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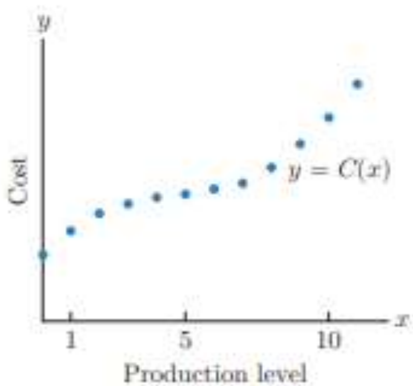
Theme:
Limit of a function Continuity of a function. Differential calculus of functions of one variable. Analysis of economic indicator interrelationships

Part 2. Differential calculus of the function of one variable

COST FUNCTION - функція витрат – C(x)

Applications of Derivatives to Business and Economics



In recent years, economic decision making has become more and more mathematically oriented. Faced with huge masses of statistical data, depending on hundreds or even thousands of different variables, business analysts and economists have increasingly turned to mathematical methods to help them describe what is happening, predict the effects of various policy alternatives, and choose reasonable courses of action from the myriad of possibilities. Among the mathematical methods employed is calculus. In this section we illustrate just a few of the many applications of calculus to business and economics. All our applications will center on what economists call the *theory of the firm*. In other words, we study the activity of a business (or possibly a whole industry) and restrict our analysis to a time period during which background conditions (such as supplies of raw materials, wage rates, and taxes) are fairly constant. We then show how derivatives can help the management of such a firm make vital production decisions.

Management, whether or not it knows calculus, utilizes many functions of the sort we have been considering. Examples of such functions are

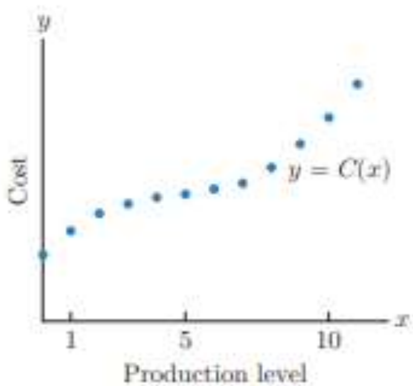
$C(x)$ = cost of producing x units of the product,

$R(x)$ = revenue generated by selling x units of the product,

$P(x) = R(x) - C(x)$ = the profit (or loss) generated by producing and (selling x units of the product.)

REVENUE – ДОХІД – R(x)

Applications of Derivatives to Business and Economics



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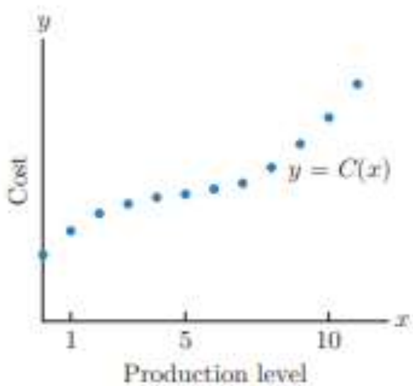
PROFIT – ПРИБУТОК – P(x)

Applications of Derivatives to Business and Economics



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C(x), R(x) and P(x)

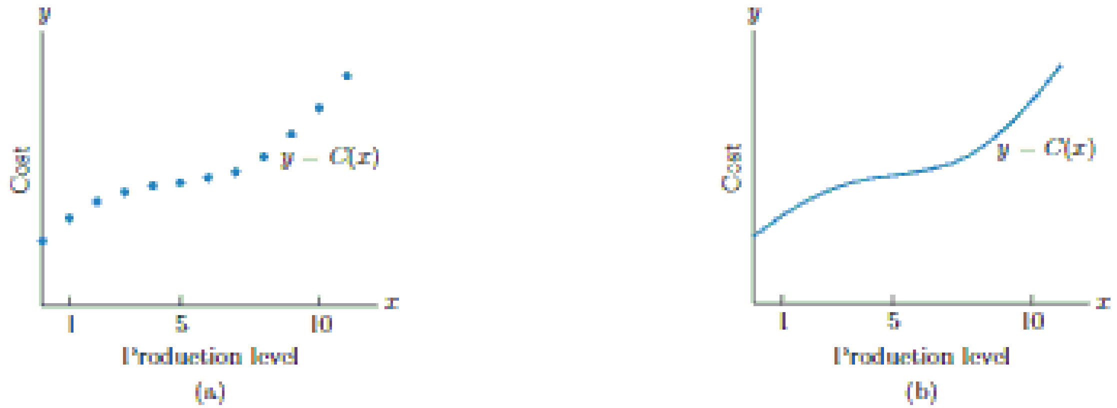

Figure 1 A cost function.

Note that the functions $C(x)$, $R(x)$, and $P(x)$ are often defined only for nonnegative integers, that is, for $x = 0, 1, 2, 3, \ldots$. The reason is that it does not make sense to speak about the cost of producing -1 cars or the revenue generated by selling 3.62 refrigerators. Thus, each function may give rise to a set of discrete points on a graph, as in Fig. 1(a). In studying these functions, however, economists usually draw a smooth curve through the points and assume that $C(x)$ is actually defined for all positive x. Of course, we must often interpret answers to problems in light of the fact that x is, in most cases, a nonnegative integer.

REVENUE – ДОХІД

Revenue Functions In general, a business is concerned not only with its costs, but also with its revenues. Recall that, if $R(x)$ is the revenue received from the sale of x units of some commodity, then the derivative $R'(x)$ is called the *marginal revenue*. Economists use this to measure the rate of increase in revenue per unit increase in sales.

If x units of a product are sold at a price p per unit, the total revenue $R(x)$ is given by

$$R(x) = x \cdot p.$$

If a firm is small and is in competition with many other companies, its sales have little effect on the market price. Then, since the price is constant as far as the one firm is concerned, the marginal revenue $R'(x)$ equals the price p [that is, $R'(x)$ is the amount that the firm receives from the sale of one additional unit]. In this case, the revenue function will have a graph as in Fig.

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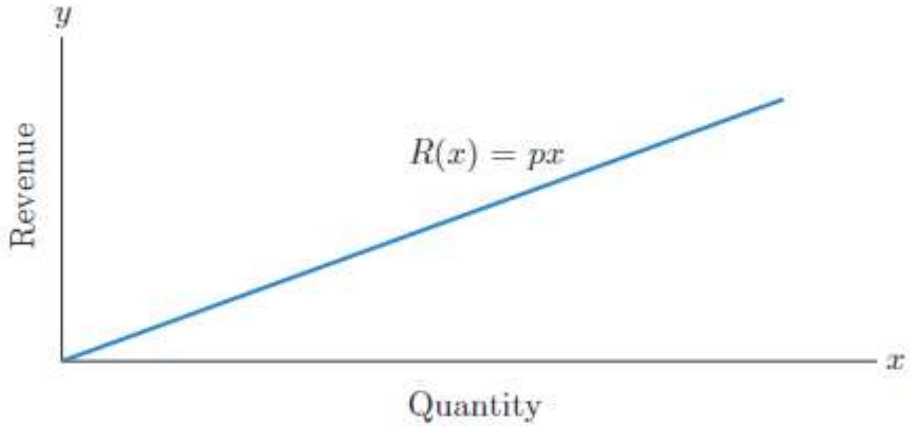


Figure A revenue curve.

An interesting problem arises when a single firm is the only supplier of a certain product or service, that is, when the firm has a monopoly. Consumers will buy large amounts of the commodity if the price per unit is low and less if the price is raised.

REVENUE – ДОХІД

EXAMPLE

Maximizing Revenue The demand equation for a certain product is $p = 6 - \frac{1}{2}x$ dollars. Find the level of production that results in maximum revenue.

SOLUTION In this case, the revenue function $R(x)$ is

$$R(x) = x \cdot p = x\left(6 - \frac{1}{2}x\right) = 6x - \frac{1}{2}x^2$$

dollars. The marginal revenue is given by

$$R'(x) = 6 - x.$$

The graph of $R(x)$ is a parabola that opens downward. (See Fig. 6.) It has a horizontal tangent precisely at those x for which $R'(x) = 0$—that is, for those x at which marginal revenue is 0. The only such x is $x = 6$. The corresponding value of revenue is

$$R(6) = 6 \cdot 6 - \frac{1}{2}(6)^2 = 18 \text{ dollars.}$$

Thus, the rate of production resulting in maximum revenue is $x = 6$, which results in total revenue of 18 dollars.

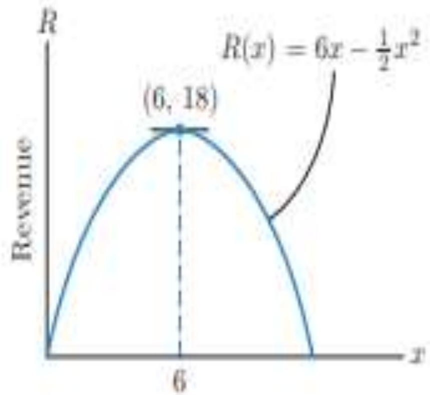


Figure Maximizing revenue.

PROFIT – ПРИБУТОК

EXAMPLE **Maximizing Profits** Suppose that the demand equation for a monopolist is $p = 100 - .01x$ and the cost function is $C(x) = 50x + 10{,}000$. Find the value of x that maximizes the profit and determine the corresponding price and total profit for this level of production. (See Fig. 9.)

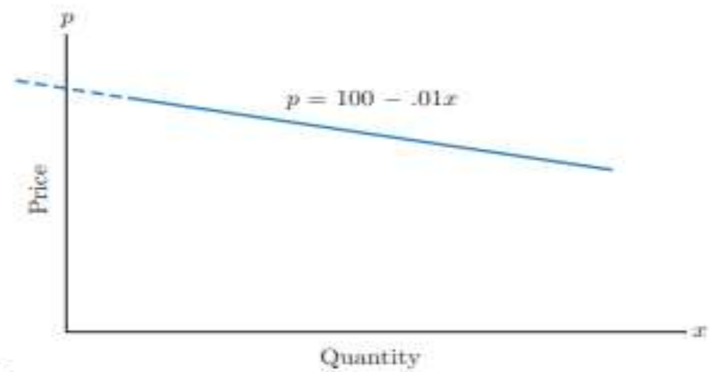


Figure A demand curve.

SOLUTION The total revenue function is

$$R(x) = x \cdot p = x(100 - .01x) = 100x - .01x^2.$$

Hence, the profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 100x - .01x^2 - (50x + 10{,}000) \\ &= -.01x^2 + 50x - 10{,}000. \end{aligned}$$

The graph of this function is a parabola that opens downward. (See Fig.) Its highest point will be where the curve has zero slope, that is, where the marginal profit $P'(x)$ is zero. Now,

$$P'(x) = -.02x + 50 = -.02(x - 2500).$$

COST FUNCTION - функція витрат

EXAMPLE

Marginal Cost Analysis Suppose that the cost function for a manufacturer is given by $C(x) = (10^{-6})x^3 - .003x^2 + 5x + 1000$ dollars.

(a) Describe the behavior of the marginal cost.

(b) Sketch the graph of $C(x)$.

SOLUTION The first two derivatives of $C(x)$ are given by

$$C'(x) = (3 \cdot 10^{-6})x^2 - .006x + 5$$
$$C''(x) = (6 \cdot 10^{-6})x - .006.$$

Let us sketch the marginal cost $C'(x)$ first. From the behavior of $C'(x)$, we will be able to graph $C(x)$. The marginal cost function $y = (3 \cdot 10^{-6})x^2 - .006x + 5$ has as its graph a parabola that opens upward. Since $y' = C''(x) = .000006(x - 1000)$, we see that the parabola has a horizontal tangent at $x = 1000$. So the minimum value of $C'(x)$ occurs at $x = 1000$. The corresponding y-coordinate is

$$(3 \cdot 10^{-6})(1000)^2 - .006 \cdot (1000) + 5 = 3 - 6 + 5 = 2.$$

The graph of $y = C'(x)$ is shown in Fig. 2. Consequently, at first, the marginal cost decreases. It reaches a minimum of 2 at production level 1000 and increases thereafter.

COST FUNCTION - функція витрат

This answers part (a). Let us now graph $C(x)$. Since the graph shown in Fig. 2 is the graph of the derivative of $C(x)$, we see that $C'(x)$ is never zero, so there are no relative extreme points. Since $C'(x)$ is always positive, $C(x)$ is always increasing (as any cost curve should). Moreover, since $C'(x)$ decreases for x less than 1000 and increases for x greater than 1000, we see that $C(x)$ is concave down for x less than 1000, is concave up for x greater than 1000, and has an inflection point at $x = 1000$. The graph of $C(x)$ is drawn in Fig. 3. Note that the inflection point of $C(x)$ occurs at the value of x for which marginal cost is a minimum.

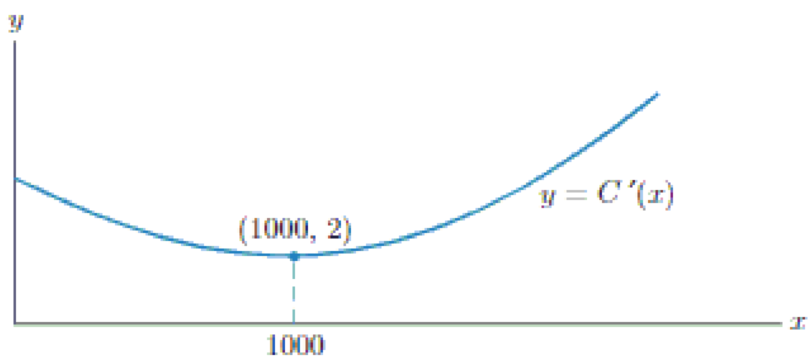


Figure 2 A marginal cost function.

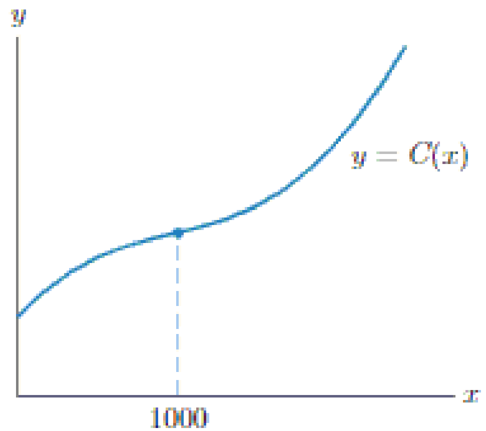


Figure 3 A cost function.

EXERCISES

1. **Minimizing Marginal Cost** Given the cost function $C(x) = x^3 - 6x^2 + 13x + 15$, find the minimum marginal cost. $1

2. **Minimizing Marginal Cost** If a total cost function is $C(x) = .0001x^3 - .06x^2 + 12x + 100$, is the marginal cost increasing, decreasing, or not changing at $x = 100$? Find the minimum marginal cost.

3. **Maximizing Revenue Cost** The revenue function for a one-product firm is

$$R(x) = 200 - \frac{1600}{x+8} - x.$$

Find the value of x that results in maximum revenue. 32

4. **Maximizing Revenue** The revenue function for a particular product is $R(x) = x(4 - .0001x)$. Find the largest possible revenue. $R(20{,}000) = 40{,}000$ is maximum possible.

