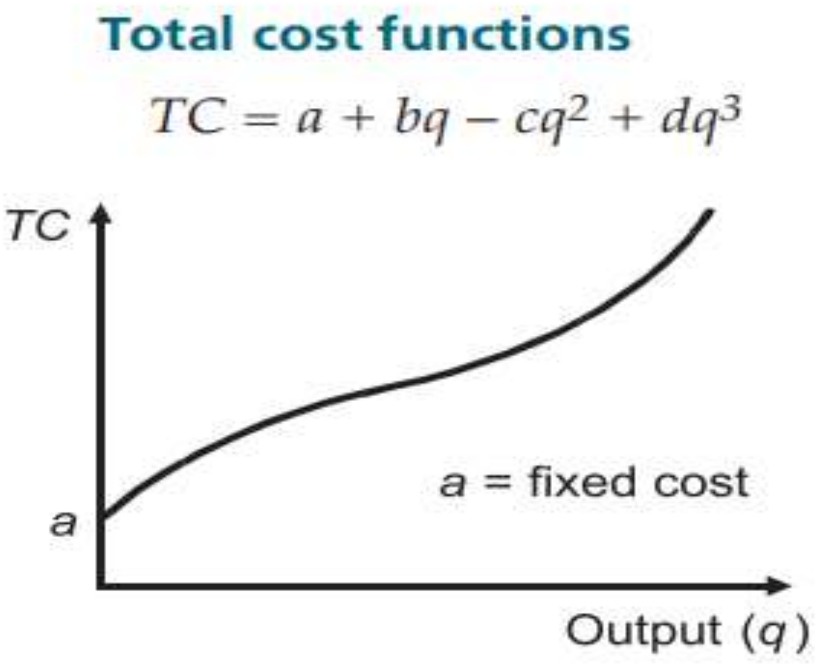
Total cost functions

$$TC = a + bq - cq^2 + dq^3$$



Economic function in MATH

A firm's total cost of production, **TC**, depends on its output, q. The **TC** function may include a constant term, which represents fixed costs, **FC**. The part of total cost that varies with **q** is called variable cost, **VC**.



Economic function in MATH

A firm's total cost of production, TC, depends on its output, Q. The TC function may include a constant term, which represents fixed costs, FC. The part of total cost that varies with Q is called variable cost, VC. We have, then, that TC = FC + VC. Average cost per unit of output is found by dividing by Q. We can find average total cost, denoted AC, which is given by AC = TC ÷ Q, together with average variable cost AVC = VC ÷ Q and average fixed cost AFC = FC ÷ Q.

Remember...

FC is the constant term in TC.

$VC = TC - FC$

$AC = TC/Q$

$AVC = VC/Q$

$AFC = FC/Q$

Economic function in MATH

For a firm with total cost given by

$$TC = 120 + 45Q - Q^2 + 0.4Q^3$$

identify its AC, FC, VC, AVC and AFC functions. List some values of TC, AC and AFC, correct to the nearest integer. Sketch the total cost function and, on a separate graph, the AC and AFC functions.

$$TC = 120 + 45Q - Q^2 + 0.4Q^3$$

$$AC = TC/Q = 120/Q + 45 - Q + 0.4Q^2$$

$$FC = 120 \text{ (the constant term in TC)}$$

$$VC = TC - FC = 45Q - Q^2 + 0.4Q^3$$

$$AVC = VC/Q = 45 - Q + 0.4Q^2$$

$$AFC = FC/Q = 120/Q$$

Some possible values for Q and for each of the terms in the total cost function are shown in the table. The corresponding TC, AC and AFC values are calculated and are plotted in figures 1 Notice that when $Q = 0$ the first terms in AC and in AFC involve dividing by zero. To avoid the problem of an infinite result, the smallest value of Q for which AC and AFC are calculated is 0.3.

Economic function in MATH

Q	0	0.3	1	3	5	8	10	12	15
$45Q$	0	13.5	45	135	225	360	450	540	675
Q^2	0	0.09	1	9	25	64	100	144	225
$0.4Q^3$	0	0.0108	0.4	10.8	50	204.8	400	691.2	1350

Correct to the nearest integer

TC	120	133	164	257	370	621	870	1207	1920
AC		445	164	86	74	78	87	101	128
AFC		400	120	40	24	15	12	10	8

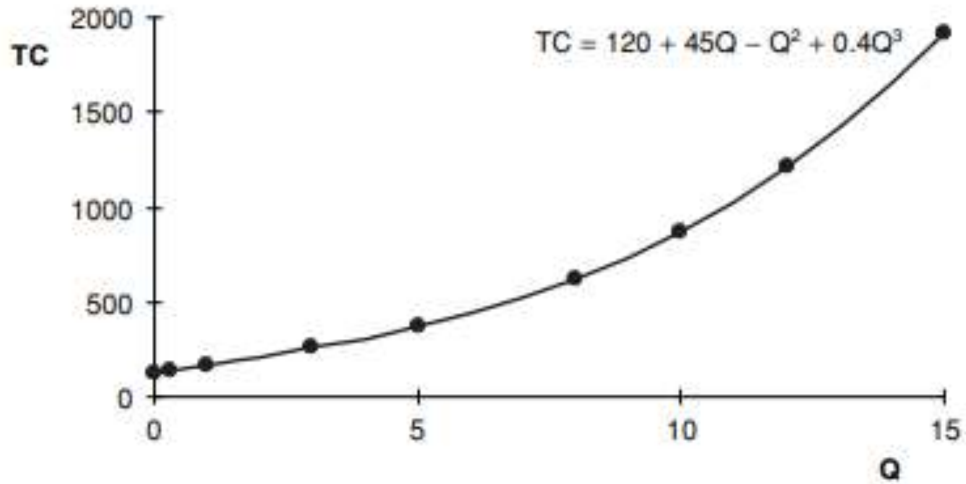


Figure 1

1. Sets

The belonging of the element ***x*** to the set ***X*** is symbolized in the following way:

$$x \in X$$

If ***x*** does not enter in the set ***X*** it is written as

$$x \notin X$$

1. Sets

It is advisable to introduce into consideration a set containing not a single element.

Such a set is called empty and it is denoted

by the symbol $\varnothing$.

2. Sets of real numbers or sets on the real axis

1. Sets of the form

$$(a, b), (-\infty, b), (a, +\infty) \quad \text{and} \quad (-\infty, +\infty)$$

consisting, respectively, of all $x \in R$ such that

$$a < x < b, \quad x < b, \quad x > a,$$

and ***x*** is arbitrary, are called ***open intervals*** (sometimes simply *intervals*).

2. Sets of real numbers or sets on the real axis

1. Sets of the form

$(a,b),(-\infty,b),(a,+\infty)$ and $(-\infty,+\infty)$

consisting, respectively, of all $x \in R$ such that

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and ***x*** is arbitrary, are called ***open intervals*** (sometimes simply *intervals*).

$$(a;b) \Leftrightarrow \{ x \in R \mid a < x < b \}$$

2. Sets of real numbers or sets on the real axis

2. Sets of the form

$$[a, b]$$

consisting of all $x \in R$ such that $a \le x \le b$ are called ***closed intervals* or *segments*.**

2. Sets of real numbers or sets on the real axis

2. Sets of the form

$[a,b]$

consisting of all $x \in R$ such that $a \le x \le b$ are called ***closed intervals* or *segments*.**