5. **Cost and Profit** A one-product firm estimates that its daily total cost function (in suitable units) is $C(x) = x^3 - 6x^2 + 13x + 15$ and its total revenue function is $R(x) = 28x$. Find the value of x that maximizes the daily profit. 5

Lecture plan

1. A derivative. The elementary rules of derivatives calculation

2. The table of basic derivatives

3. The derivative of the composite function

4. The physical meaning of a derivative

5. The economic meaning of a derivative

6. The geometric meaning of a derivative: the equations of the tangent and the normal line

7. The derivative of the implicit function

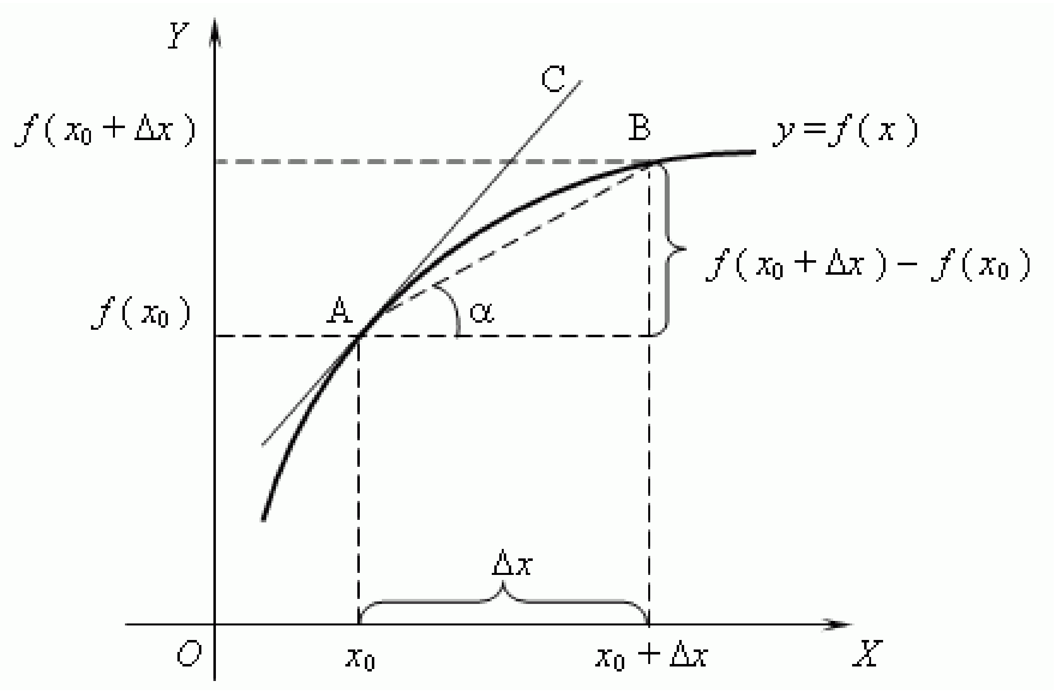
8. The derivative of the power exponential function (logarithmic differentiation)

9. The derivative of the function given in parametric form

1. A derivative.
The elementary rules of derivatives calculation

Definition. The limit of the ratio of the function increment to the argument increment, while the last one approaches zero is called ***a derivative*** of the function $y = f(x)$ at a point x and is denoted as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



The elementary rules of derivatives calculation

Let the functions $f(x)=u$ *and* $g(x)=v$ *have their derivatives* u' *,* v' *at some definite point. Then*

– the constant multiplier can be taken out from a sign of the derivative, i.e.

$$(Cu)' = Cu'$$

– the derivative of the algebraic sum (difference) of functions is equal to the algebraic sum (difference) of derivatives of components, i.e.

$$(u \pm v)' = u' \pm v'$$

– the derivative of the product is equal to the sum of products of the derivative of the first multiplier by the second one without changing and the derivative of the second multiplier by the first one without changing, i.e.

$$(u \cdot v)' = u' \cdot v + v' \cdot u$$

– derivative of a fraction is equal to a fraction whose denominator is the square of the denominator of the given fraction, and the numerator is the difference between the product of the denominator by the derivative of the numerator and the product of the numerator by the derivative of the denominator, i.e.

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} \qquad \text{if} \qquad v \neq 0$$

2. The table of basic derivatives of the simplest elementary functions

$$(C)' = 0$$

$$\left(a^x\right)' = a^x \cdot \ln a$$

$$\left(x^n\right)' = nx^{n-1}$$

$$\left(e^x\right)' = e^x$$

$$(x)' = 1$$

$$(\ln x)' = \frac{1}{x}$$

$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

2. The table of basic derivatives of the simplest elementary functions

$$(\sin x)' = \cos x$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\cos x)' = -\sin x$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(tgx)' = \frac{1}{\cos^2 x}$$

$$(arctgx)' = \frac{1}{1+x^2}$$

$$(ctgx)' = -\frac{1}{\sin^2 x}$$

$$(arcctgx)' = -\frac{1}{1+x^2}$$

Example 1. Find the derivative of the function:

$$y = 4x^6 + \frac{1}{2x^3} - 3\sqrt[4]{x^5} + \frac{2}{3}$$

Solution. We transform each summand using the formulas for the power function:

$$\frac{1}{x^n} = x^{-n} \qquad \sqrt[m]{x^n} = x^{\frac{n}{m}}$$

Then we obtain:

$$\frac{1}{2x^3} = \frac{1}{2}x^{-3} \qquad 3\sqrt[4]{x^5} = 3x^{\frac{5}{4}}$$

$$y' = \left(4x^6 + \frac{1}{2x^3} - 3\sqrt[4]{x^5} + \frac{2}{3}\right)' = \left| \begin{array}{l} \textit{we apply the property} \\ (u \pm v)' = u' \pm v' \end{array} \right| =$$

$$= \left(4x^6\right)' + \left(\frac{1}{2}x^{-3}\right)' - \left(3x^{\frac{5}{4}}\right)' + \left(\frac{2}{3}\right)' =$$

Now, using the table, we calculate the derivative:

$$= \left| \begin{array}{l} C' = 0, (Cu)' = Cu' \\ \left(x^n\right)' = nx^{n-1} \end{array} \right| =$$

$$= 6 \cdot 4x^5 + \frac{1}{2} \cdot (-3) \cdot x^{-3-1} - 3 \cdot \frac{5}{4} x^{\frac{5}{4}-1} + 0 = 24x^5 - \frac{3}{2}x^{-4} - \frac{15}{4}x^{\frac{1}{4}}$$

Example 2. Find the derivatives of the function: $y = ctgx \cdot x^5$

Solution.

$$y' = \left[ctgx \cdot x^5\right]' = \left| (u \cdot v)' = u' \cdot v + u \cdot v' \right| =$$

$$= (ctgx)' \cdot x^5 + ctgx \cdot \left(x^5\right)' = \left| \begin{matrix} (ctgx)' = -\dfrac{1}{\sin^2 x} \\ \left(x^n\right)' = nx^{n-1} \end{matrix} \right| =$$

$$= \left(-\frac{1}{\sin^2 x}\right) \cdot x^5 + ctgx \cdot 5x^4 = -\frac{x^5}{\sin^2 x} + ctgx \cdot 5x^4$$

Example 3. Find the derivative of the function:

$$y = \frac{arctgx}{2x + x^3}$$

Solution.

$$y' = \left(\frac{arctgx}{2x + x^3} \right)' = \left| \left(\frac{u}{v} \right)' = \frac{u' \cdot v - u \cdot v'}{v^2} \right| =$$

$$= \frac{(arctgx)' \cdot \left(2x + x^3\right) - arctgx \cdot \left(2x + x^3\right)'}{\left(2x + x^3\right)^2} =$$

$$= \frac{\frac{1}{x^2 + 1} \cdot \left(2x + x^3\right) - arctgx \cdot \left(2 + 3x^2\right)}{\left(2x + x^3\right)^2}$$

3. The derivative of the composite function

The derivative of the composite function is equal to the product of the derivative of the given function with respect to the intermediate argument by the derivative of the intermediate argument with respect to the independent variable.

$$y = f(g(x))$$

$$u = g(x)$$

$$y' = f'(u)$$

$$u' = g'(x)$$

$$y'_x = f'_u \cdot u'_x$$

4. The physical meaning of a derivative

If the function $y = f(x)$ is the equation of the way,

then $y' = f(x)'$ is the velocity

and $y'' = f(x)''$ is the acceleration.

5. The economic meaning of a derivative

An elasticity of function is defined by the formula:

$$E_x(y) = \frac{x}{y} \cdot y'_x$$

Elasticity of a function $y = f(x)$ approximately shows the change of one variable (y) if the other one (x) is changed within 1 %.

If $|E_x(y)| > 1$, then the function is elastic;

if $|E_x(y)| = 1$, then the function is neutral;

if $|E_x(y)| < 1$, then the function is inelastic.

5. The economic meaning of a derivative

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P-elasticity of Q: $\varepsilon = \frac{\text{\% change in quantity Q}}{\text{\% change in price P}}$

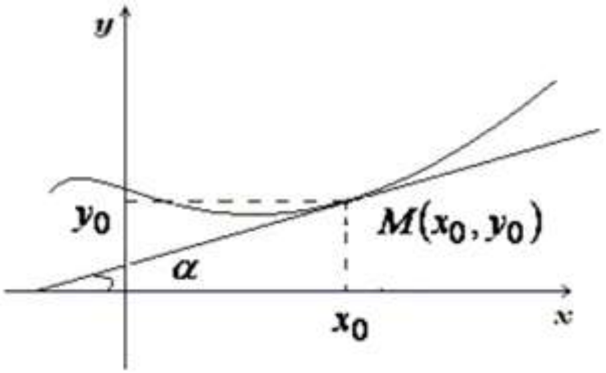
$\lvert\varepsilon\rvert > 1$	elastic	Q changes more than P
$\lvert\varepsilon\rvert = 1$	unit elastic	Q changes like P
$\lvert\varepsilon\rvert < 1$	inelastic	Q changes less than P

Suppose price rises by 1%. If the elasticity of supply is 0.5, quantity rises by .5%; if it is 1, quantity rises by 1%; if it is 2, quantity rises by 2%.

Let’s give an example.
Price elasticity of demand (**PED**) is a measure used in economics to show the responsiveness, or elasticity, of the quantity demanded of a good or service to a change in its price. More precisely, it gives the percentage change in quantity demanded in response to a one percent change in price.
In general, the demand for a good is said to be ***inelastic*** (or *relatively inelastic*) when the PED is less than one (in absolute value): that is, changes in price have a relatively small effect on the quantity of the good demanded.
The demand for a good is said to be ***elastic*** (or *relatively elastic*) when its PED is greater than one (in absolute value): that is, changes in price have a relatively large effect on the quantity of a good demanded.

6. The geometric meaning of a derivative: the equations of the tangent and the normal line

The derivative of a function at the given point is numerically equal to the slope of the tangent to the curve at this point. Therefore *the equation of a non-vertical tangent line* to the curve $y = f(x)$ at the point x_0 is as follows:



$$y - y_0 = f'(x_0)(x - x_0)$$

where $f'(x_0) = tg\alpha$,

α is the angle of the slope of the tangent to the curve at the point $M(x_0, y_0)$.

The equation of a vertical tangent line is $x = x_0$

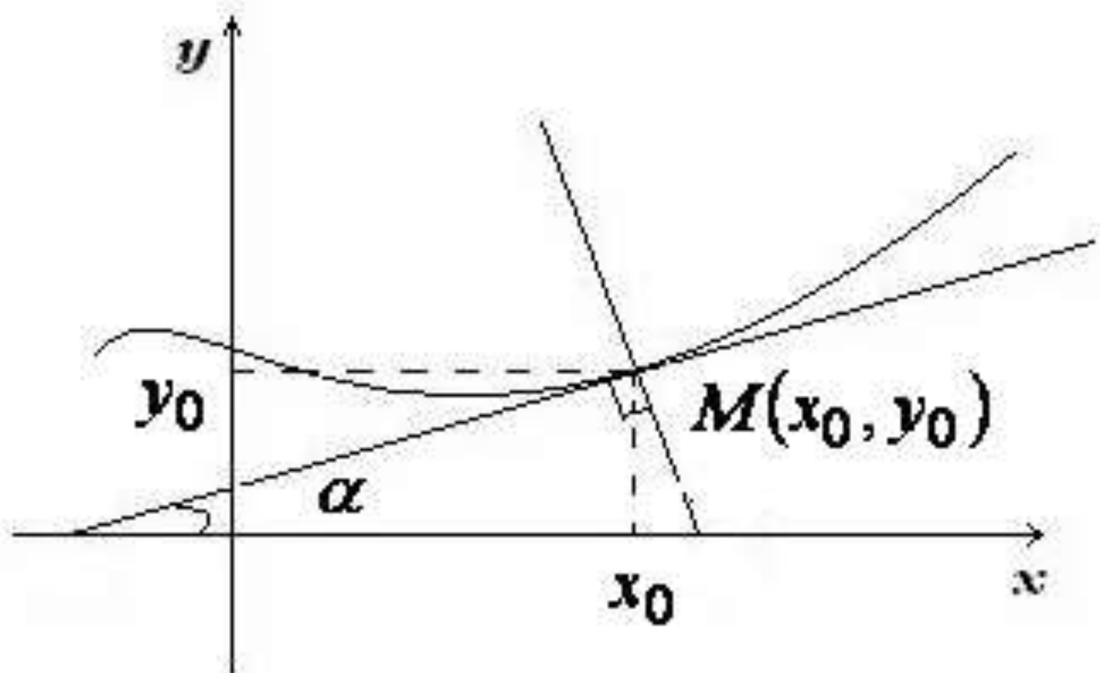
The normal line to the curve at the point $M(x_0, y_0)$ is a perpendicular to the tangent drawn to this curve at the given point. The equation of a non-horizontal normal line looks like

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$$

The equation of

a horizontal normal is

$$y = y_0$$



Example. Find the equations of the tangent and the normal line to the curve $y = x^2 + 5$ at the point with the abscissa $x_0 = 1$

Solution. The ordinate of the tangency point is $y_0 = 1^2 + 5 = 6$

The slope of the tangent is $k = f'(x_0) = f'(1)$

Find it:

$$f'(x) = y' = (x^2 + 5)' = 2x \qquad f'(1) = 2 \cdot 1 = 2$$

then

$$k_{\tan g} = 2$$

The equation of the tangent is as follows:

$$y - y_0 = f'(x_0) \cdot (x - x_0)$$

$$y - 6 = f'(1) \cdot (x - 1)$$

$$y-6=2(x-1)$$
$$y-6=2x-2$$
$$y=2x-2+6$$
$$y=2x+4$$

The slope of the normal line is

$$k_{norm}=-\frac{1}{f'(x_0)}=-\frac{1}{f'(1)}=-\frac{1}{k_{\tan g}}=-\frac{1}{2}$$

The equation of the normal line is

$$y-y_0=-\frac{1}{f'(x_0)}(x-x_0)$$
$$y-y_0=-\frac{1}{f'(1)}(x-x_0)$$

$$y - y_0 = k_{norm} \cdot (x - x_0)$$

$$y - 6 = -\frac{1}{2}(x - 1)$$

$$y - 6 = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{1}{2} + 6$$

$$y = -0.5x + 6.5$$

7. The derivative of the implicit function

An implicit function $F(x, y) = 0$ is a function that is defined implicitly by a relation between its argument and its value.

In order to find the derivative y' of the function implicitly given by the equation $F(x, y) = 0$ it is necessary to differentiate both the parts of the identity $F(x, y(x)) = 0$ with respect to the variable x using the rule of differentiation of a composite function. Then the obtained equation should be solved for y'

Example. Find the derivative of the function given by the following equation:

$$x^2 + y^2 = 9$$

Solution. Differentiating the given identity with respect to x we obtain the following:

$$\left(x^2\right)' + \left(y^2\right)' = (9)'$$

$$2x + 2yy' = 0$$

Let's express y'

$$2yy' = -2x$$

$$y' = -x / y$$

8. The derivative of the power exponential function

Let the function $y = f(x)$ have its derivative $y' = f'(x)$ rather difficult to calculate using previously described methods and formulas, but its Napierian logarithm $\ln f(x)$ is the function that can be easily differentiated. Then, in order to find the derivative, we should use the method of logarithmic differentiating including sequential taking the logarithm of the initial function $\ln y = \ln f(x)$ and then its differentiating as an implicit function.