$$[a;b] \Leftrightarrow \left\{ x \in R \,\middle|\, a \le x \le b \right\}$$

2. Sets of real numbers or sets on the real axis

3. Sets of the form

$$(a,b],\ [a,b),\ (-\infty,b],\ [a,+\infty)$$

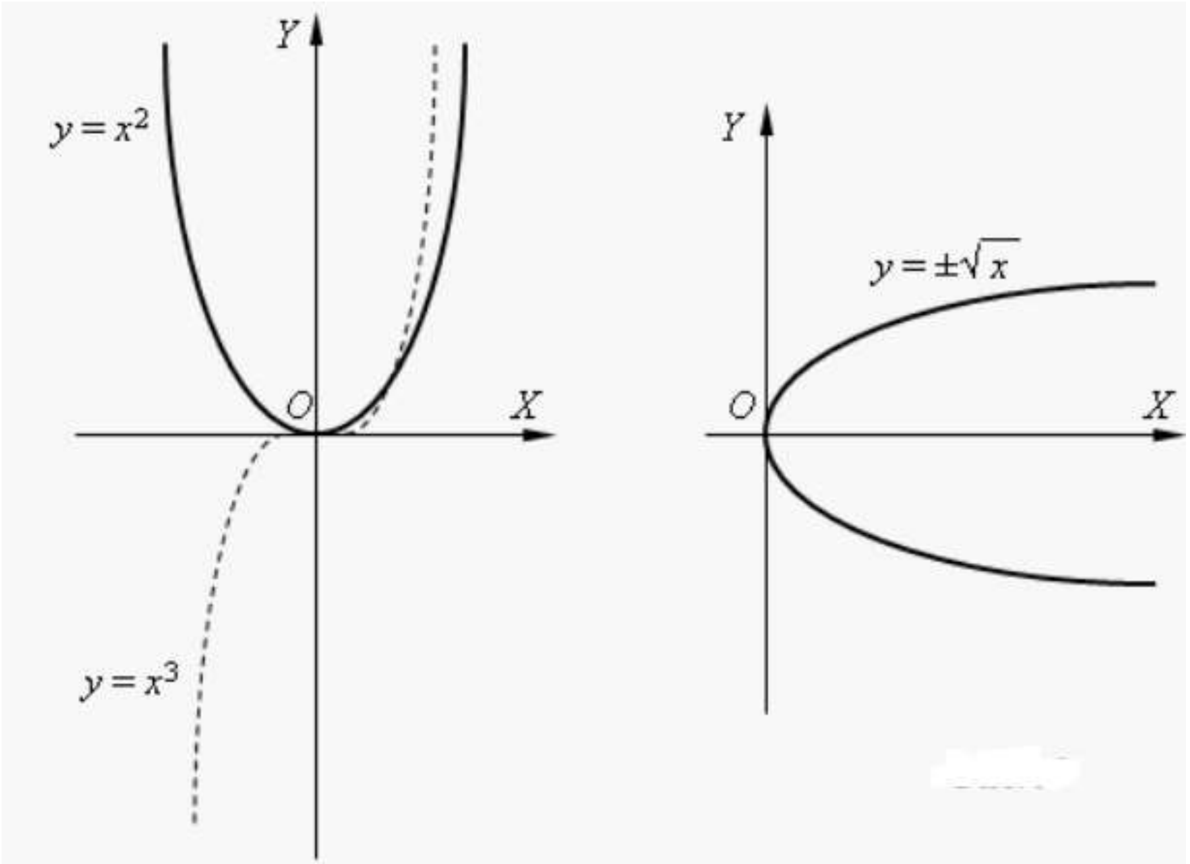
consisting of all ***x*** such that

$$a < x \le b \qquad a \le x < b \qquad x \le b \qquad x \ge a$$

are called ***half-open intervals***.

4. A function

Definition. If for every value of a variable *x* belonging to some set *X* ($x \in X$) there corresponds by the rule *f* a definite number $y \in Y$ then it is said that on the set *X* the *function* $y = f(x)$ is given.



Example

$f : X \to Y$ or $y = f(x)$

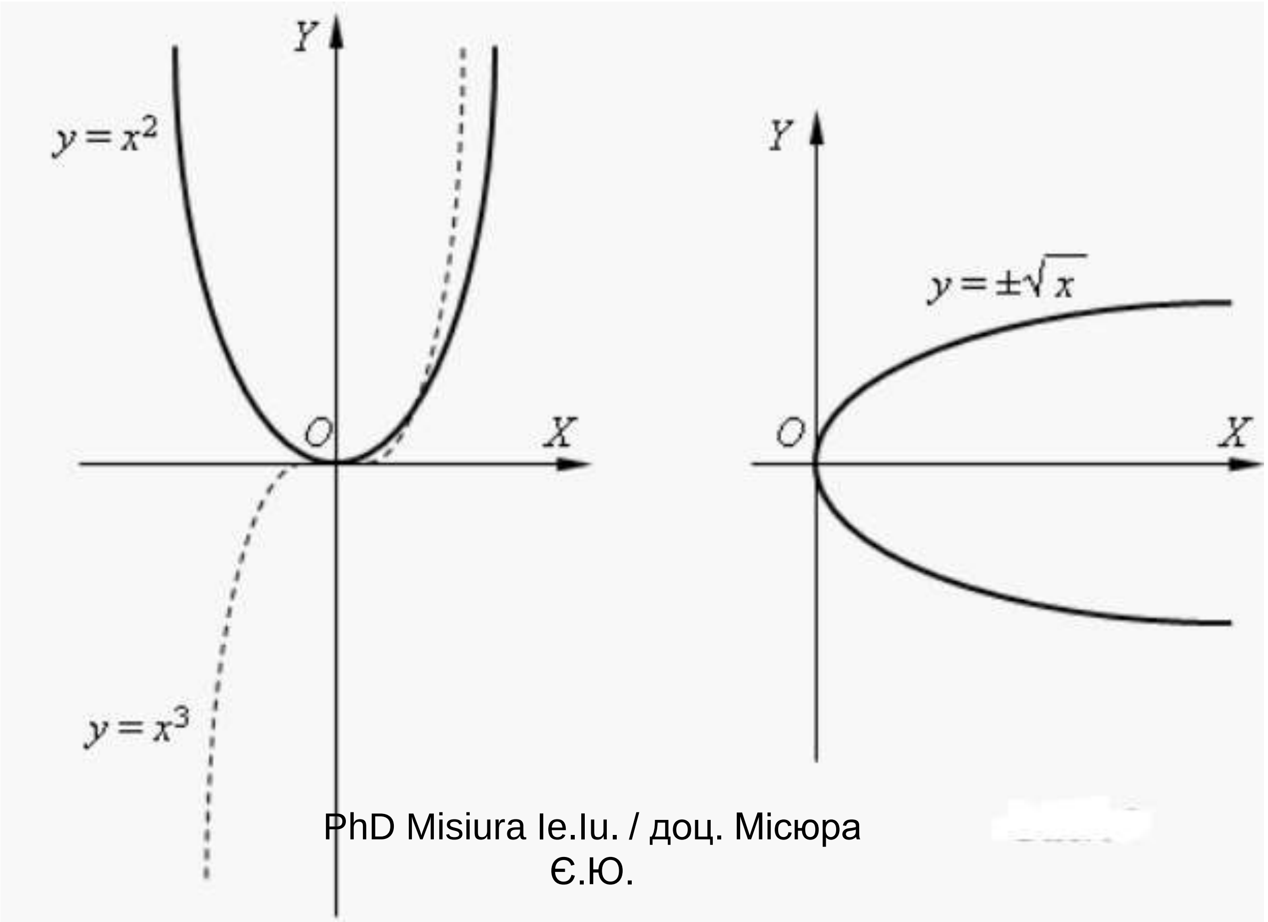
Please, remember

a domain of the function definition,

a range of the function values.

Definitions. The set ***X*** is called the *domain of the function definition,* the set ***Y*** is *a range of the function values.*

Definitions. Thus ***x*** *is called an independent variable or an argument,* ***y*** *is a function.*



The function $y = \sin x$

Its domain of definition is the whole numerical axis

$$X = \left(-\infty; +\infty\right)$$

$$D(f)$$

The function $y = \sin x$

Its domain of definition is the whole numerical axis

$$X = \left(-\infty; +\infty\right) \qquad D(f)$$

and its range is the segment

$$Y = \left[-1; 1\right] \qquad E(f)$$

Please, remember

a single valued function,

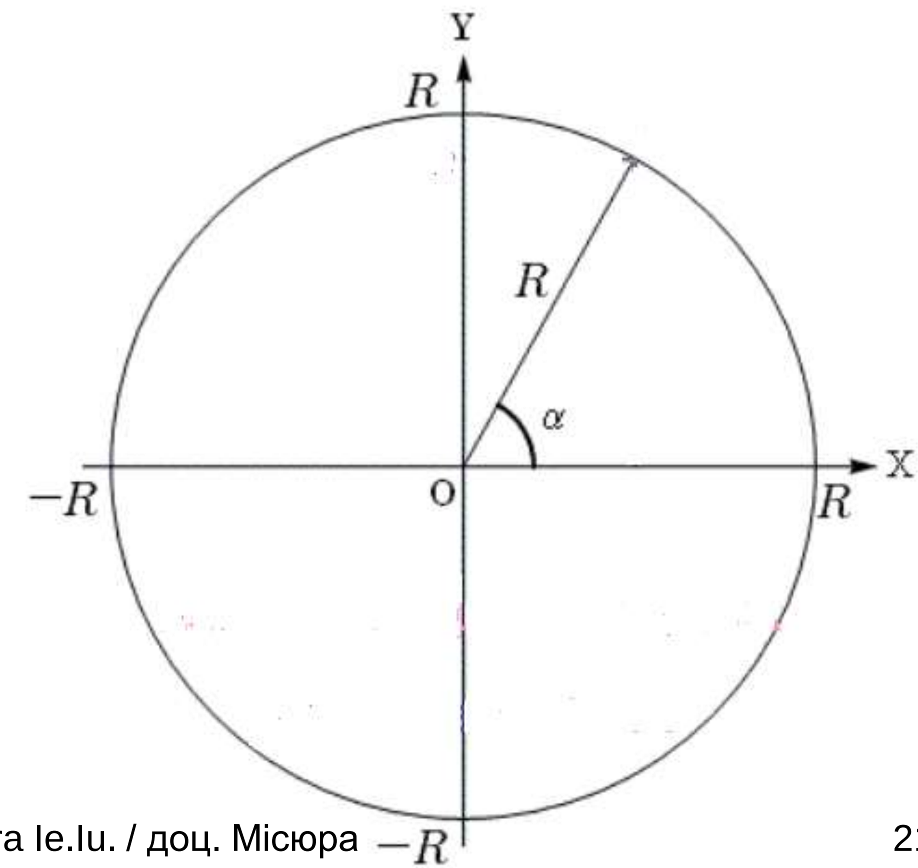
a multivalued function.