Thus, if $\ln y = \ln f(x)$, then $$\frac{y'}{y} = (\ln f(x))'$$

whence we find $$y' = y \cdot (\ln f(x))'$$

Example. Find the derivative of the following function: $y = x^{\sin x}$

Solution. There is no formula to differentiate the given function in the table. Therefore we use the method of logarithmic differentiation. Let's take the natural logarithm of this function:

$$\ln y = \ln x^{\sin x}$$

Let's use the following property:

$$\ln a^{b} = b \cdot \ln a$$

We obtain

$$\ln y = \sin x \cdot \ln x$$

We obtain

$$\ln y = \sin x \cdot \ln x$$

Differentiating both parts of the equation we obtain the following:

$$(\ln y)' = (\sin x \cdot \ln x)'$$

$$\frac{1}{y} \cdot y' = (\sin x)' \ln x + \sin x \cdot (\ln x)'$$

$$\frac{1}{y} \cdot y' = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$$

$$\frac{1}{y} \cdot y' = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$$

Let's multiply both parts by y

$$y' = y \cdot \left(\cos x \cdot \ln x + \sin x \cdot \frac{1}{x} \right)$$

$$y' = x^{\sin x} \cdot \left(\cos x \cdot \ln x + \sin x \cdot \frac{1}{x} \right)$$

9. The derivative of the function given in a parametric form

If the function is given in a parametric form, i.e. $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

then its derivative on ***x*** can be presented in the following way:

$$y'_x = \frac{dy}{dx} = \frac{y'_t dt}{x'_t dt} = \frac{y'_t}{x'_t}$$

$$y'_x = \frac{y'_t}{x'_t}$$

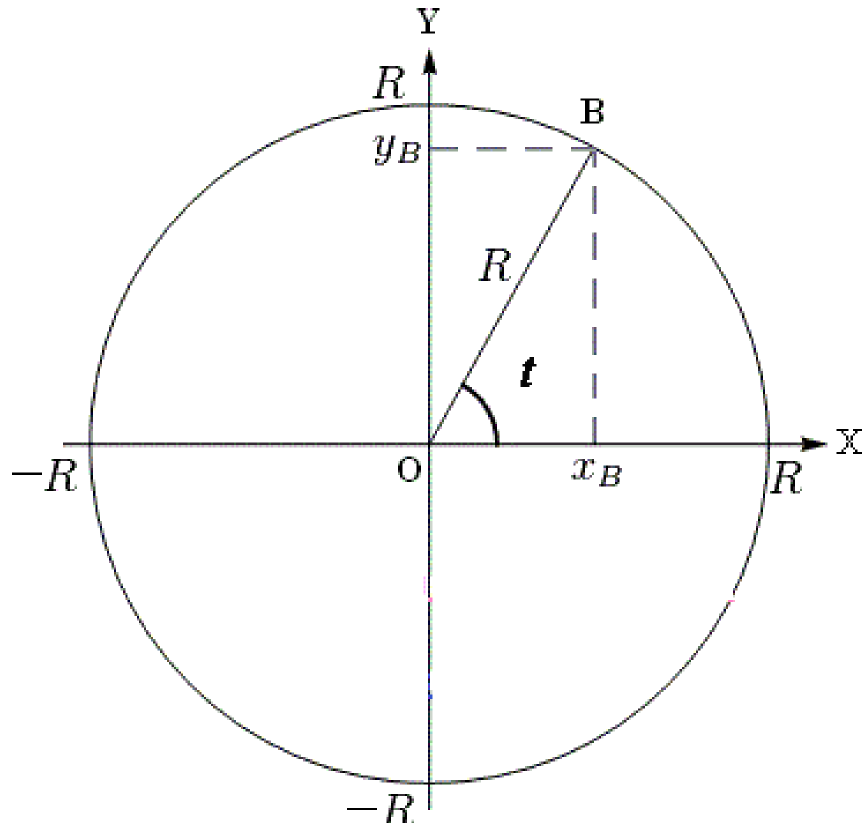
Example. Find the derivative y'_x , if the function is given in a parametric form $\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}$

Solution.

$$y'_t = \left(R\sin t\right)'_t = R\cos t$$

$$x'_t = \left(R\cos t\right)'_t = -R\sin t$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{R\cos t}{-R\sin t} = -\frac{x}{y}$$

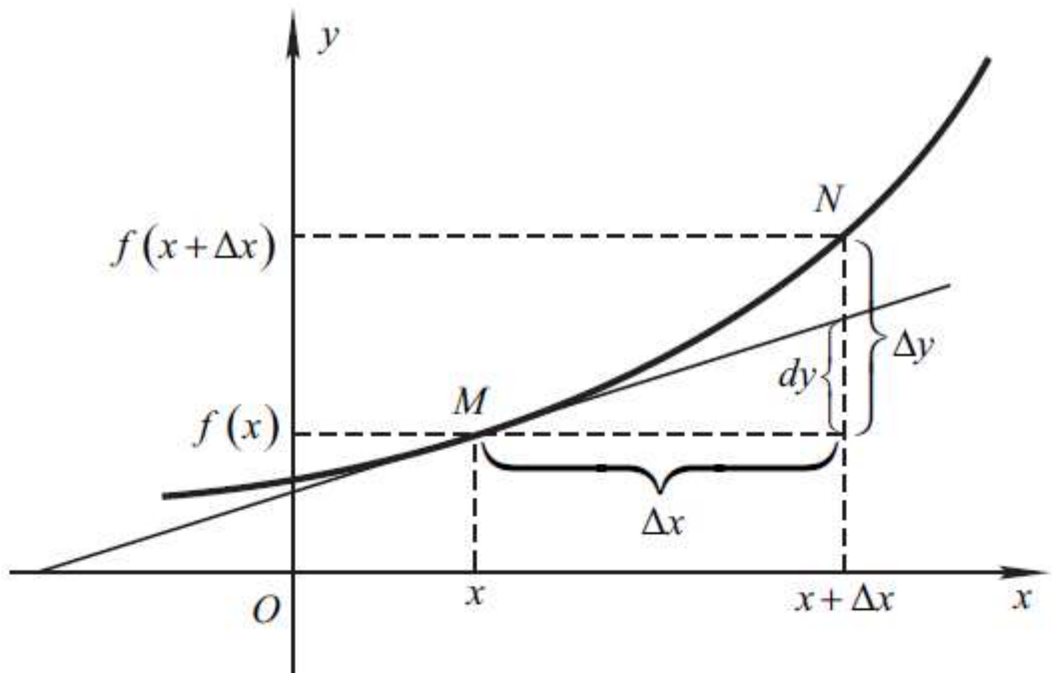


10. Differential of a function

The function $y = f(x)$ is considered to be differential at the given point x if the increment Δy of this function at the point x corresponding to be increment of the argument Δx can be shown as:

$$\Delta y = A \cdot \Delta x + \alpha \cdot \Delta x$$

where ***A*** is some value, not depending on Δx, and α is a function of the argument Δx being infinitesimal as Δx approaches to zero.



Definition. The main part of the function increment $A\cdot\Delta x$ linear relatively Δx , is called the *differential of the function* and designated as

$$dy = A\cdot\Delta x$$

The differential of the variable x is equal to its increment $dx = \Delta x$

therefore

$$dy = y'\cdot dx$$

$$y' = \frac{dy}{dx}$$

11. Derivatives and differentials of the higher orders

Let the function $y = f(x)$ be differentiable in some interval (a,b)
Generally, the value of the derivative $f'(x)$ depends on *x*, i.e.
the derivative $f'(x)$ is also a function of *x.* If this function is
differentiable at some point *x* of the interval (a,b), i.e. has
the derivative at this point, then this derivative is called the *second derivative* (or second order derivative) and designated as:

$$y'' = (y')' = f''(x)$$

The same way we can introduce the concept of the third order derivative then the concept of the fourth order derivative, etc.

The second order differential is called the differential of the differential function, i.e.

$$d(dy) = d(y'dx) = y''dx^2 = d^2 y \qquad d^2 y = y''dx^2$$

Example. Find the third order derivative of the function $y = x\,ln\,x$

Solution.

$$y' = \left(x\,ln\,x\right)' = ln\,x + x\cdot\frac{1}{x} = ln\,x + 1$$

$$y'' = \left(ln\,x + 1\right)' = \frac{1}{x}$$

$$y''' = \left(y''\right)' = \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

Functions investigation by means of derivatives

Lecture plan

1. **The greatest and the least values of a function on the interval**
2. **Conditions of the function monotonicity. Extremums**
3. **Concavity up and Concavity down of a Curve. Inflection Points.**
4. **Asymptotes of curves**
5. **The general plan for investigating a function and constructing its plot**

1. The greatest and the least values of a function on the interval

Let the function $y = f(x)$ be continuous on the interval $[a,b]$
Then, according to Weierstrass' theorem, it achieves the greatest and the least values in this interval. These values can be achieved either on the borders of the interval or at internal points, being the extremums of the function. From this fact we conclude the following plan for defining the greatest and the least values of a function:

1) Define all the critical points, belonging to the given interval $[a,b]$

2) Calculate the values of the function at the found critical points and on the borders of the interval

3) Choose the greatest and the least values from the obtained values. The chosen values are required ones.

Example 1. Find the greatest and the least values of the function

$f(x) = x^3 - 3x^2 + 1$ on the interval $[-1, 4]$

Solution. Let’s find all the critical points using the condition

$$f'(x) = 0$$

We have the derivative:

$$f'(x) = \left(x^3 - 3x^2 + 1\right)' = 3x^2 - 6x$$

Let’s equate it to 0 and find the critical points:

$$f'(x) = 0 \qquad 3x^2 - 6x = 0 \qquad 3x(x-2) = 0$$

We obtain

$$3x(x-2) = 0$$

$$x = 0$$

$$x = 2$$

Thus, the given function has two stationary points $x_1 = 0$ and $x_2 = 2$ inside the interval $[-1, 4]$

Let's calculate the function values at these points and on the borders of the interval:

$$f(0) = 0^3 - 3\cdot 0^2 + 1 = 1$$

$$f(2) = 2^3 - 3\cdot 2^2 + 1 = 8 - 12 + 1 = -3$$

$$f(-1) = (-1)^3 - 3\cdot(-1)^2 + 1 = -1 - 3 + 1 = -3$$

$$f(4) = 4^3 - 3\cdot 4^2 + 1 = 64 - 48 + 1 = 17$$

As we see, the function takes the greatest value on the right border of the interval $[-1, 4]$ and the least value is taken at the internal point $x = 2$ and on the left border of the interval.

$$f_{least} = f(2) = f(-1) = -3$$

$$f_{greatest} = f(4) = 17$$

TASK. Find the greatest and the least values of the function $y = x^4 - 8x^2 + 3$ on the interval $\left[-2; 2\right]$

2. Conditions of the function monotonicity. Extremums

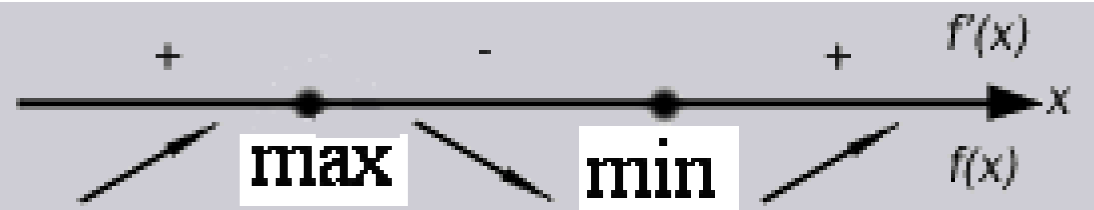
Theorem 1. Let the function $y = f(x)$ be continuous and differentiable on the interval $[a,b]$. In order to the function be constant on $[a,b]$ it is necessary and sufficiently that

$$f'(x) = 0 \qquad \forall x \in (a,b)$$

Theorem 2. Let the function $y = f(x)$ be continuous and differentiable on the interval $[a,b]$, then:

a) if $f'(x) > 0 \quad \forall x \in (a,b)$, then $f(x)$ increases;

b) if $f'(x) < 0 \quad \forall x \in (a,b)$, then $f(x)$ decreases.



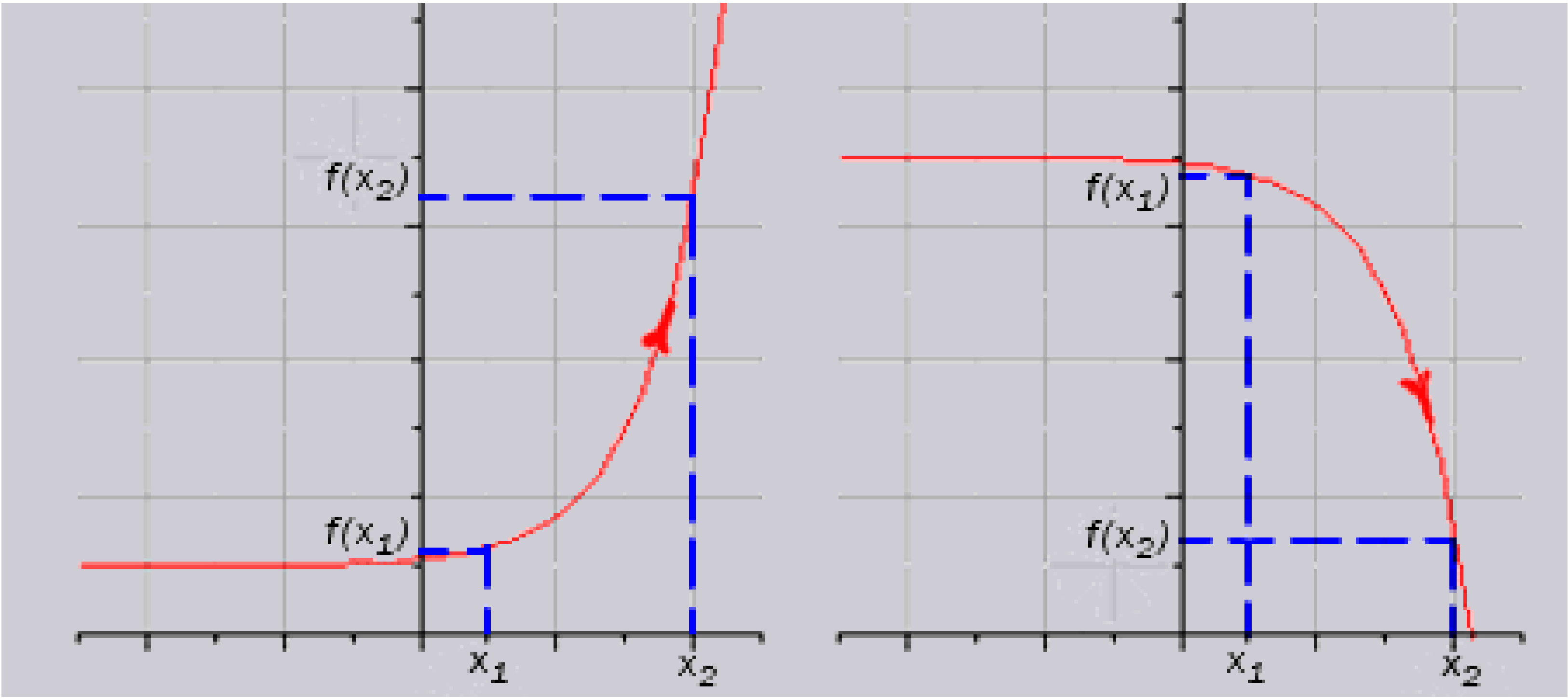
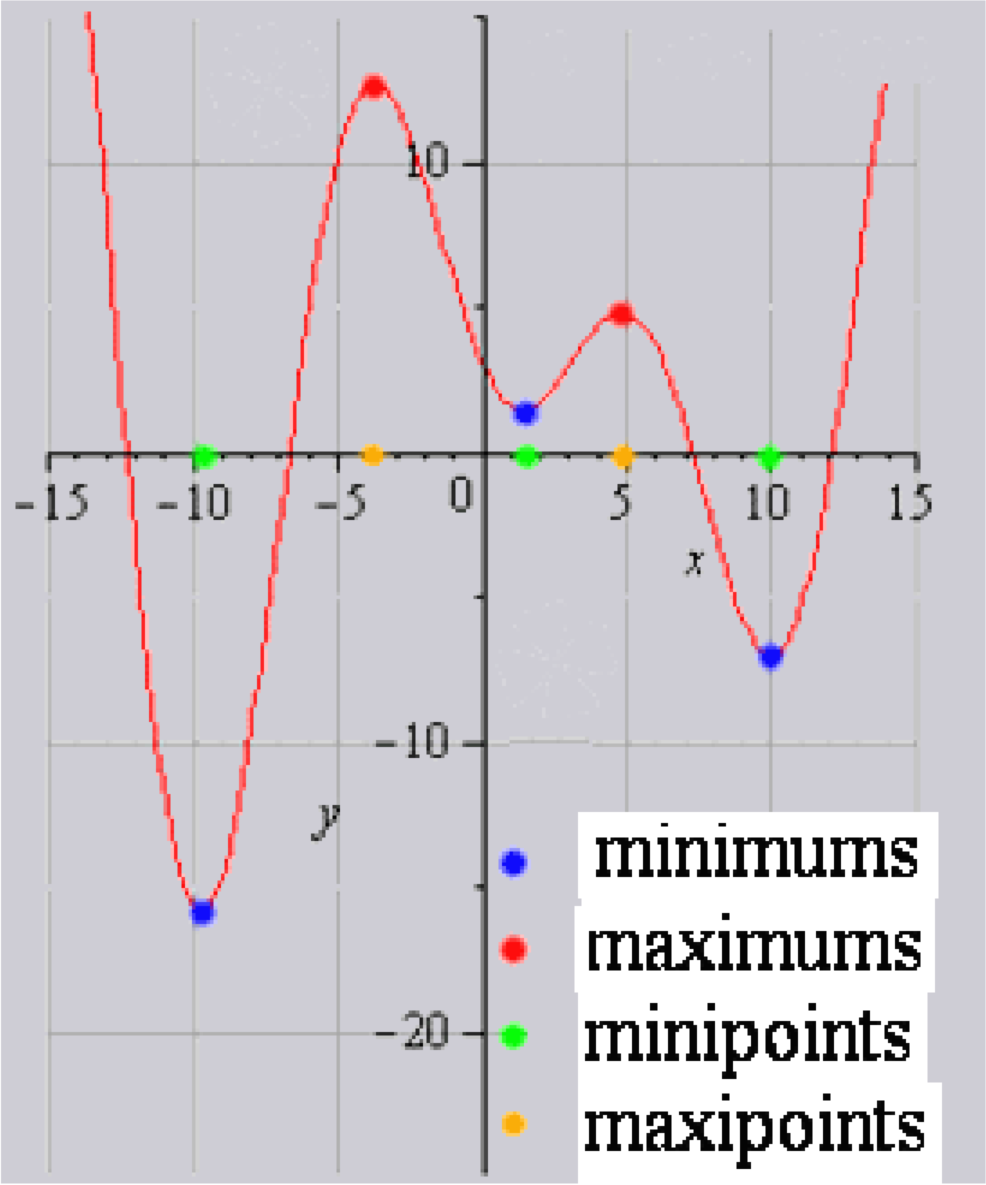


Fig. 1. The increasing function and the decreasing function

The point x_0 is called *the point of the maximum (minimum)* of the function $y = f(x)$, if there is such a neighborhood $(x_0 - \delta, x_0 + \delta)$, in which $f(x_0)$ is the greatest (smallest) value among the values of all the points of this interval, i.e. $f(x_0) \geq f(x)$ or $f(x_0) \leq f(x)$

The points of the maximum and minimum of a function are called the points of the *extremum* of this function.



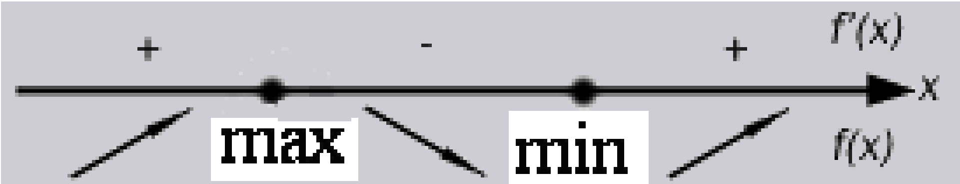
10
-10
-20
-15
-10
-5
0
5
10
15
x
y
minimums
maximums
minipoints
maxipoints

Theorem 4. *(The necessary condition for existence of an extremum).* If the continuous function $y = f(x)$ has an extremum at the point $x = x_0$, then the derivative of the function at this point is either equal to zero or does not exist.

Points, in which the derivative is equal to zero or does not exist, are called *critical.*

Theorem 5. *(The sufficient condition for existence of an extremum of the function by the first derivative).*

Let x_0 be a critical point. Then, if the function $y = f(x)$ has its derivative $f'(x)$ in some neighborhood of the point x_0 and if the derivative $f'(x)$ changes its sign from plus to minus at passing through the point $x = x_0$, then the function has a maximum at this point, and at changing the sign from minus to plus it has a minimum.



Theorem 6. (*The sufficient condition for existence of an extremum of the function by the second derivative).*

If the function $y = f(x)$ in some neighborhood of the point x_0 is continuous, has the second derivative and $f'(x_0) = 0$,

$f''(x_0) \neq 0$ then, if $f''(x_0) > 0$, the function has the minimum

at the point x_0, if $f''(x_0) < 0$, the function has the maximum

at the point x_0.

Example. Let's determine the intervals of monotonicity and the extremums of the function $y = \frac{x^3}{x^2 - 1}$

For this purpose it is necessary to find the first derivative of the function and to determine points, in which it is to zero or does not exist.

Let's find the first derivative of the function:

$$y' = \left(\frac{x^3}{x^2 - 1} \right)' = \frac{3x^2 \cdot (x^2 - 1) - 2x \cdot x^3}{(x^2 - 1)^2} = \frac{3x^4 - 3x^2 - 2x^4}{(x^2 - 1)^2} =$$

$$= \frac{x^4 - 3x^2}{(x^2 - 1)^2} = \frac{x^2 \cdot \left(x^2 - 3\right)}{(x^2 - 1)^2} = \frac{x^2 \cdot \left(x - \sqrt{3}\right) \cdot \left(x + \sqrt{3}\right)}{(x^2 - 1)^2}$$

Let’s find the critical points:

$$y' = 0 \qquad\qquad y' = \frac{x^2 \cdot \left(x-\sqrt{3}\right)\cdot\left(x+\sqrt{3}\right)}{(x^2-1)^2}$$

$$x^2\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)=0 \qquad\qquad x^2-1\neq 0$$

$$x=0 \qquad x=-\sqrt{3} \qquad x=\sqrt{3} \qquad\qquad x\neq -1,\ x\neq 1$$

Let’s determine the intervals of monotonicity. These points divide a range into 6 intervals, on each of which the first derivative keeps its sign.

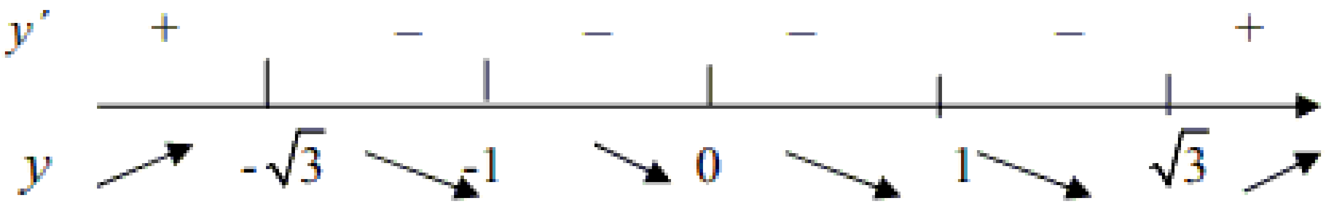
If $x\in\left(-\infty;-\sqrt{3}\right)\cup\left(\sqrt{3};\infty\right)$ then $y'>0$, i.e. the function increases.

If $x\in\left(-\sqrt{3};-1\right)\cup\left(-1;0\right)\cup\left(0;1\right)\cup\left(1;\sqrt{3}\right)$ then $y'<0$

i.e. the function decreases.

Let’s find the critical points:

$x = 0$ $\quad x = -\sqrt{3}$ $\quad x = \sqrt{3}$ $\quad x \neq -1,\ x \neq 1$



Let’s determine the intervals of monotonicity. These points divide a range into 6 intervals, on each of which the first derivative keeps its sign.

If $x \in \left(-\infty; -\sqrt{3}\right) \cup \left(\sqrt{3}; \infty\right)$ then $y' > 0$, i.e. the function increases.

If $x \in \left(-\sqrt{3}; -1\right) \cup (-1; 0) \cup (0; 1) \cup \left(1; \sqrt{3}\right)$ then $y' < 0$

i.e. the function decreases.

Hence, the point $x=-\sqrt{3}$ is maximum and the point $x=\sqrt{3}$ is minimum. The values of the functions at these points are equal to

$$y_{\max}\left(-\sqrt{3}\right)=\frac{\left(-\sqrt{3}\right)^{3}}{\left(-\sqrt{3}\right)^{2}-1}=-\frac{3\sqrt{3}}{3-1}=-\frac{3\sqrt{3}}{2}$$

$$y_{\min}\left(\sqrt{3}\right)=\frac{\left(\sqrt{3}\right)^{3}}{\left(\sqrt{3}\right)^{2}-1}=\frac{3\sqrt{3}}{3-1}=\frac{3\sqrt{3}}{2}$$

TASK

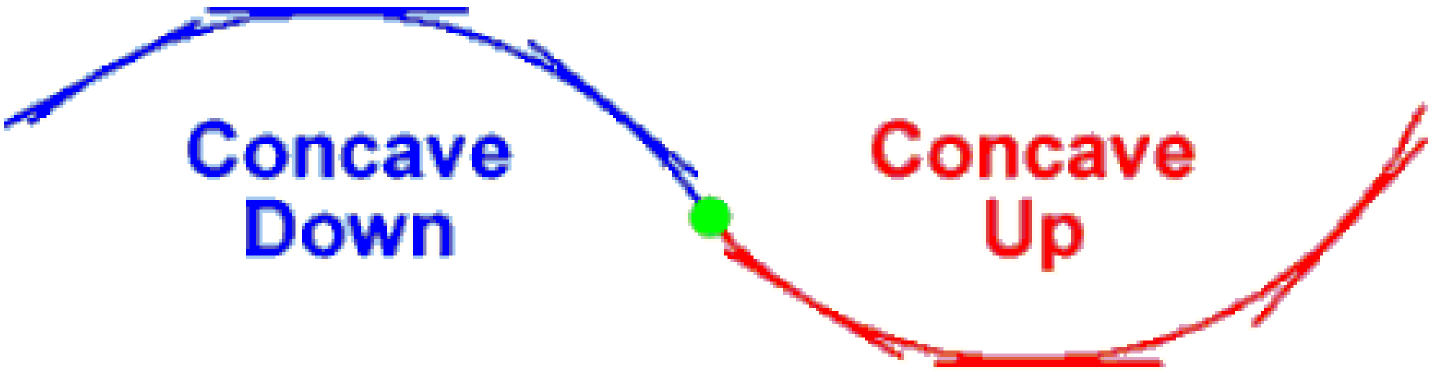
Find the intervals of monotonicity and the extremums of the function