Definition. If for each value of variable the $x \in X$ there is more than one correspondent value $y \in Y$ (and even an indefinite set of them), then this function is called ***multivalued*** as against *single valued function* defined above.

$$x^2 + y^2 = R^2$$

It's the equation of a circle

R is a radius



Please, remember

all names of methods of representing functions

Methods of Representing Functions

1) Analytical method

In this case a formula is indicated by means of which it is possible to compute $f(x)$ for any $x \in X$. For instance, $y = 6x^3$, ***x*** is the infinite interval $-\infty < x < +\infty$

Methods of Representing Functions

2) Tabular method

We simply write down a sequence of values of the independent variable and in the values of the function corresponding to them we also indicate the method of computing ***y*** for intermediate values of ***x***, using the values given in table. Let’s consider the table.

Methods of Representing Functions

2) Tabular method

Table of values of the independent and dependent variables

x	…	-3	-2	-1	0	1	2	3	…
y	…	-27	-8	-1	0	1	8	27	…

$$y = x^3$$

Methods of Representing Functions

3) Graphic method

The graph of a function (in rectangular Cartesian coordinates) is the locus of all points in which abscissas are values of the independent variable and ordinates are the corresponding values of the function.

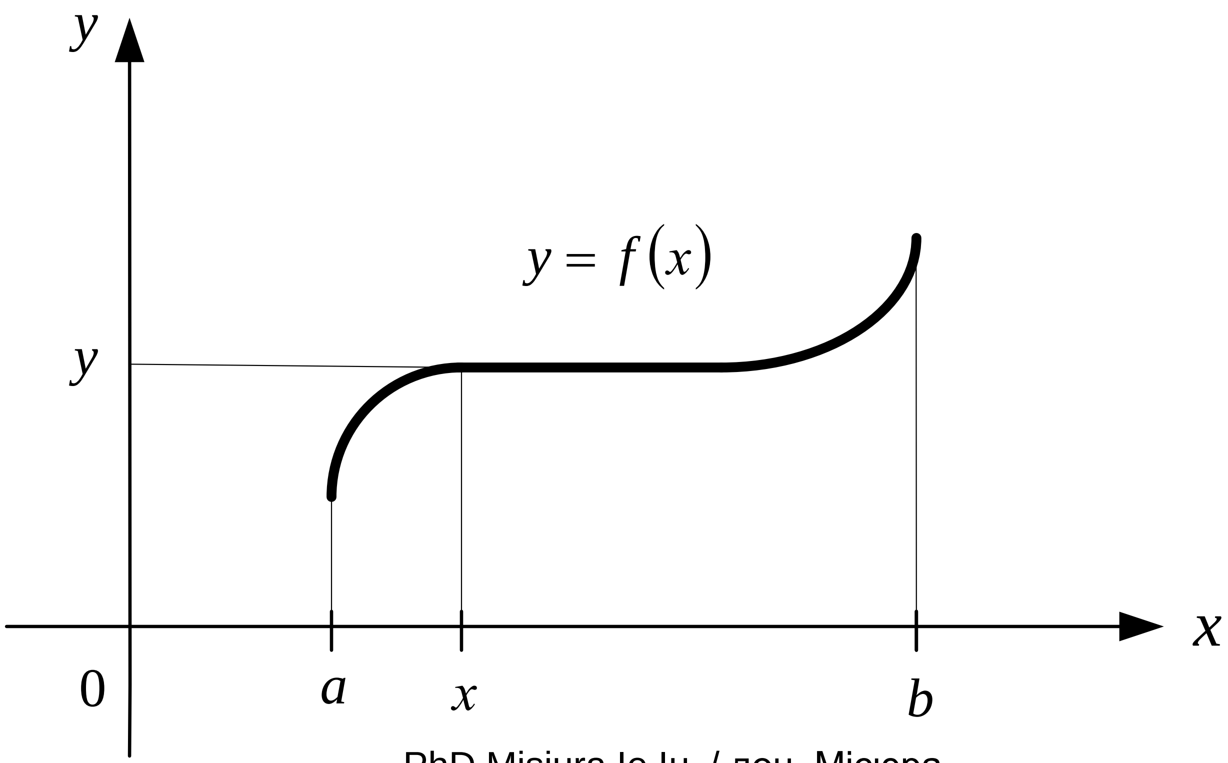
Methods of Representing Functions

3) Graphic method

In other words, if we take the abscissa equal to the value of the independent variable and the ordinate of the corresponding point of the graph is equal to the value of the function corresponding to that the value of the independent variable. When plotting a graph we can take similar or different scales along the coordinate axes.