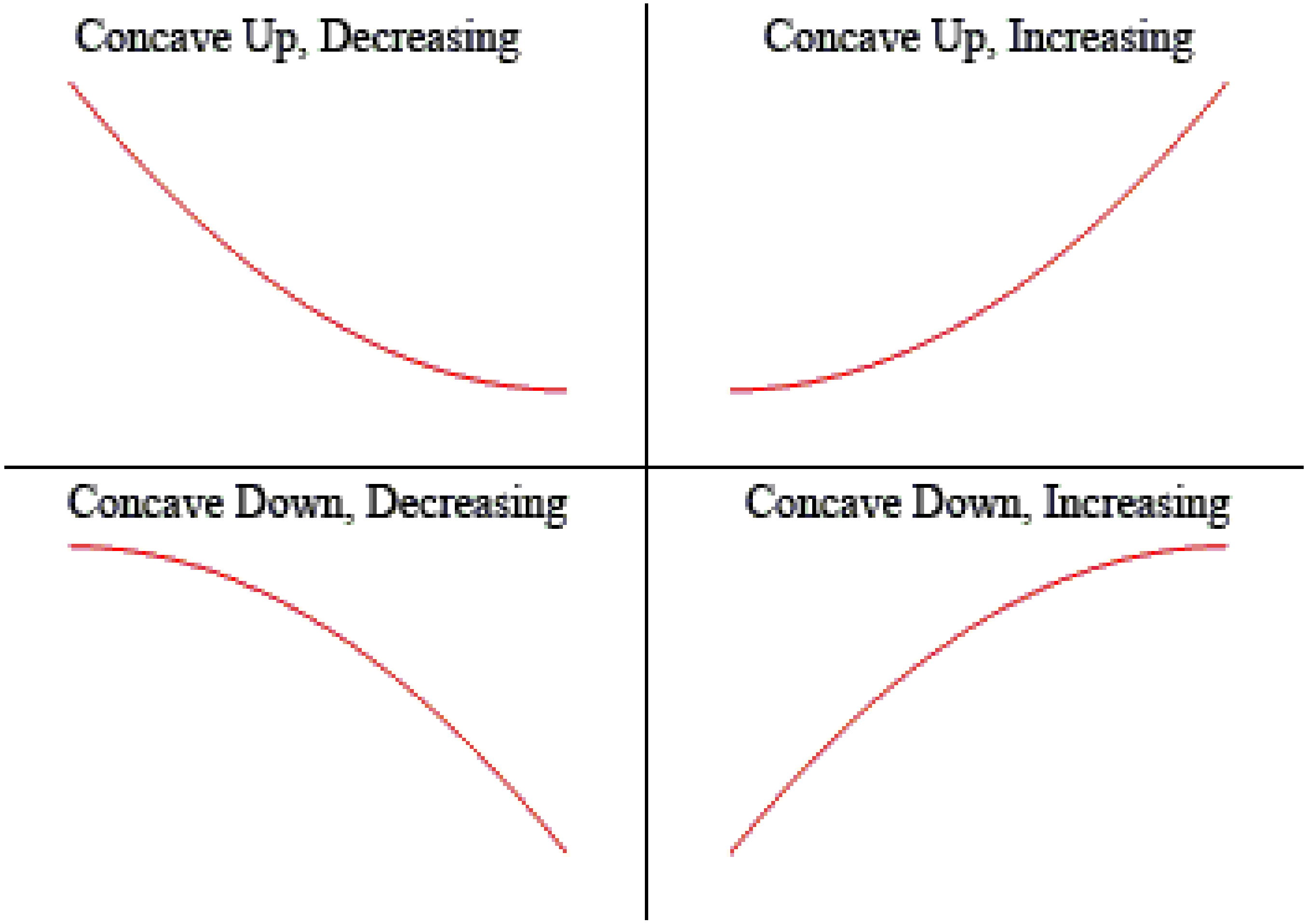
$$y = \frac{1}{3}x^3 - 2x^2 + 3x + 2$$

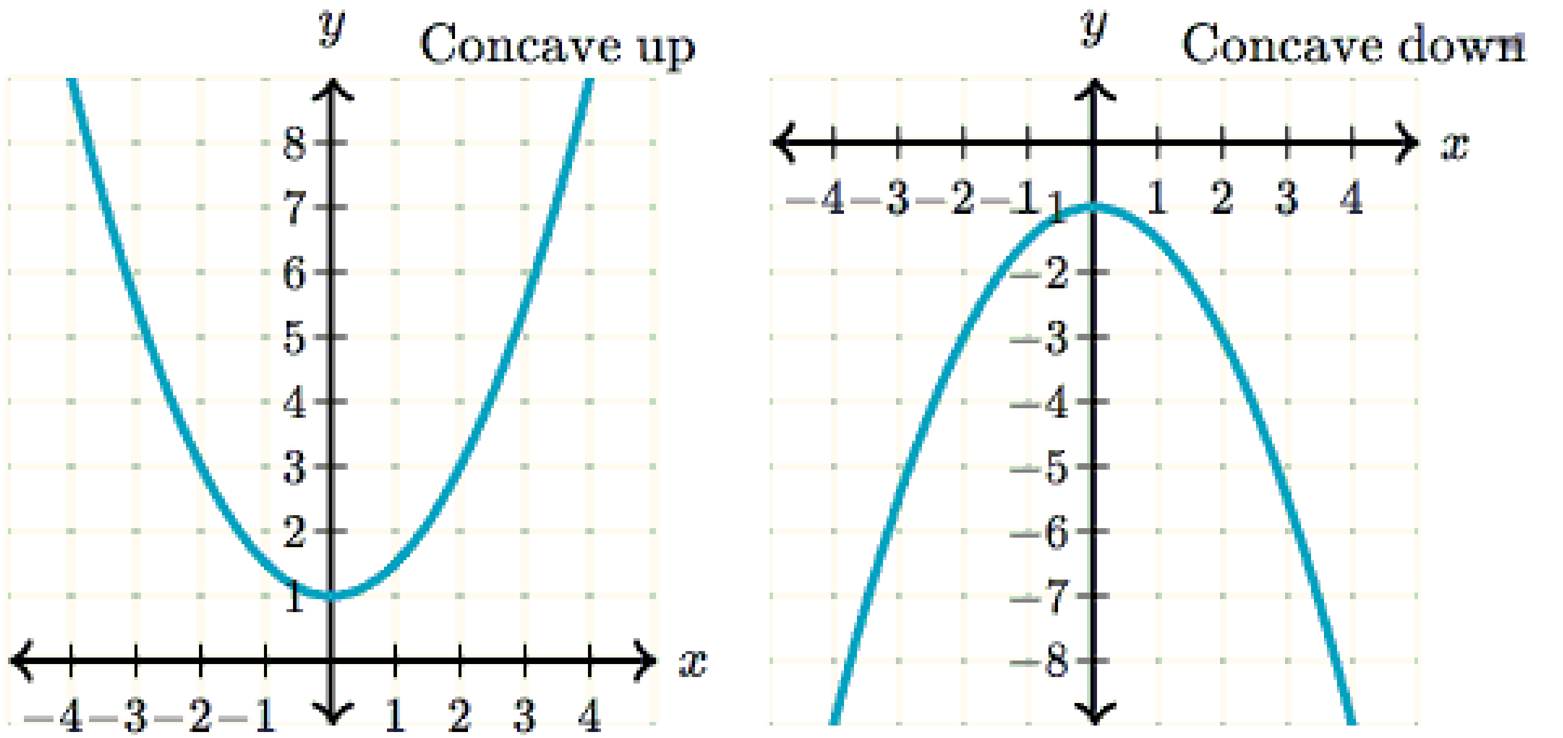
$$y = x\sqrt{1 - x^2}$$

3. Concavity of a Curve. Points of Inflection

The curve $y = f(x)$ is called ***concave down*** at the point x_0 if in some neighborhood of this point $(x_0 - \delta, x_0 + \delta)$ it is located below the tangent, drawn at the point x_0. If the curve is located above the tangent, it is called ***concave up.***



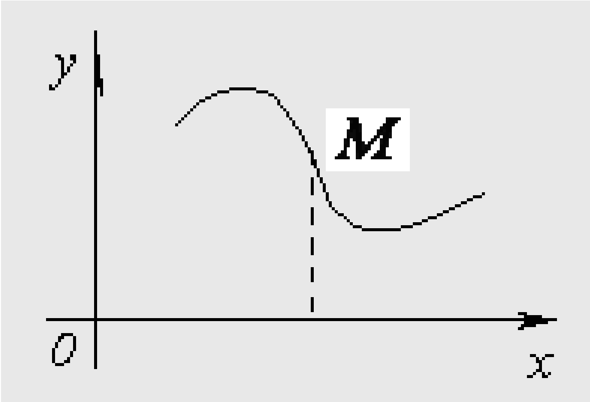

Concave Up, Decreasing
Concave Up, Increasing
Concave Down, Decreasing
Concave Down, Increasing

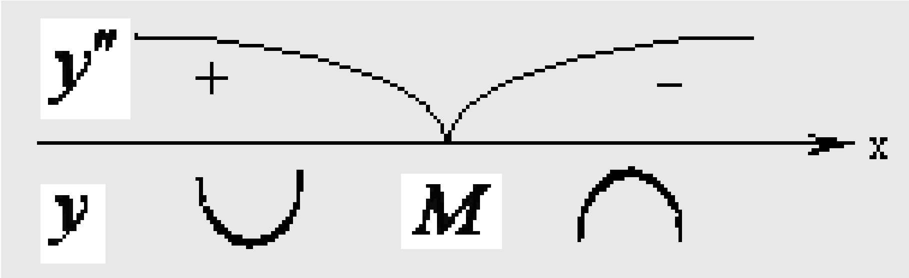


Concave up
y
x
8
7
6
5
4
3
2
1
−4 −3 −2 −1 1 2 3 4
Concave down
y
x
−4 −3 −2 −1 1 2 3 4
−1
−2
−3
−4
−5
−6
−7
−8

Theorem 1. If the function $y = f(x)$ in some neighborhood of the point x_0 is doubly continuously differentiable and $f''(x_0) \neq 0$ then the necessary and sufficient condition of the curve concavity down at the point x_0 is the requirement $f''(x_0) < 0$ concavity up - $f''(x_0) > 0$

The point $M(x_1, f(x_1))$ is called *the inflection point* of the given curve $y = f(x)$, if there is such neighborhood of the point x_1 that while $x > x_1$, in this neighborhood the convexity of the curve is directed to one direction, and while $x < x_1$ in this neighborhood the concavity to the other direction.





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concavity up - $f''(x_0) > 0$

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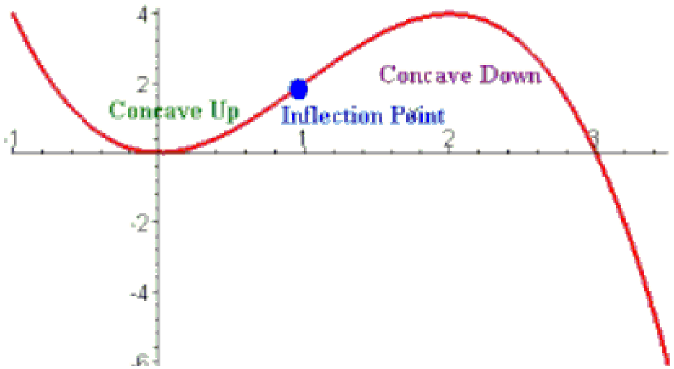

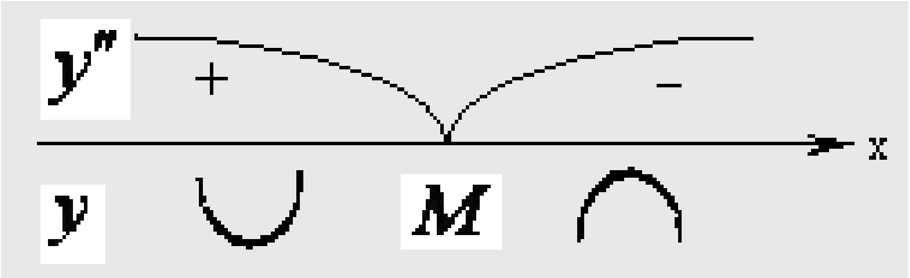

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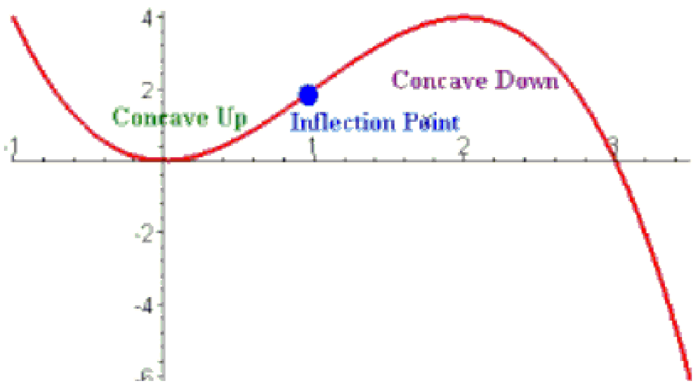
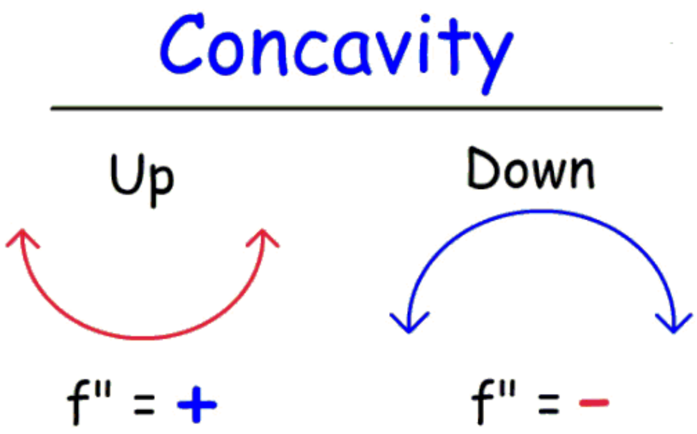
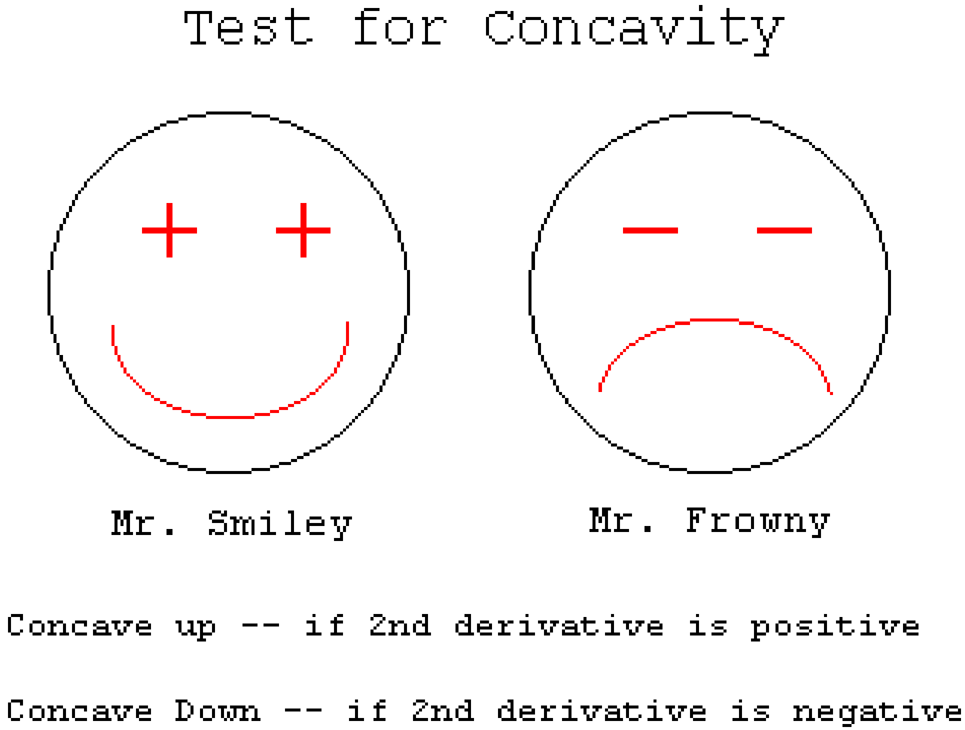




TEST for Concavity



In order to let the point $x = x_0$ be the inflection point of the given curve it is necessary that the second derivative at this point will be either equal to zero $f''(x_0) = 0$ or will not exist.

Example. To define the intervals of concavity up (or down of the function $y = \dfrac{x^3}{x^2 - 1}$ and the inflection points we find the second derivative:

$$y'' = (y')' = \left(\frac{x^4 - 3x^2}{(x^2 - 1)^2} \right)' =$$

$$= \frac{(4x^3 - 6x)(x^2 - 1)^2 - (x^4 - 3x^2)4x(x^2 - 1)}{(x^2 - 1)^4} =$$

$$=\frac{\left(x^2-1\right)\left((4x^3-6x)(x^2-1)-(x^4-3x^2)4x\right)}{(x^2-1)^4}=$$

$$=\frac{4x^5-4x^3-6x^3+6x-4x^5+12x^3}{(x^2-1)^3}=$$

$$=\frac{2x^3+6x}{\left(x^2-1\right)^3}=\frac{2x\left(x^2+3\right)}{\left(x^2-1\right)^3}$$

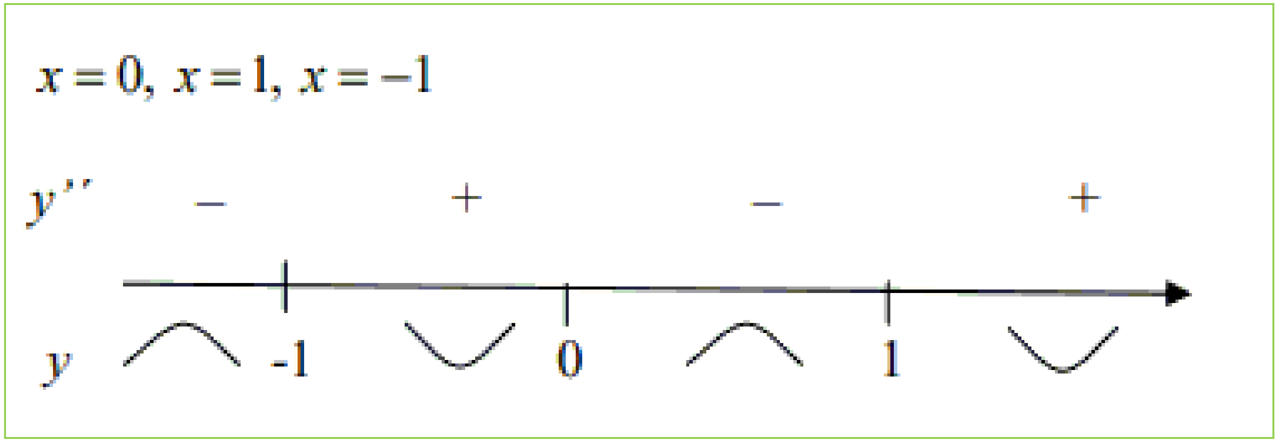
If $y''=0$ then $y''=\frac{2x(x^2+3)}{(x^2-1)^3}=0$

$x=0$ $x^2+3\neq 0$ $x\neq -1,\ x\neq 1$

We obtain that at $x \in (-1;0) \cup (1;+\infty)$ the graph is concave up and the graph is concave down at $x \in (-\infty;-1) \cup (0;1)$

The point with coordinates $x = 0$ and $y = \frac{0^3}{0^2 - 1} = 0$

i.e. the origin, is a point of the inflection.



TASK

Find the intervals of concavity of the function:

$$y = \frac{x}{x^2 - 6x - 16}$$

4. Asymptotes of curves

The straight line $x = x_0$ is called a *vertical asymptote* if

$$\lim_{x \to x_0 \pm 0} f(x) = \pm\infty$$

At a finding vertical asymptotes the break points of the function are investigated. In these points these limits are calculated.