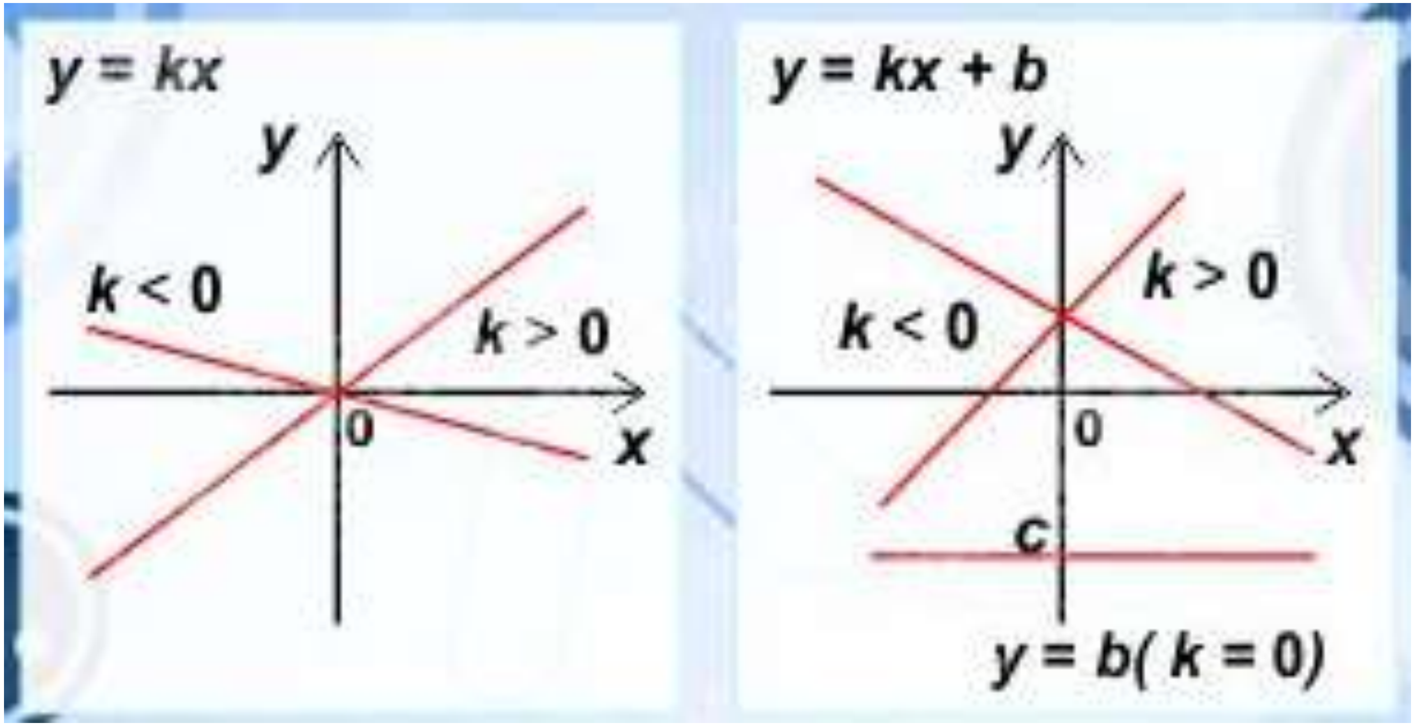
Methods of Representing Functions

3) Graphic method



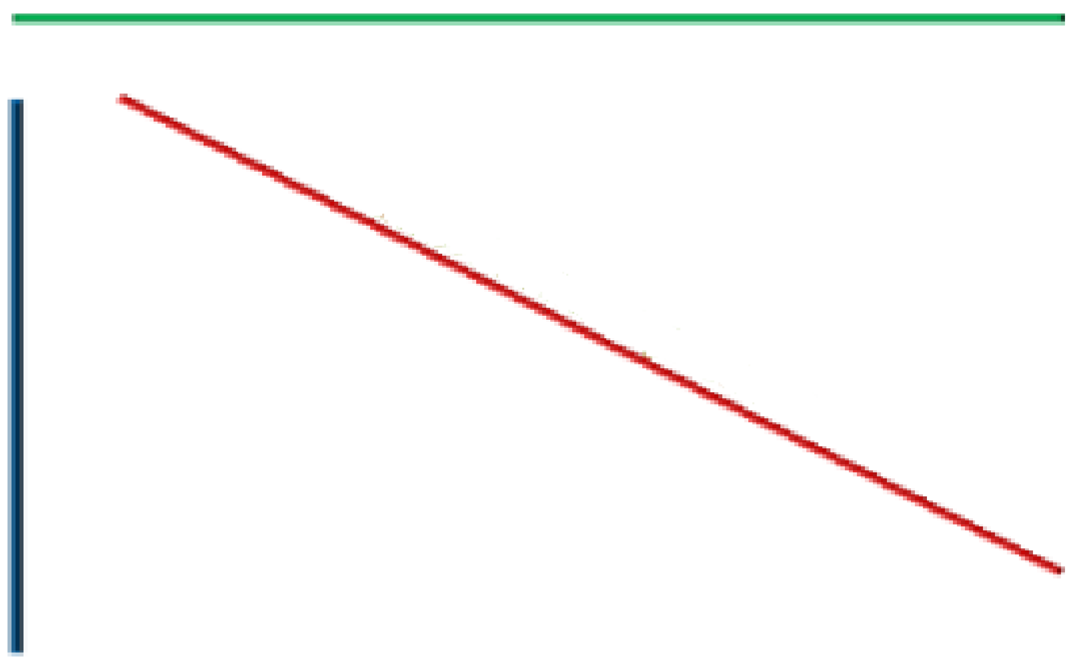
Classification of one Argument Functions

Straight line

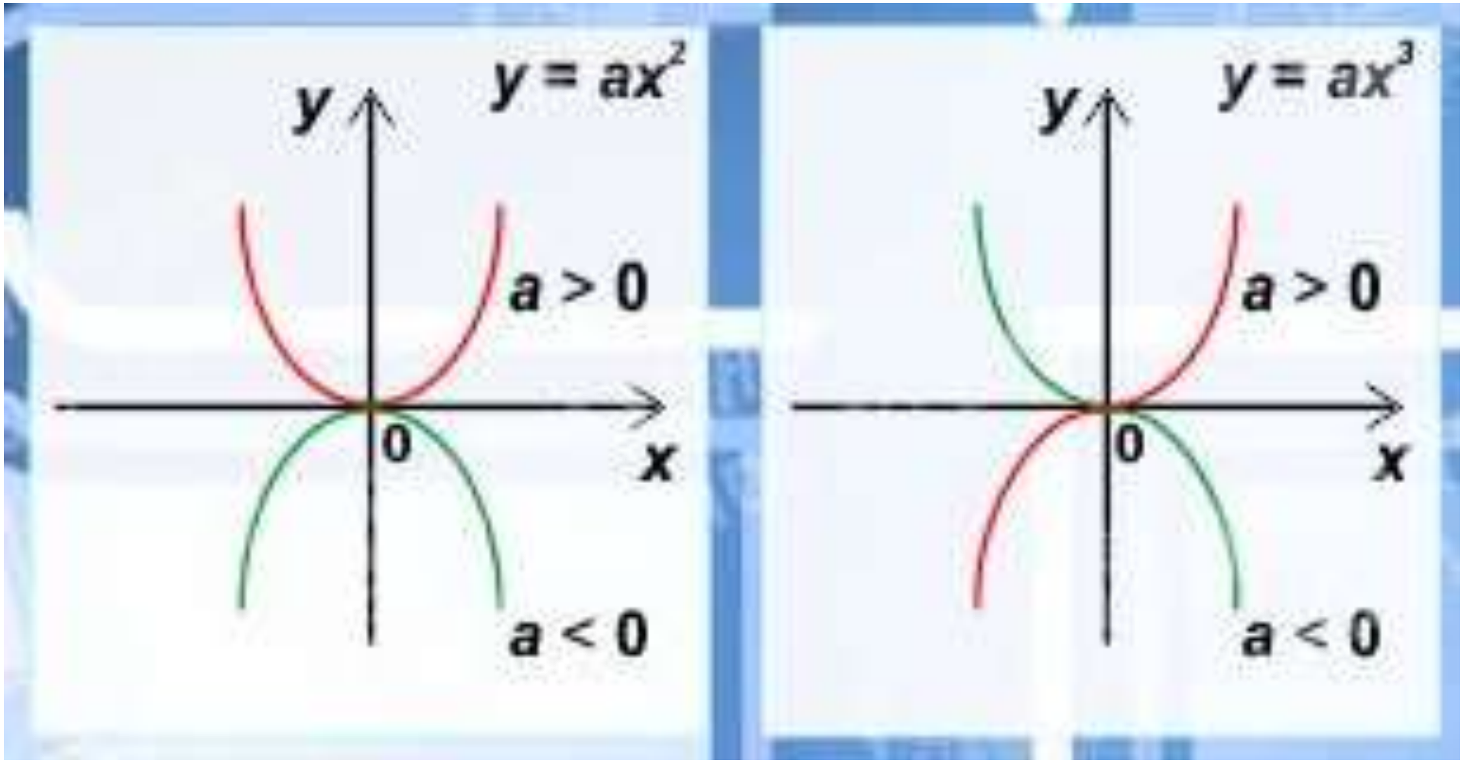


Straight line (inclined, horizontal and vertical)

Straight line (inclined, horizontal and vertical)

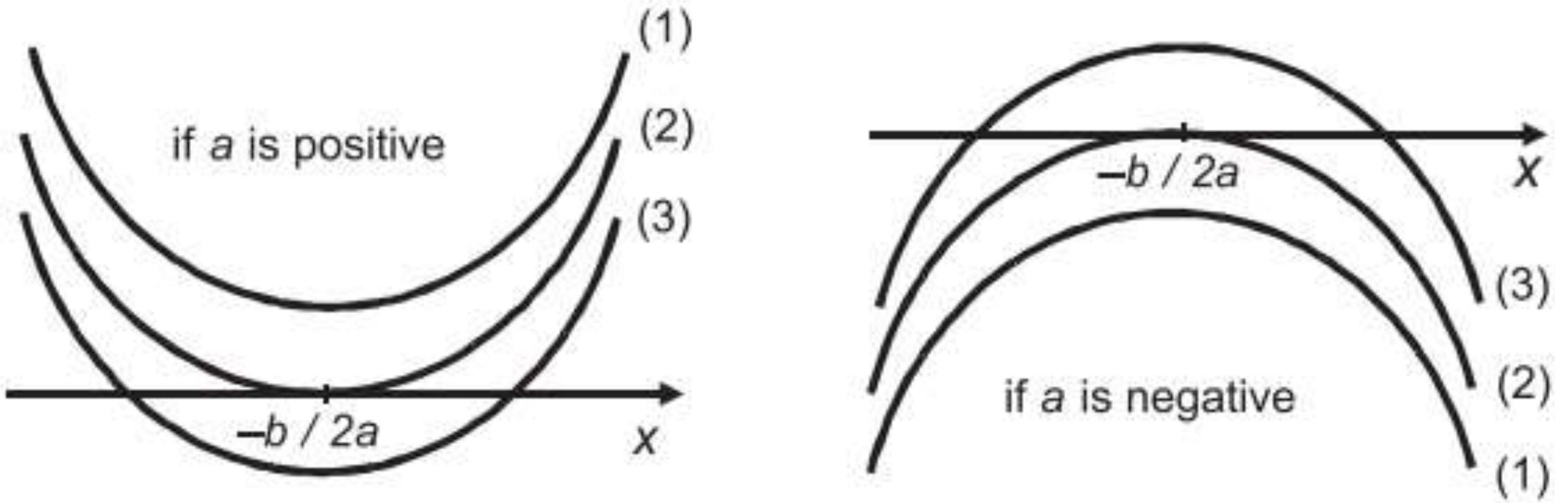


Parabola



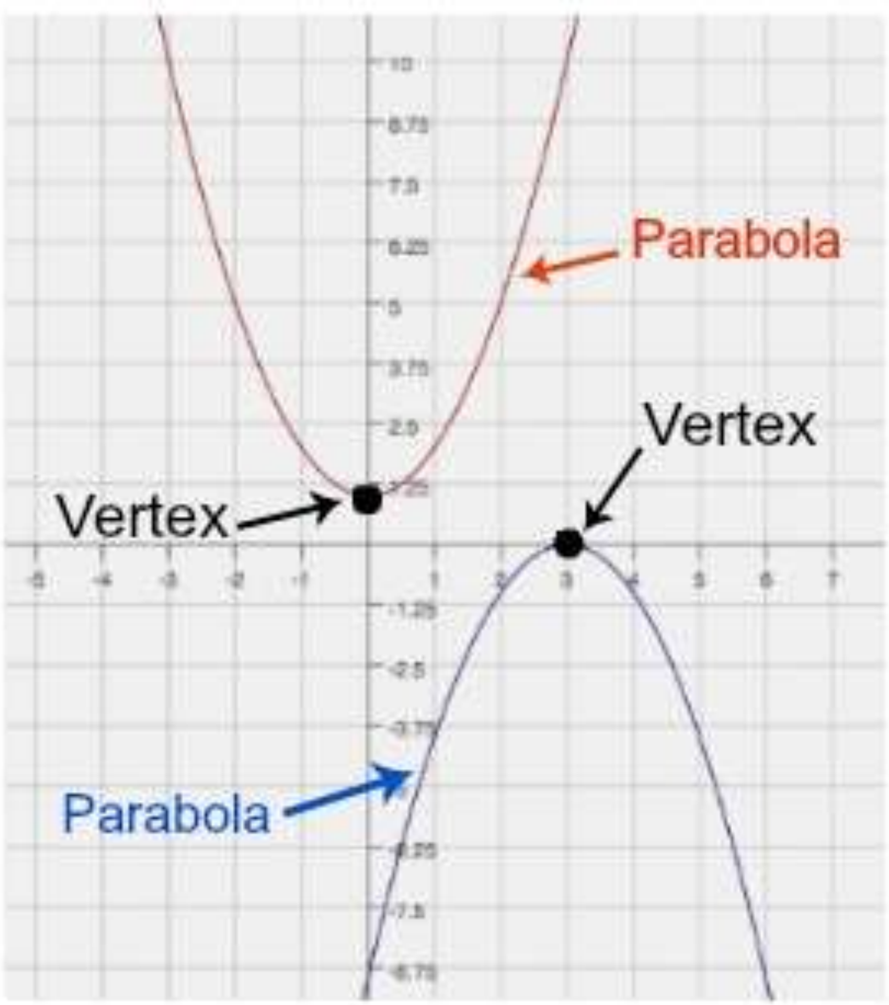
Parabola

Quadratic functions $y = ax^2 + bx + c$

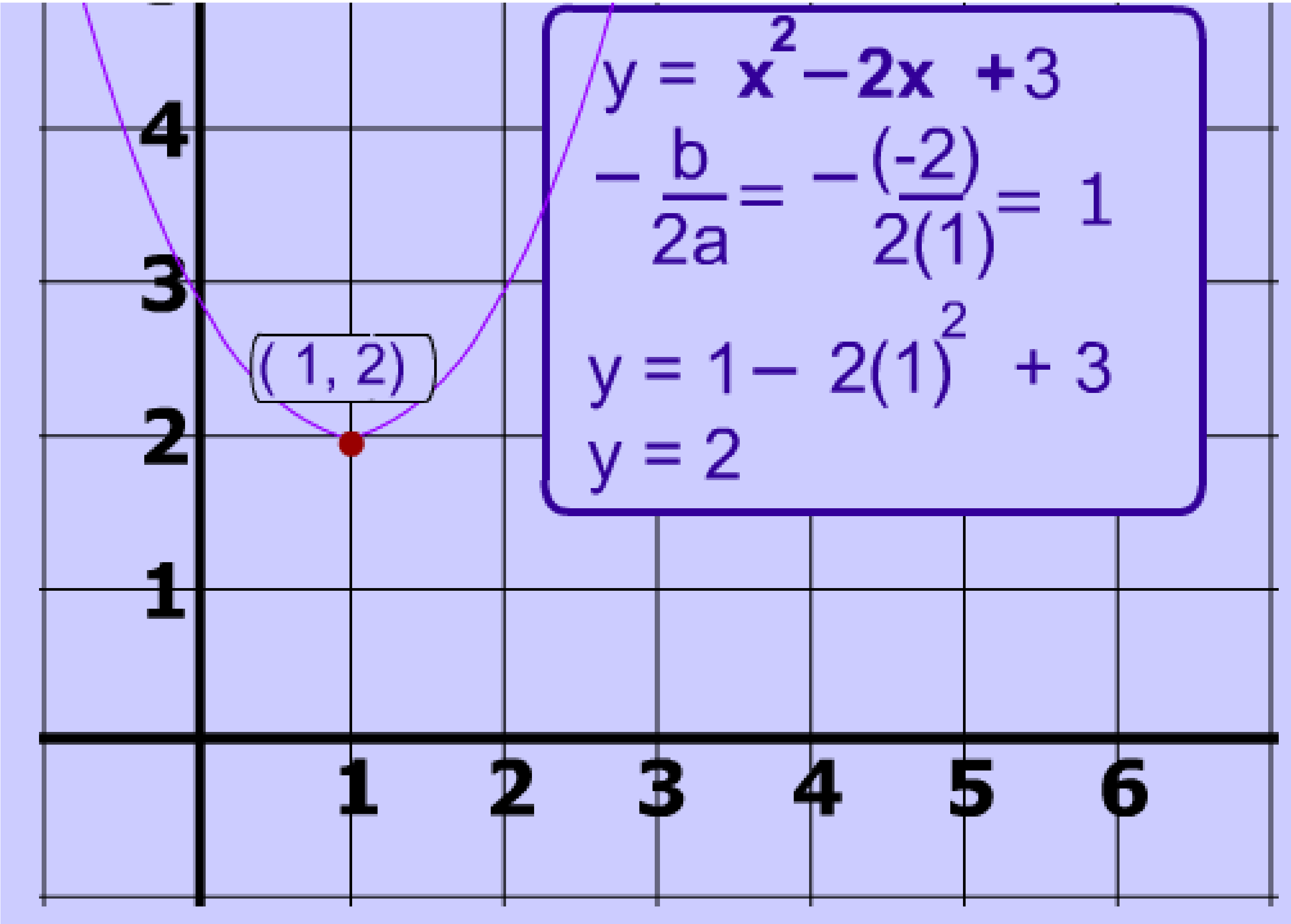


(1) $b^2 - 4ac < 0$; (2) $b^2 - 4ac = 0$; (3) $b^2 - 4ac > 0$

Parabola



Parabola



1) The square trinomial $P_2(x) = ax^2 + bx + c$, where $D \geq 0$ ($D = b^2 - 4ac$ is the discriminant of $ax^2 + bx + c = 0$), can be presented as a product of linear factors:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

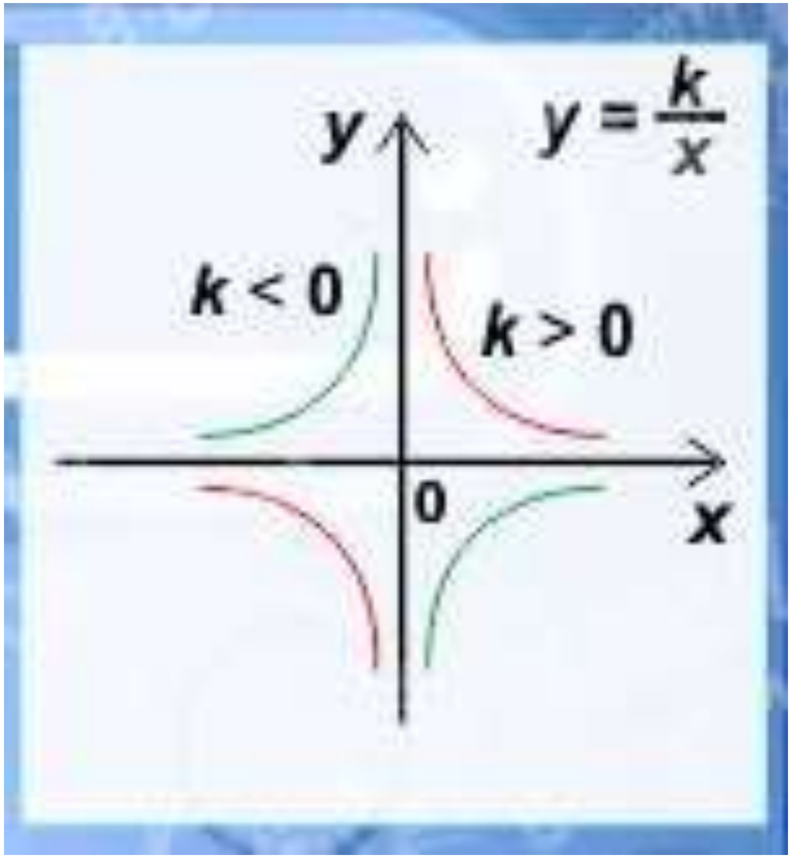
where x_1 and x_2 are roots of the square trinomial.