The asymptote of the plot of the function $y = f(x)$ is a straight line, possessing such a property that the distance between the line and the point on the curve tends to zero while the point on the curve tends to infinity $(x \to \pm\infty)$

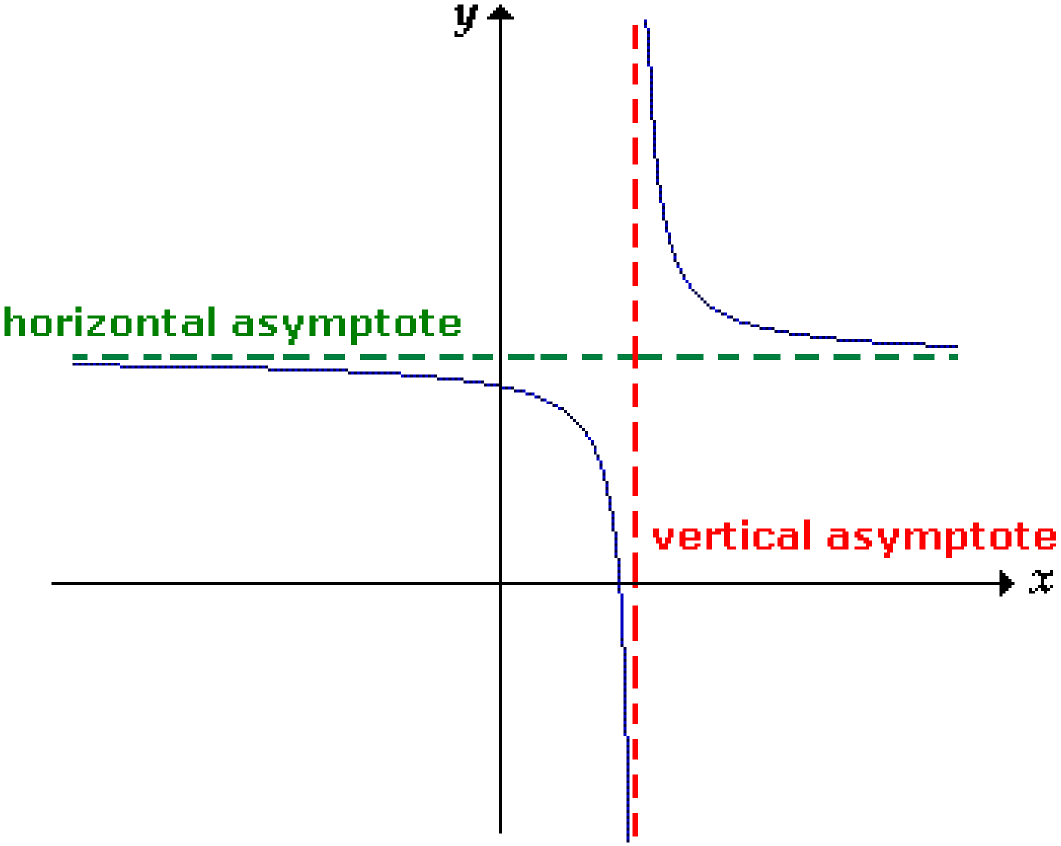
The equation of the *oblique asymptote* looks like $y = kx + b$

In particular, if $k = 0$, the asymptote is horizontal. If the inclined asymptote exists, k and b have to be calculated according to the following formulas:

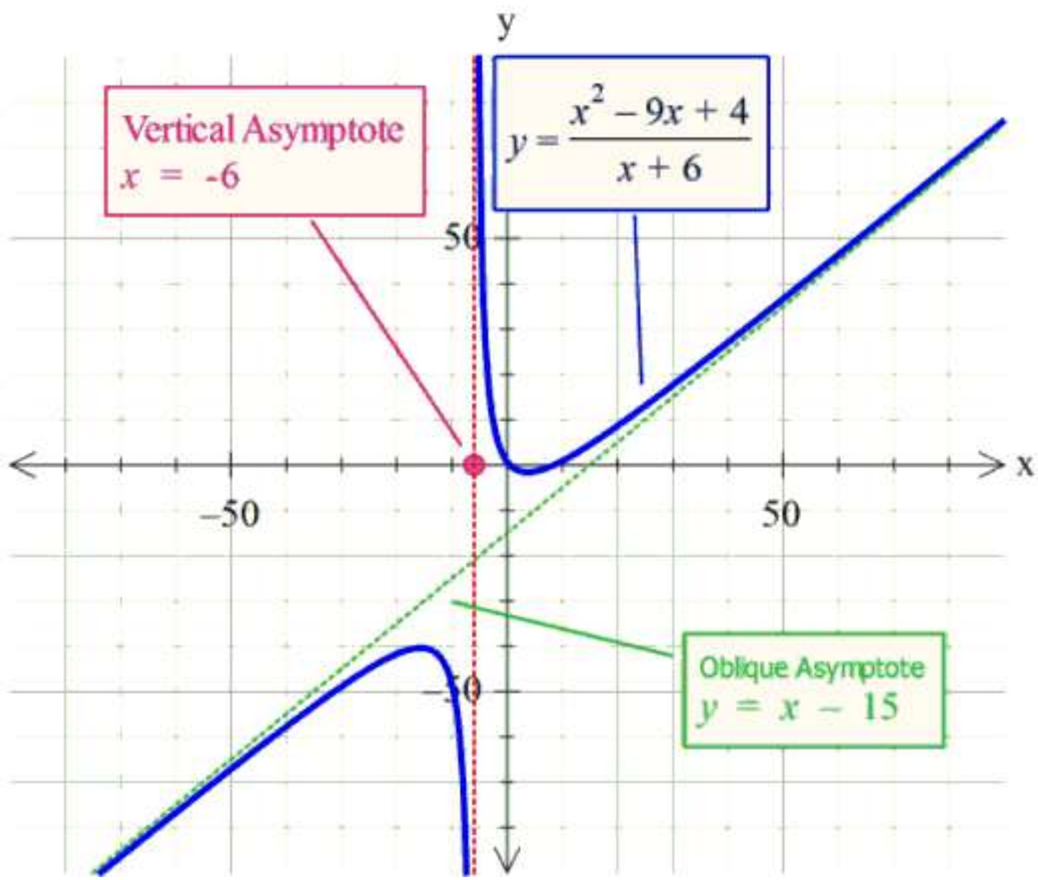
$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} \qquad b = \lim_{x \to \pm\infty} \left(f(x) - k \cdot x \right)$$

If, at least, one limit does not exist, the curve has no inclined asymptotes. The asymptotes can vary while $x \to +\infty$ and $x \to -\infty$

GRAPH and ASYMPTOTES



GRAPH and ASYMPTOTES



Example. Let's research the continuity of the function $y=\dfrac{x^3}{x^2-1}$

$x=1$ $x=-1$

are break points of the function, since

$$\lim_{x\to 1\pm 0}\frac{x^3}{x^2-1}=\frac{(1\pm 0)^3}{(1\pm 0)^2-1}=\frac{1}{1-1}=\frac{1}{0}=+\infty$$

$$\lim_{x\to -1\pm 0}\frac{x^3}{x^2-1}=\frac{(-1\pm 0)^3}{(-1\pm 0)^2-1}=\frac{1}{1-1}=\frac{1}{0}=+\infty$$

and therefore $x=1$ and $x=-1$ are vertical asymptotes.

The inclined asymptote has the equation $y = kx + b$

where $k = \lim\limits_{x \to \pm\infty} \dfrac{f(x)}{x}$ $\quad b = \lim\limits_{x \to \pm\infty} \left(f(x) - k \cdot x \right)$

If $k = 0$ then the asymptote $y = b$ is called horizontal.

Let's find the inclined asymptotes:

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^3}{x\left(x^2 - 1\right)} = \lim_{x \to \pm\infty} \frac{x^2}{x^2 - 1} = \left|\frac{\infty}{\infty}\right| =$$

$$= \lim_{x \to \pm\infty} \frac{\left(x^2\right)'}{\left(x^2 - 1\right)'} = \lim_{x \to \pm\infty} \frac{2x}{2x - 0} = \lim_{x \to \pm\infty} \frac{2x}{2x} = \lim_{x \to \pm\infty} 1 = 1$$

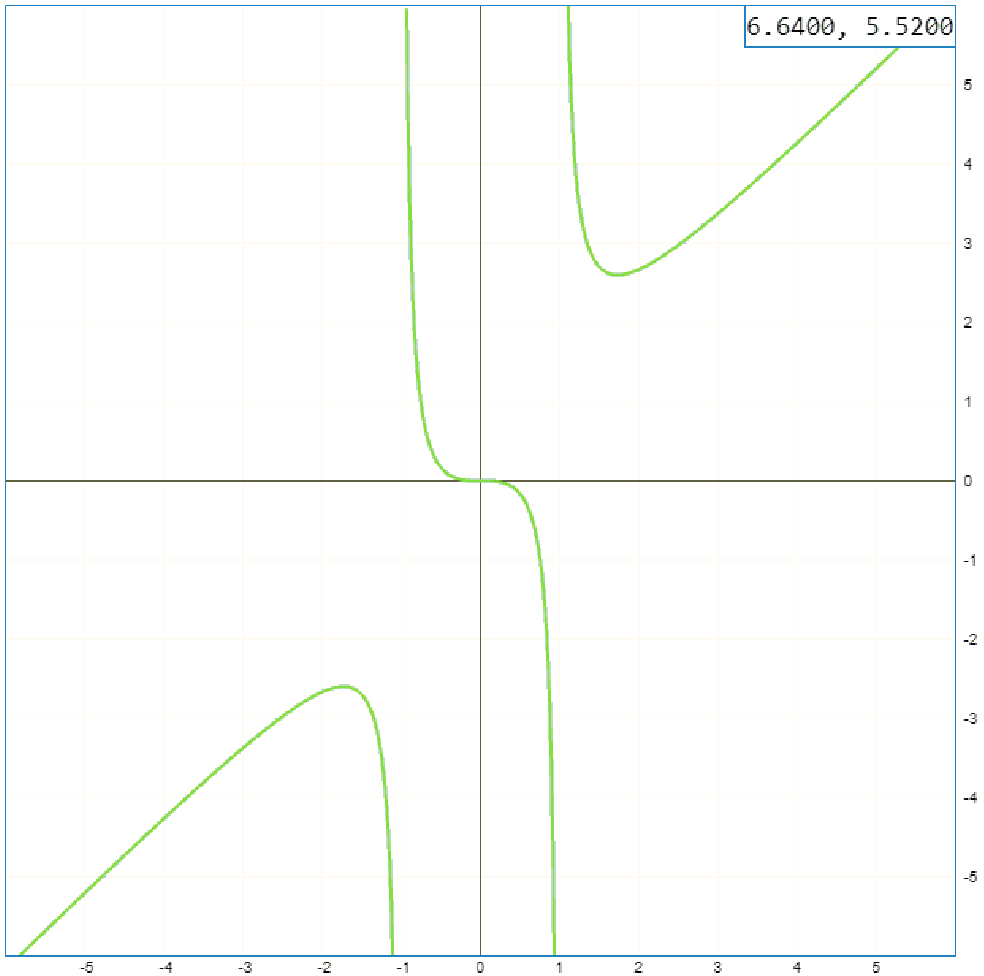
$$b=\lim_{x\to\pm\infty}\left(f(x)-k\cdot x\right)=\lim_{x\to\pm\infty}\left(\frac{x^3}{x^2-1}-x\right)=\lim_{x\to\pm\infty}\left(\frac{x^3-x^3+x}{x^2-1}\right)=$$

$$=\lim_{x\to\infty\pm}\frac{x}{x^2-1}=\left|\frac{\infty}{\infty}\right|=\lim_{x\to\infty\pm}\frac{(x)'}{\left(x^2-1\right)'}=\lim_{x\to\infty\pm}\frac{1}{2x}=0$$

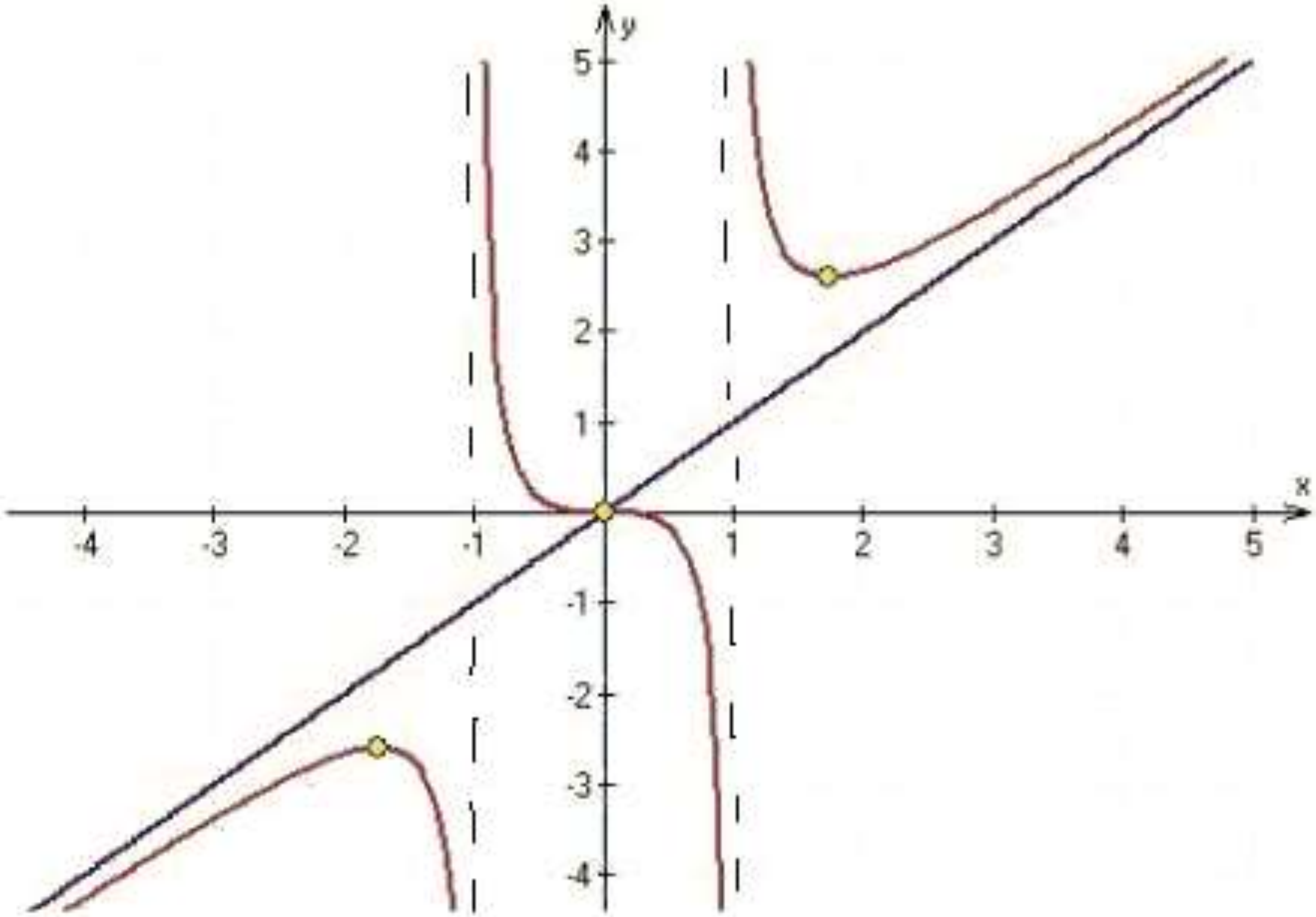
Hence, this function has an inclined asymptote