2) ***Vieta*** theorem

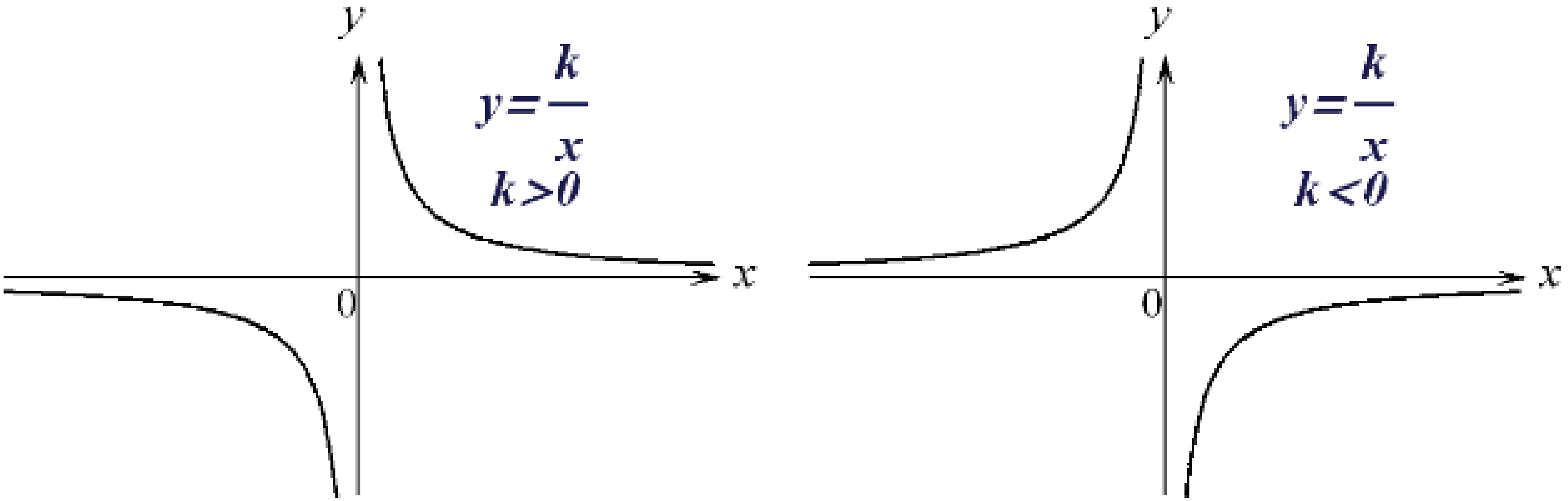
$$x_1 + x_2 = -\frac{b}{a}$$

$$x_1 \cdot x_2 = \frac{c}{a}$$

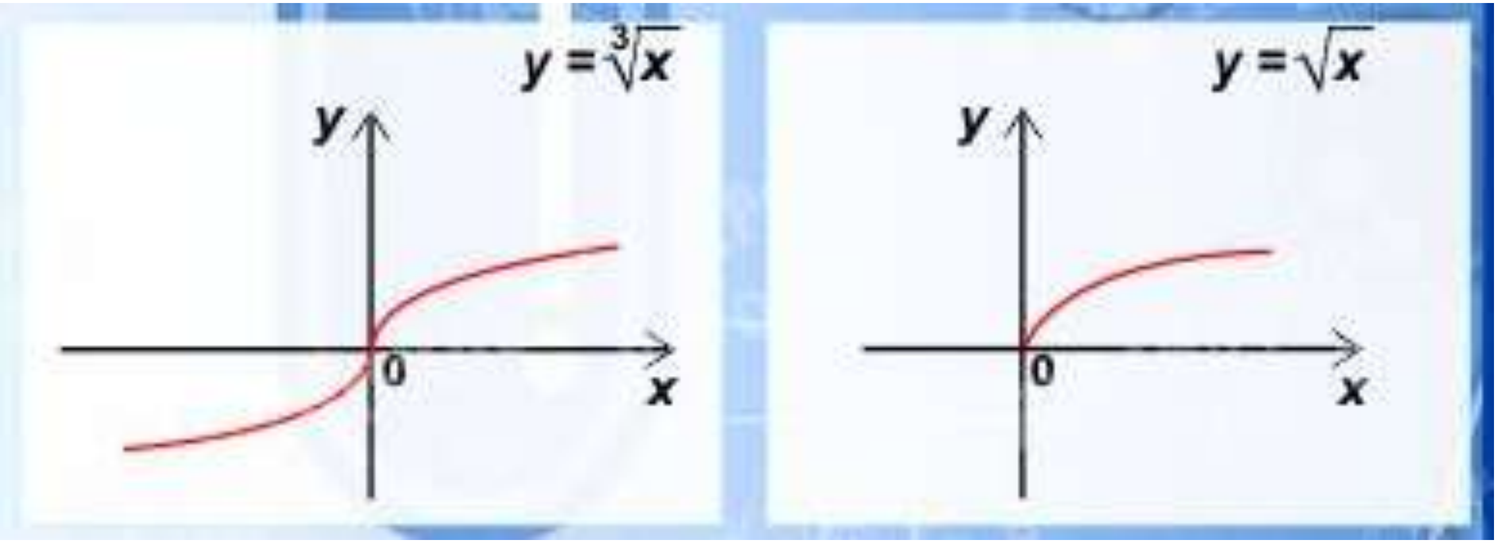
Hyperbola



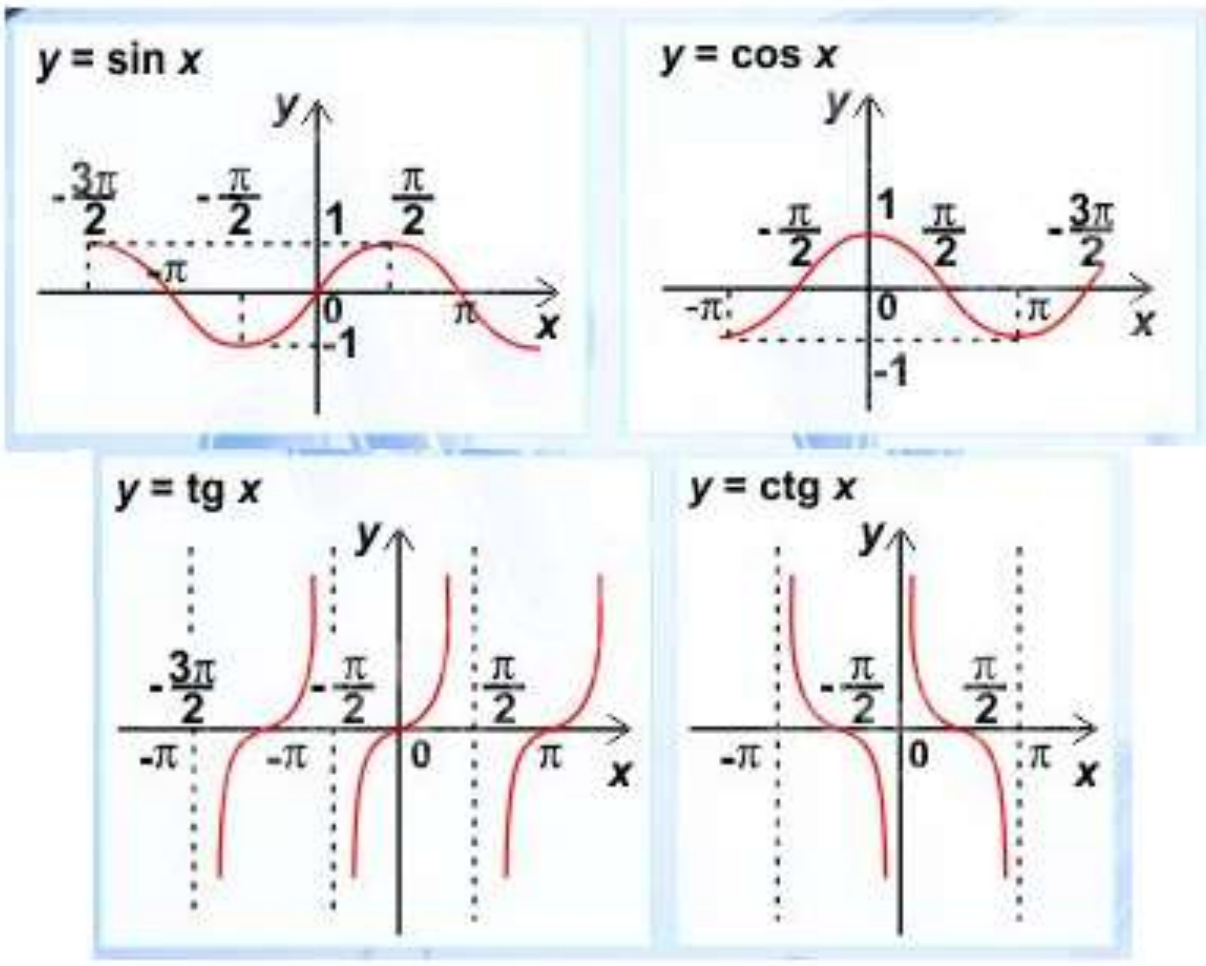
Hyperbola



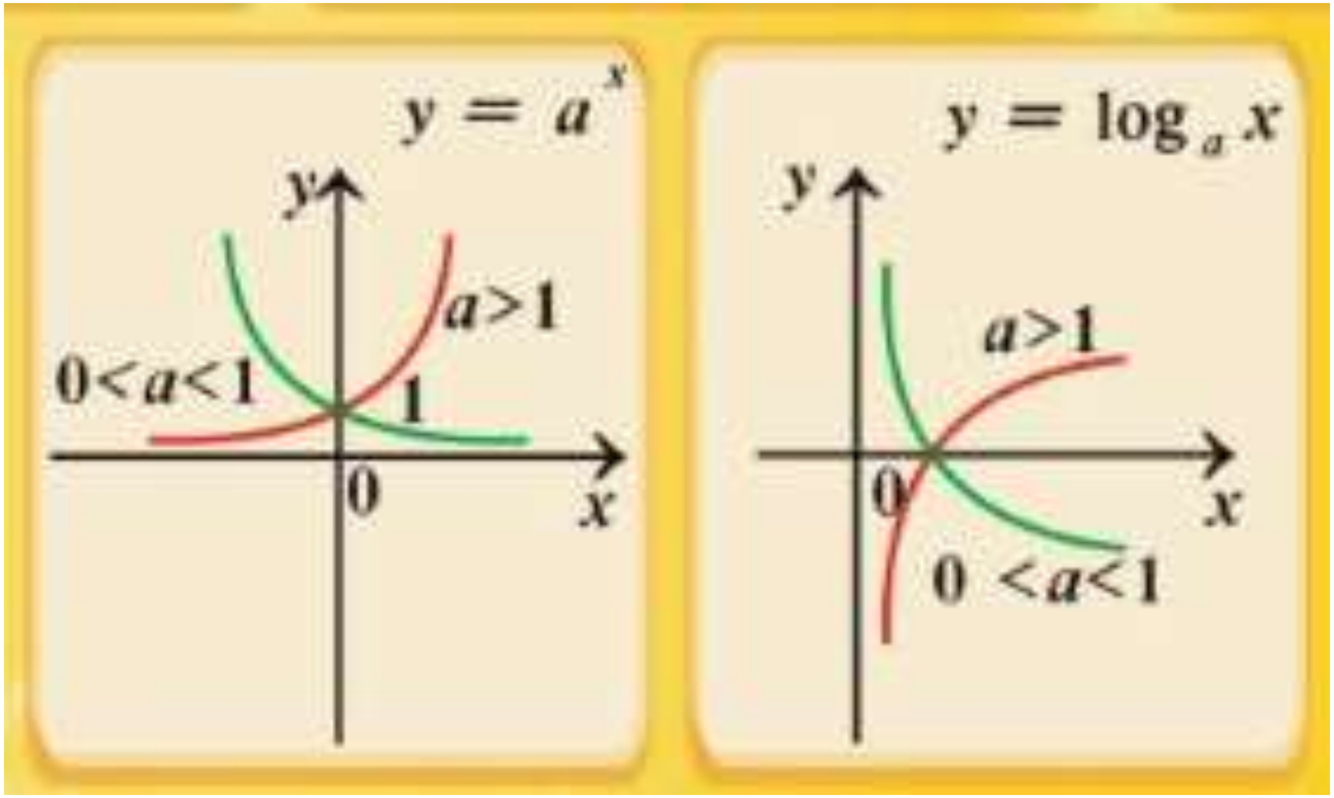
Roots



Trigonometric functions



Expontial and logarithmic functions



Types of functions

Composite Function

$$y = F(x) = f(\varphi(x))$$

It is said that $F(x)$ as a function of ***x*** is a composite function ("a function of a function") formed of the function f and φ. The function $u = \varphi(x)$ entering the expression $f(\varphi(x))$ is referred to as the intermediate variable.

Types of functions

Implicit Function $$F(x,y)=0$$

We can say that an implicit function ***y*** (which can be multiple-valued) of an independent variable ***x*** is a function in which values are found from an equation connecting ***x*** and ***y*** not solved in ***y***.

$$y=\frac{1}{2}\sin y+x$$

Types of functions

Inverse Function

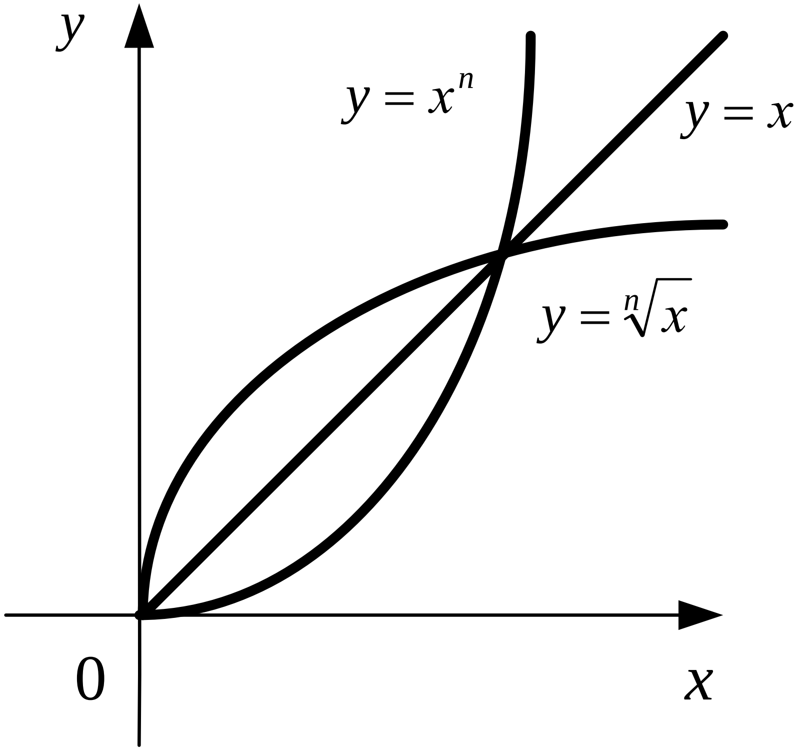
One of the methods of implicit representation of a function is representing the function $y = f(x)$ by means of the relationship $x = \varphi(y)$

$$y = x^n$$

$$y = \sqrt[n]{x}$$

Types of functions

Inverse Function



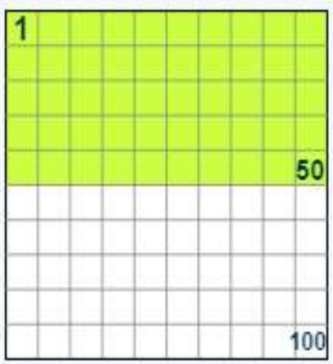
Simple and compound interest in economic studies (percent)

Percent

When we say "Percent" we are really saying "per 100"

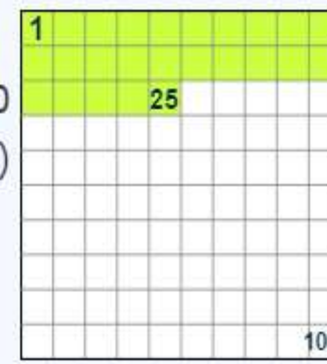
One percent (**1%**) means 1 per 100.

1% of this line is shaded green: it is very small isn't it?



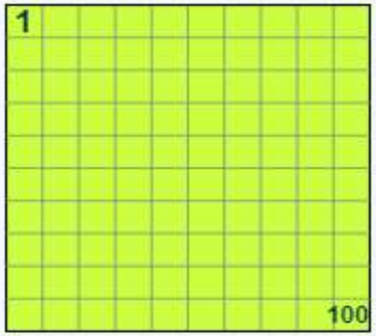
50% means 50 per 100
(50% of this box is green)

25% means 25 per 100
(25% of this box is green)



Examples:

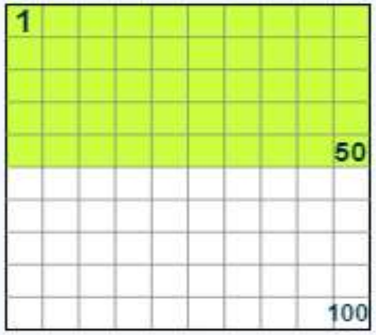
100% means all



Example:

100% of **80** is $\frac{100}{100} \times 80 = 80$

50% means half

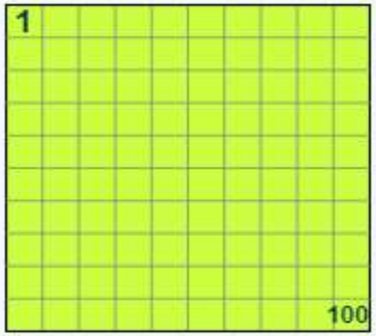


Example:

50% of **80** is $\frac{50}{100} \times 80 = 40$

Examples:

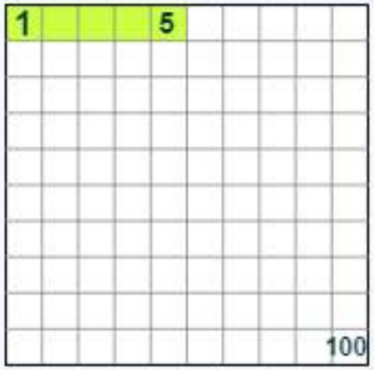
100% means all



Example:

100% of **80** is $\frac{100}{100} \times 80 = 80$

5% means $\frac{5}{100}$ths



Example:

5% of **80** is $\frac{5}{100} \times 80 = 4$

Because "Percent" means "per 100" think:

"this should be divided by 100"

So **75%** really means $\frac{75}{100}$

And **100%** is $\frac{100}{100}$, or exactly **1**

(100% of any number is just the number, unchanged)

And **200%** is $\frac{200}{100}$, or exactly **2**

(200% of any number is twice the number)

A **Percent** can also be expressed as a ***Decimal*** or a ***Fraction***

A Half can be written...