$$y=kx+b \qquad y=1\cdot x+0 \qquad y=x$$

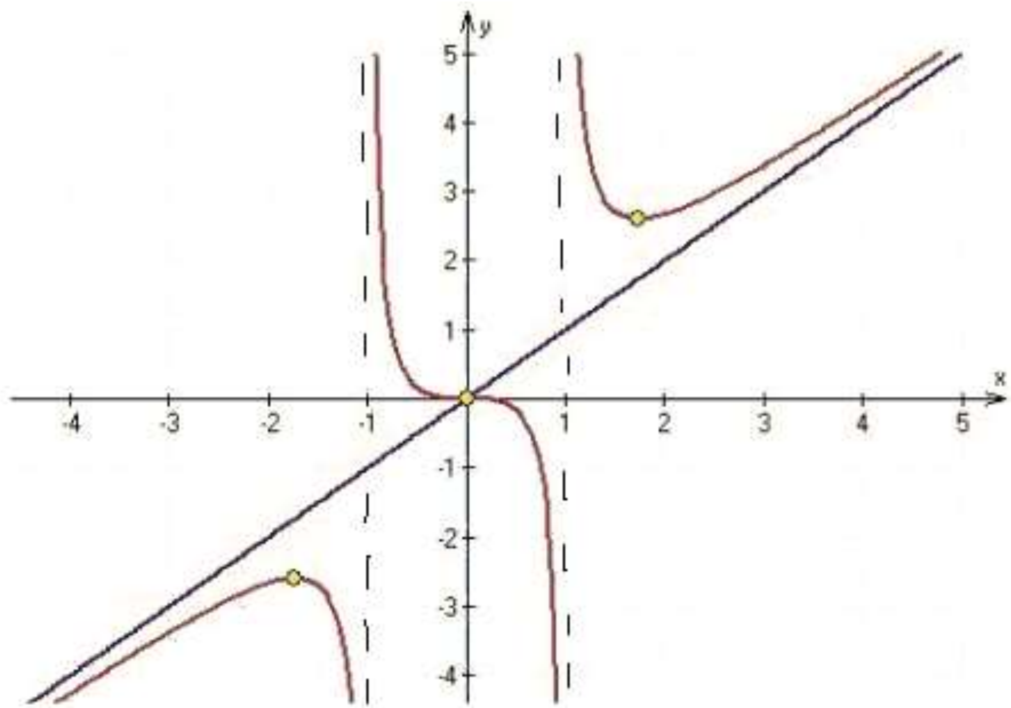
GRAPH and ASYMPTOTES

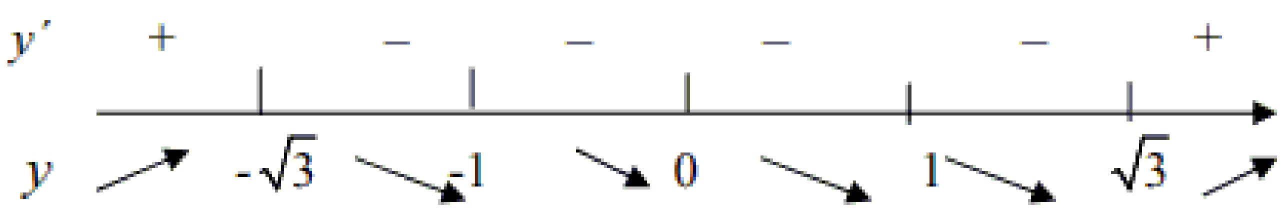


GRAPH and ASYMPTOTES

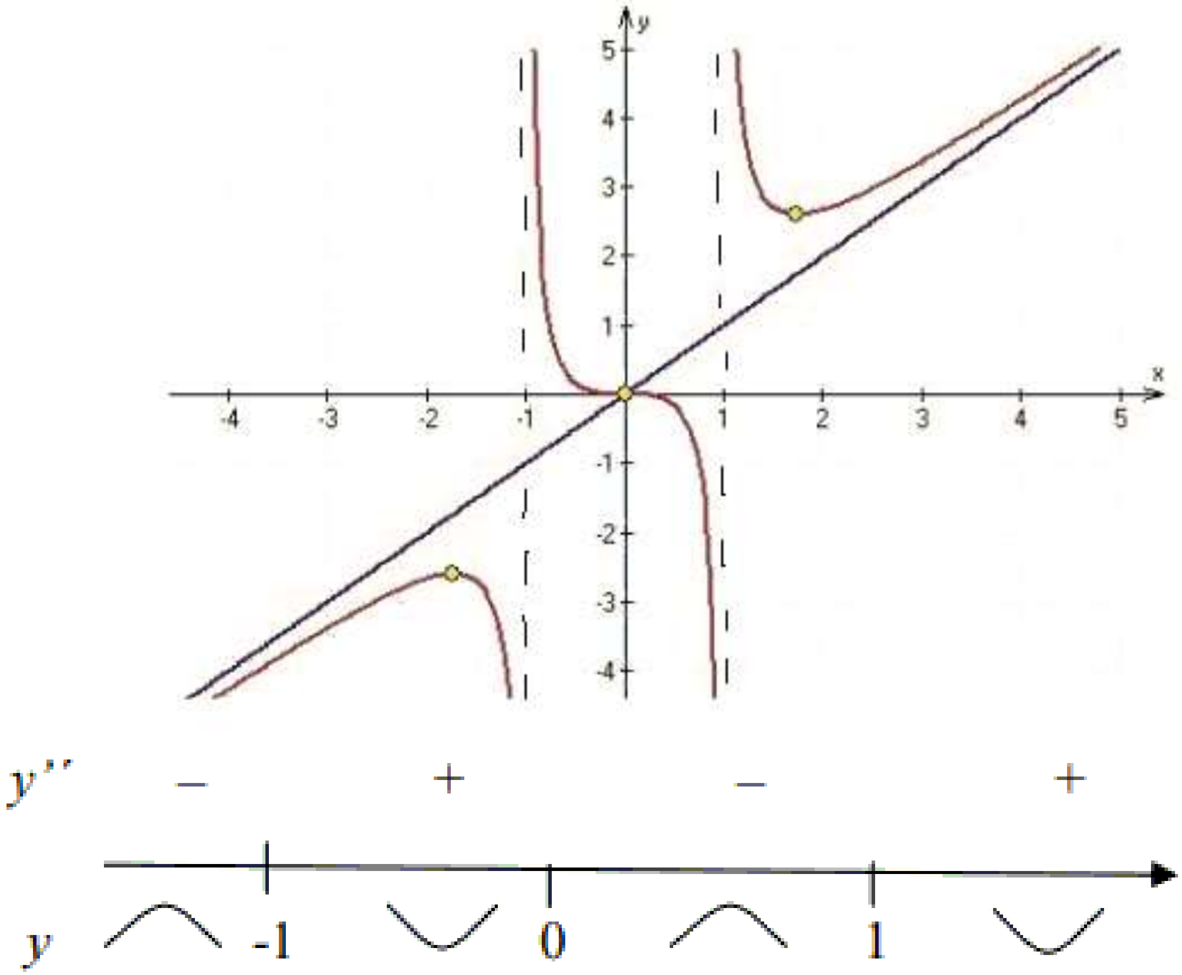


GRAPH and ASYMPTOTES

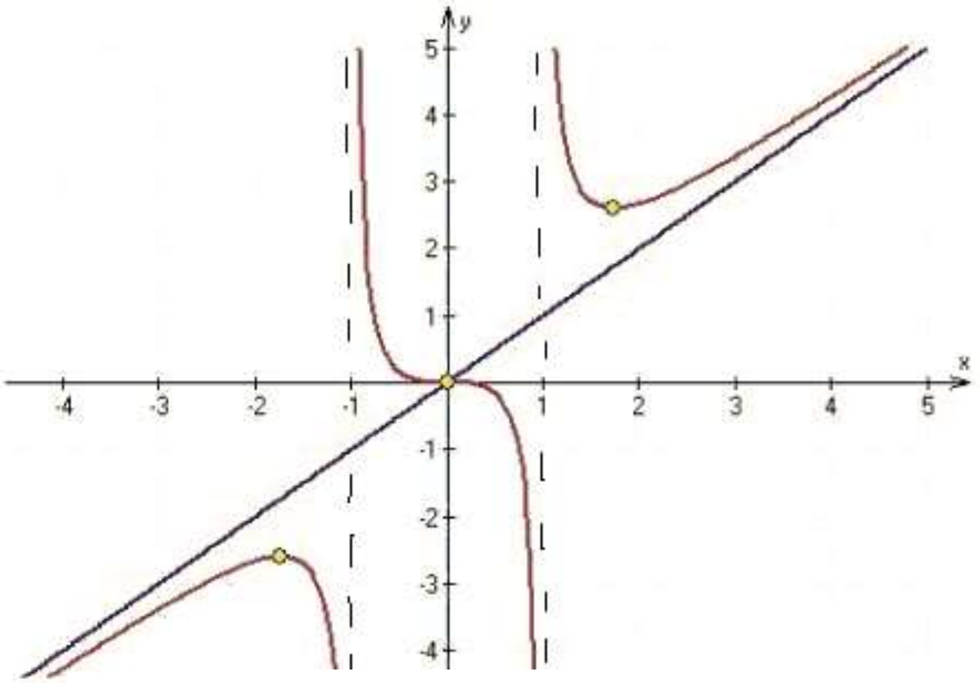




GRAPH and ASYMPTOTES



GRAPH and ASYMPTOTES



$$x = 1,\ x = -1$$

$$y = x$$

TASK

Find asymptotes of functions:

$$y = \frac{x}{x^2 - 4}$$

$$y = \frac{x^2 - 2x + 2}{x - 1}$$

$$y = \frac{1}{x^2 + 3}$$

5. The General Plan for Investigating a Function and Constructing Its Plot

1) Definition of existence of the domain of the function.

2) Investigation of the function on continuity. Finding break points of the function and defining their character. Determining vertical asymptotes.

3) Investigation of the function on parity and oddness.

4) Investigation of the function on periodicity.

5) Definition of inclined and horizontal asymptotes.

6) Investigation of the function for extremums. Finding intervals of monotonicity of the function.

7) Finding inflection points of the function, intervals of convexity and concavity.

8) Definition of intersection points with the coordinate axes.

9) Investigation of the function behaviour at infinity.

10) Plot a graph of the function.

Example. Investigate and plot the graph of the function: $y = \dfrac{x^3}{x^2 - 1}$

Let's determine the domain of the function existence:

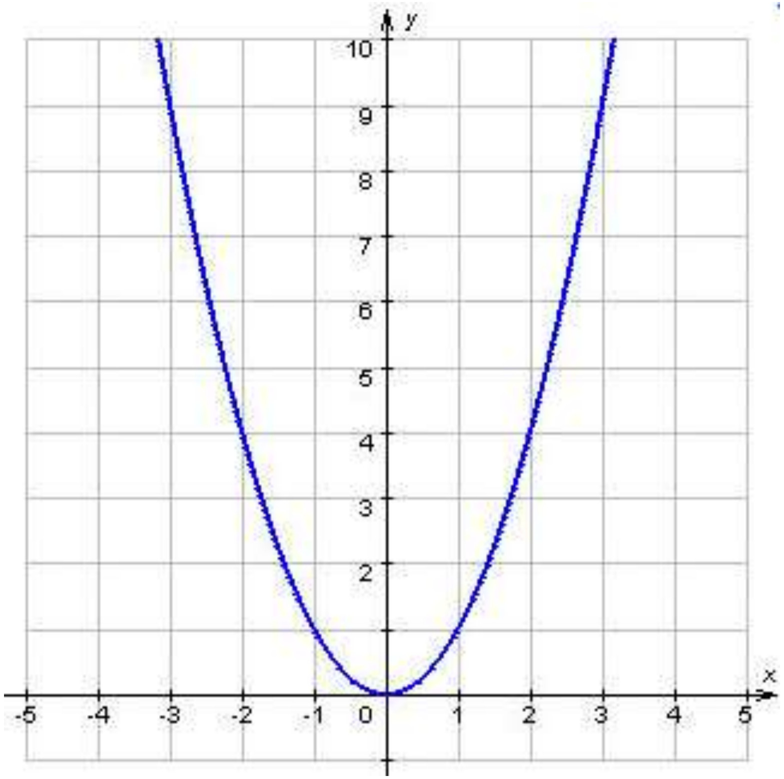
$$(-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$$

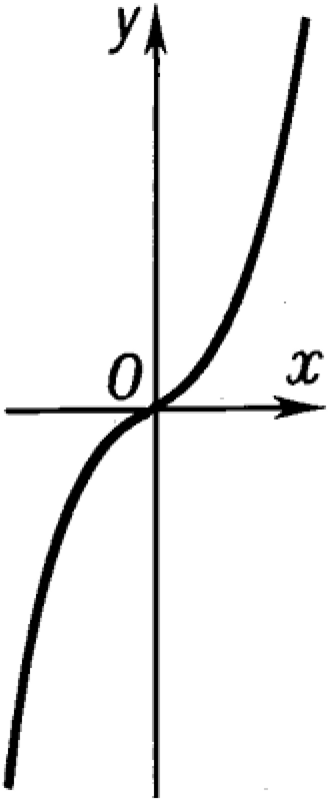
Let's investigate the function on parity:

If $f(-x) = f(x)$ then a function is parity.

If $f(-x) = -f(x)$ then a function is oddness.

If $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$ then it's a function of a general view.





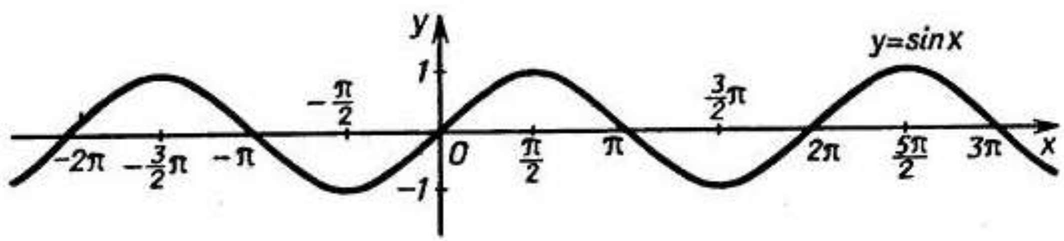
Y is the axis of symmetry

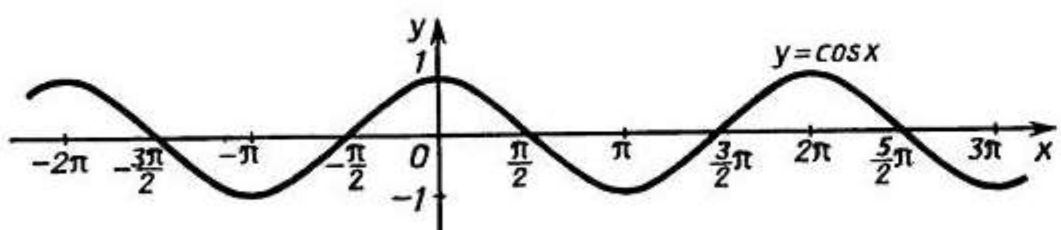
Origin is the point of symmetry

Let’s check it.

$$f(-x)=\frac{(-x)^3}{(-x)^2-1}=\frac{-x^3}{x^2-1}=-\frac{x^3}{x^2-1}=-f(x)$$

This function is nonperiodic, i.e. there is no such value ***T*** that the equality $$f(x+T)=f(x),\quad \forall x\in D(f)$$