As a percentage:	50%
As a decimal:	0,5
As a fraction:	$^{1}/_{2}$

Example: 15% of 200 apples are bad. How many apples are bad?

$$15\% = \frac{15}{100}$$

$$\text{And } \frac{15}{100} \times 200 = 15 \times \frac{200}{100}$$

$$= 15 \times 2$$

$$= \mathbf{30\ apples}$$

30 apples are bad

Example: A Skateboard is reduced 25% in price.
The old price was $120.
Find the new price.

First, find 25% of $120:

$$25\% = \frac{25}{100}$$

$$\text{And } \frac{25}{100} \times \$120 = \mathbf{\$30}$$

25% of $120 is $30

So the **reduction** is $30

Take the reduction from the original price

$$\$120 - \$30 = \$90$$

The Price of the Skateboard in the sale is **$90**

Compound interest formula

Compound interest is interest calculated on top of the original amount including any interest accumulated so far. The compound interest formula is:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

Where:

- A represents the final amount
- P represents the original principal amount
- r is the interest rate over a given period
- n represents the number of times the interest rate is applied

Compound interest

Compound interest is similar to simple interest in that the interest is added on annually.

The difference between the two is that simple interest is a fixed amount of interest that is added on every year, whereas with compound interest the amount you are calculating interest on, changes every year.

The interest is calculated for the first year and is then added on to the original amount to give you the amount after the first year.

The interest for the second year is then calculated from the amount after the first year, which then gives you a different amount of interest gained from the first year.

Example

Calculate the amount of compound interest Jane will have earned on $£6000$ at 2.8% for 3 years.

Method 1

Year 1

$2.8\%\ of\ 6000$

$= 0.028 \times 6000$

$= £168$

Year 2

$2.8\%\ of\ 6168$

$= 0.028 \times 6168$

$= £172.70$

Year 3

$2.8\%\ of\ 6340.70$

$= 0.028 \times 6340.70$

$= £177.54$

Amount after year 3: $£6340.70 + £177.54 = £6518.24$

Total amount of compound interest earned

$= £6518.24 - £6000 = £518.24$

Example

Calculate the amount of compound interest Jane will have earned on $\pounds 6000$ at 2.8% for 3 years.

Method 2

$$A = P(1 + \frac{r}{100})^n$$

This is a much quicker method.

As the interest is going up by $2.8\%\, p.\,a.$ this means that each year the amount is 102.8% of the previous year. Therefore:

$102.8\%\, of\, 6000$

$= 1.028 \times 6000$

$= \pounds 6168$

However, this only gives you the amount after year 1. To get the amount after year 3:

$(1.028)^3 \times 6000$

$= \pounds 6518.24$

Now try this:

Question

Calculate the compound interest earned on $£8000$ at 2.2% per annum for 5 years.

Question

Calculate the compound interest earned on $£8000$ at 2.2% per annum for 5 years.

answer

$$(1.022)^5 \times 8000$$

$$= £8919.58$$

Total amount of interest earned $= £8919.58 - £8000 = £919.58$

Answer: $£919.58$

Compound interest

Compound interest means that each time interest is paid onto an amount saved or owed, the added interest also receives interest from then on.

Put simply, compound interest changes the amount of money in the bank each time and a new calculation has to be worked out.

Examples

Calculate the interest on borrowing £40 for 3 years if the compound interest rate is 5% per year.

- Year 1: $£40 + 5\% = £40 + £2 = £42$
- Year 2: $£42 + 5\% = £42 + £2.10 = £44.10$
- Year 3: $£44.10 + 5\% = £44.10 + £2.21 = £46.31$

As this is a problem involving money, all rounding should be done to 2 decimal places to represent the pence.

Note that the example above is exactly the same as the example for simple interest, but the answers are different as compound interest changes the amount each period.

Compound Interest Formula

$$A = P(1 + \frac{r}{k})^{kt}$$

$\boldsymbol{A}$ is the balance in the account after $\boldsymbol{t}$ years.

$\boldsymbol{P}$ is the starting balance of the account (also called initial deposit, or principal).

$\boldsymbol{r}$ is the annual interest rate in decimal form.

$\boldsymbol{k}$ is the number of compounding periods in one year.

- If the compounding is done annually (once a year), $\boldsymbol{k = 1}$.
- If the compounding is done semiannually (twice a year), $\boldsymbol{k = 2}$.
- If the compounding is done quarterly, $\boldsymbol{k = 4}$.
- If the compounding is done monthly, $\boldsymbol{k = 12}$.
- If the compounding is done daily, $\boldsymbol{k = 365}$.

Example

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit $3,000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

In this example,

$P = \$3000$	the initial deposit
$r = 0.06$	6% annual interest rate
$k = 12$	12 months in 1 year
$t = 20$	since we're looking for how much we'll have after 20 years

$$A = P(1 + \tfrac{r}{k})^{kt}$$

So $A = 3000(1 + \frac{0.06}{12})^{12\cdot 20} = \$9930.61$ (round your answer to the nearest penny).

Example

You know that you will need \$40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

In this example, we're looking for P.

$r = 0.04$	4%
$k = 4$	4 quarters in one year
$t = 18$	Since we know the balance in 18 years
$A = \$40,000$	The amount we have in 18 years

In this case, we're going to have to set up the equation and solve for P.

$$\begin{aligned} 40000 &= P(1+\tfrac{0.04}{4})^{4\cdot 18} \\ 40000 &= P(2.0471) \\ \frac{40000}{2.0471} &= P \\ \$19539.84 &= P \end{aligned}$$

So you would need to deposit \$19,539.84 now to have \$40,000 in 18 years.

Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better. If your calculator allows it, do all your calculations without rounding in the calculator and only round the final answer.

Example

To see why not over-rounding is so important, suppose you were investing $1,000 at 5% interest compounded monthly for 30 years.

$P = \$1000$	the initial deposit
$r = 0.05$	5%
$k = 12$	12 months in one year
$t = 30$	since we're looking for the amount after 30 years

If we first compute $\frac{r}{k}$, we find $\frac{0.05}{12} = 0.00416666666667$

Here is the effect of rounding this to different values:

$\frac{r}{k}$ rounded to:	Gives A to be:	Error
0.004	\$4208.59	\$259.15
0.0042	\$4521.45	\$53.71
0.00417	\$4473.09	\$5.35
0.004167	\$4468.28	\$0.54
0.0041667	\$4467.80	\$0.06
no rounding	\$4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough—$5 off of $4,500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

Annual Percentage Yield

The **annual percentage yield** is the actual percent a quantity increases in one year. It can be calculated as

$$APR = (1 + \frac{r}{k})^k - 1$$

Notice this is equivalent to finding the value of \$1 after 1 year and subtracting the original dollar.

Example

Bank A offers an account paying 1.2% compounded quarterly. Bank B offers an account paying 1.1% compounded monthly. Which is offering a better rate?

We can compare these rates using the annual percentage yield—the actual percent increase in a year.

Bank A: $APR = (1 + \frac{0.012}{4})^4 - 1 = 0.012054 = 1.2054\%$

Bank B: $APR = (1 + \frac{0.011}{12})^{12} - 1 = 0.011056 = 1.1056\%$

Bank B's monthly compounding is not enough to catch up with Bank A's better APR. Bank A offers a better rate.

Example

If you invest $2,000 at 6% compounded monthly, how long will it take the account to double in value?

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

$P = \$2000$	the initial deposit
$r = 0.06$	6% annual rate
$k = 12$	12 months in one year

So our general equation is $A = 2000(1 + \frac{0.06}{12})^{12t}$.

Example

So our general equation is $A = 2000(1 + \frac{0.06}{12})^{12t}$.

We also know that we want our ending amount to be double of $2,000, which is $4,000, so we're looking for t so that $A = 4000$. To solve this, we set our equation for A equal to 4000.

$$
\begin{aligned}
4000 &= 2000(1 + \tfrac{0.06}{12})^{12t} && \text{Divide both sides by 2000} \\
2 &= (1.005)^{12t} && \text{To solve for the exponent, take the log of both sides} \\
log(2) &= log(1.005)^{12t} && \text{Use the exponent property of logs on the right side}
\end{aligned}
$$

Example

$$\log(2) = \log(1.005)^{12t}$$

Use the exponent property of logs on the right side

$$\log(2) = 12t\,\log(1.005)$$

Now we can divide both sides by $12\log(1.005)$

$$\frac{\log(2)}{12\log(1.005)} = t$$

Approximate this to a decimal

$$t = 11.581$$

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example, if you rounded $\log(2)$ to 0.301 and $\log(1.005)$ to 0.00217, then your final answer would have been about 11.577 years.