Let’s determine the points of intersection of the graph with the coordinate axes:

with the *x*-axis: $y=0$ $$\frac{x^3}{x^2-1}=0$$

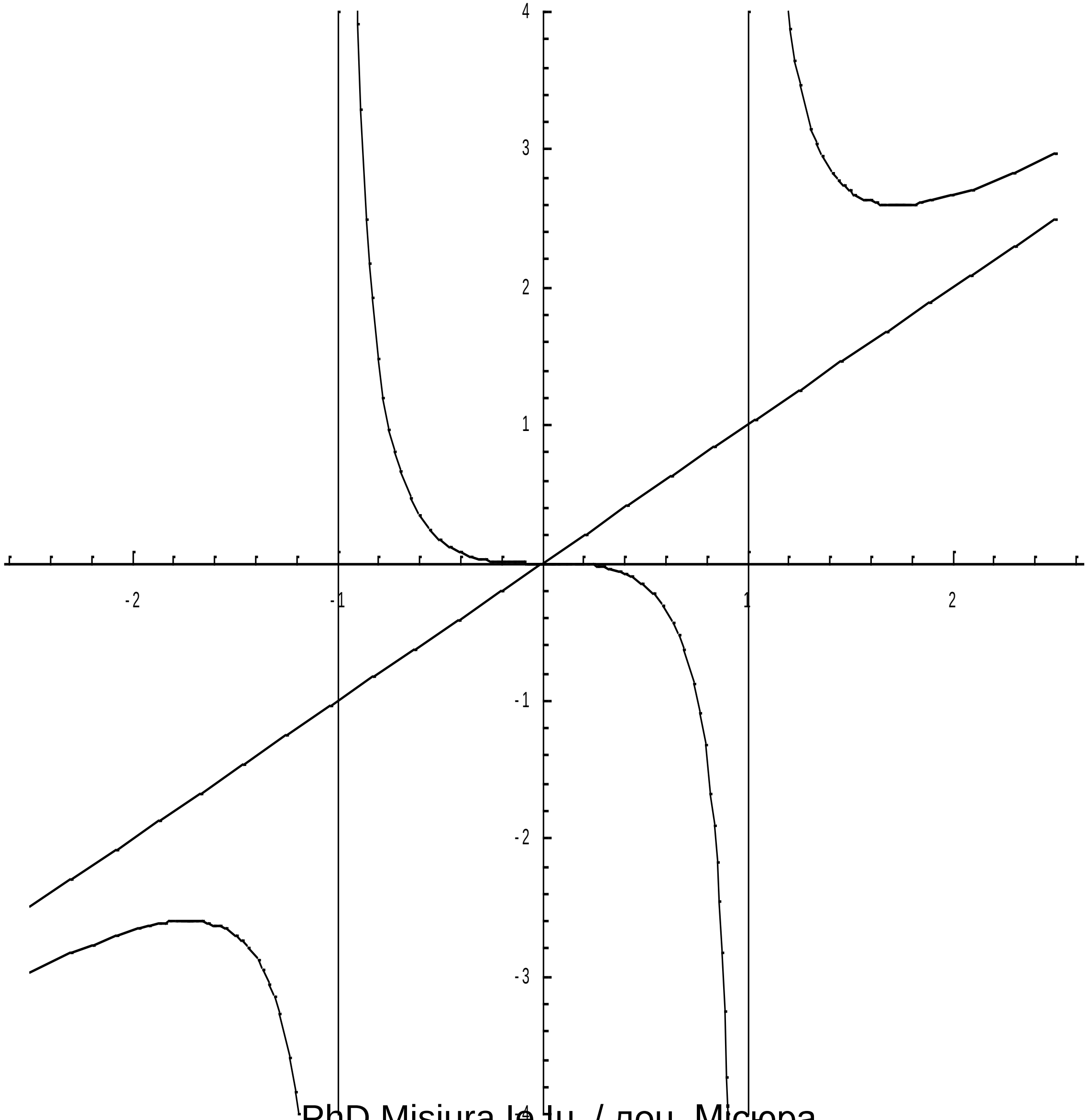
$$x^3=0$$

$$x=0$$

with the y-axis: $x=0$ $$y=\frac{0^3}{0^2-1}=0$$

This point is the origin $O(0;0)$

Let’s test the function on infinity: $$\lim_{x\to\pm\infty}\frac{x^3}{x^2-1}=\pm\infty$$



4
3
2
1
-2
-1
1
2
-1
-2
-3

The main theorems.
The application of the differential. L'Hospital's rule

Lecture plan

1. The main theorems (the mean-value theorem)
2. The application of the differential to approximate calculations
3. Evaluation of indeterminate forms by L'Hospital's rule

1. The main theorems (the mean-value theorem)

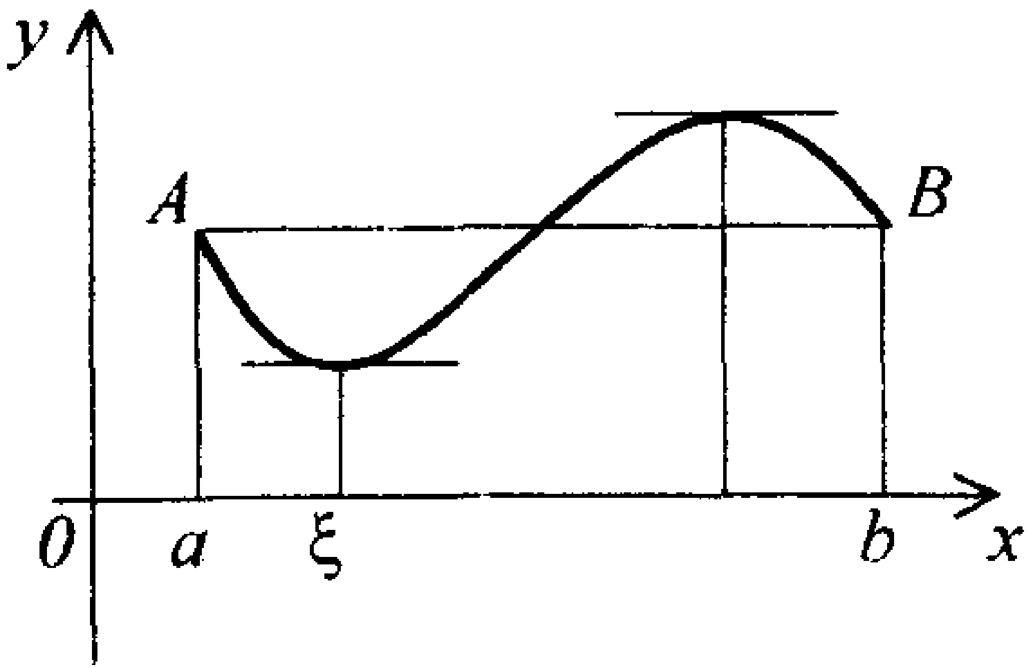
Rolle's theorem

If the function $f(x)$ is continuous on some interval $[a,b]$ and differentiable at all interior points of the interval (a,b) and $f(a)=f(b)$, then there is at least one point $x=\xi$ belonging to the interval (a,b), where the derivative of the given function vanishes, i.e. it is equal to zero,

$$f'(\xi)=0$$

Rolle's theorem

The geometrical illustration of Rolle's theorem is presented on figure 1.



Lagrange's theorem

If the function $f(x)$ is continuous on some interval $[a,b]$ and has the derivative at all interior points of the interval (a,b), then there is at least one point in the interval (a,b) that

$$\frac{f(b)-f(a)}{b-a}=f'(\xi)$$

Lagrange's theorem

Rolle's theorem is a special case of Lagrange's theorem at $f(a) = f(b)$

Cauchy's Theorem

If the functions $f(x)$ and $g(x)$ are continuous on the interval $[a,b]$, have their derivatives $f'(x)$ and $g'(x)$ at all interior points of the interval (a,b), and $g'(x) \neq 0$ then there is at least one point ξ on the interval (a,b) that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)} \quad \text{where} \quad a < \xi < b$$

Cauchy's Theorem

Cauchy's theorem is the generalization of Lagrange's theorem.

2.The application of the differential to approximate calculations

If Δx is small enough it is possible to use the differential of the function in the state of its increment , i.e.

$$f\left(x_0+\Delta x\right)-f\left(x_0\right)\approx f'\left(x_0\right)\Delta x$$

and then get an approximate value of the required according to the following formula:

$$f\left(x_0+\Delta x\right)\approx f\left(x_0\right)+f'\left(x_0\right)\Delta x$$

Example 1. Calculate an approximate value $\sqrt[3]{8,03}$

Solution. Let's consider the function $y=\sqrt[3]{x}$

Its derivative is $y'=\dfrac{1}{3\sqrt[3]{x^2}}$

Then we obtain from the previous formula:

$$\sqrt[3]{x+\Delta x}\approx\sqrt[3]{x}+\frac{1}{3\sqrt[3]{x^2}}\cdot\Delta x$$

Example 1. Calculate an approximate value $\sqrt[3]{8,03}$

We have

$$x = 8 \qquad \Delta x = 0,03$$

Then

$$\sqrt[3]{8,03} \approx \sqrt[3]{8} + \frac{1}{3\sqrt[3]{8^2}} \cdot 0,03 = 2,0025$$

TASK

Calculate:

$$arcsin\,0,49$$

$$arctg\,0,98$$

$$\sqrt{26}$$

3. Evaluation of indeterminate forms by L'Hospital's rule

L'Hospital's rule for evaluation of the indeterminate forms

$$\left\|\frac{0}{0}\right\| \quad \text{and} \quad \left\|\frac{\infty}{\infty}\right\|$$

is formulated as the following theorem:

Theorem

Theorem. Let the single-valued functions $f(x)$ and $g(x)$ be differentiable everywhere in some neighborhood of the point ***a***, i.e. at $|x-a|<e$ and $g(x)\neq 0$, then if there exists a limit (finite or infinite) of the ratio of derivatives, then the ratio of the functions has the same limit.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\left(\left\|\frac{0}{0}\right\| \quad or \quad \left\|\frac{\infty}{\infty}\right\|\right)=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Theorem

Theorem. Let the single-valued functions $f(x)$ and $g(x)$ be differentiable everywhere in some neighborhood of the point ***a***, i.e. at $|x-a|<e$ and $g(x)\neq 0$, then if there exists a limit (finite or infinite) of the ratio of derivatives, then the ratio of the functions has the same limit.

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\left(\left\|\frac{0}{0}\right\| \quad or \quad \left\|\frac{\infty}{\infty}\right\|\right)=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

Example. Calculate the limit:

$$\lim_{x\to 2}\frac{x^4-16}{3x-6}=\left[\frac{0}{0}\right]=\lim_{x\to 2}\frac{(x^4-16)'}{(3x-6)'}=\lim_{x\to 2}\frac{4x^3}{3}=\frac{4(2)^3}{3}=\frac{32}{3}=10\frac{2}{3}$$

Example. Calculate the limit:

$$\lim_{x\to 3}\frac{x^2+2x-15}{x^2-9}=\left[\frac{0}{0}\right]=\lim_{x\to 3}\frac{(x^2+2x-15)'}{(x^2-9)'}=\lim_{x\to 3}\frac{2x+2}{2x}=\lim_{x\to 3}\frac{2(x+1)}{2x}=\frac{3+1}{3}=\frac{4}{3}=1\,\frac{1}{3}$$

Example. Calculate the limit:

$$\lim_{x \to \frac{1}{2}} \frac{2x^2 - 5x + 2}{-1 + 4x - 4x^2} = \left[\frac{0}{0}\right] =$$

Example. Calculate the limit:

$$\lim_{x \to \frac{1}{2}} \frac{2x^2 - 5x + 2}{-1 + 4x - 4x^2} = \left[\frac{0}{0}\right] = \lim_{x \to \frac{1}{2}} \frac{(2x^2 - 5x + 2)'}{(-1 + 4x - 4x^2)'} =$$

Example. Calculate the limit:

$$\lim_{x\to\frac{1}{2}} \frac{2x^2-5x+2}{-1+4x-4x^2} = \left[\frac{0}{0}\right] = \lim_{x\to\frac{1}{2}} \frac{(2x^2-5x+2)'}{(-1+4x-4x^2)'} =$$

$$= \lim_{x\to\frac{1}{2}} \frac{4x-5}{4-8x} = \frac{4*\frac{1}{2}-5}{4-8*\frac{1}{2}} = \frac{-3}{0} = -\infty$$

Example. Calculate the limit:

$$\lim_{x\to 5}\frac{5-x}{3-\sqrt{2x-1}} = \left[\frac{0}{0}\right] = \lim_{x\to 5}\frac{(5-x)'}{\left(3-\sqrt{2x-1}\right)'} = \lim_{x\to 5}\frac{-1}{\frac{1}{2\sqrt{(2x-1)^3}}} =$$

$$= -2\sqrt{(2*5-1)^3} = -2\sqrt{3^6} = -2*3^3 = -2*27 = -54$$

Example. Calculate the limit:

$$\lim_{x\to+\infty}\frac{2x^2+4x-2}{6x^2-3x+8}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to+\infty}\frac{\left(2x^2+4x-2\right)'}{\left(6x^2-3x+8\right)'}=$$

$$=\left[\begin{array}{l}\left(2x^2+4x-2\right)'=4x+4\\ \left(6x^2-3x+8\right)'=12x-3\end{array}\right]=\lim_{x\to+\infty}\frac{4x+4}{12x-3}=\left(\frac{\infty}{\infty}\right)=$$

$$=\lim_{x\to+\infty}\frac{(4x+4)'}{(12x-3)'}=\left[\begin{array}{c}(4x+4)'=4\\ (12x-3)'=12\end{array}\right]=\lim_{x\to+\infty}\frac{4}{12}=\frac{4}{12}=\frac{1}{3}$$

Example. Calculate the limit:

$$\lim_{x\to\infty}\frac{2x^3+x^2+1}{3x^3+4x^2+9}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(2x^3+x^2+1)'}{(3x^3+4x^2+9)'}=\lim_{x\to\infty}\frac{6x^2+2x}{9x^2+8x}=$$

$$=\lim_{x\to\infty}\frac{x(6x+2)}{x(9x+8)}=\lim_{x\to\infty}\frac{(6x+2)'}{(9x+8)'}=\frac{6}{9}=\frac{2}{3}$$

Example. Calculate the limit:

$$\lim_{x\to 0}\frac{1-cos2x}{x^2}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(1-cos2x)'}{(x^2)'}=\lim_{x\to 0}\frac{2sin2x}{2x}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to 0}\frac{(2sin2x)'}{(2x)'}=\lim_{x\to 0}\frac{4cos2x}{2}=\frac{4\cos(2*0)}{2}=\frac{4}{2}=2$$

Example. Calculate the limit:

$$\lim_{x\to 0}\frac{tg5x}{sin3x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(tg5x)'}{(sin3x)'}=\lim_{x\to 0}\frac{\frac{5}{cos^2x}}{3cos3x}=$$

$$=\frac{5}{3cos^2 0 cos3*0}=\frac{5}{3}=1\frac{2}{3}$$

Example 2. Calculate the limit: $\lim_{x\to 1}\frac{x^3-1-\ln x}{e^x-e}$

Solution.

$$\lim_{x\to 1}\frac{x^3-1-\ln x}{e^x-e}$$

Example 2. Calculate the limit: $\lim\limits_{x\to 1}\dfrac{x^3-1-\ln x}{e^x-e}$

Solution.

$$\lim_{x\to 1}\frac{x^3-1-\ln x}{e^x-e} = \left\|\frac{0}{0}\right\| =$$

Example 2. Calculate the limit: $\lim\limits_{x\to 1}\frac{x^3-1-\ln x}{e^x-e}$

Solution.

$$\lim_{x\to 1}\frac{x^3-1-\ln x}{e^x-e}=\left\|\frac{0}{0}\right\|=\lim_{x\to 1}\frac{\left(x^3-1-\ln x\right)'}{\left(e^x-e\right)'}=$$

Example 2. Calculate the limit: $\lim\limits_{x\to 1}\dfrac{x^3-1-\ln x}{e^x-e}$

Solution.

$$\lim_{x\to 1}\frac{x^3-1-\ln x}{e^x-e}=\left\|\frac{0}{0}\right\|=\lim_{x\to 1}\frac{\left(x^3-1-\ln x\right)'}{\left(e^x-e\right)'}=$$

$$=\lim_{x\to 1}\frac{3x^2-\frac{1}{x}}{e^x}=\frac{2}{e}$$

Indetermination $0 \cdot \infty$ is transformed to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

A product $f(x) \cdot \varphi(x)$ is transformed to

$$\frac{f(x)}{\frac{1}{\varphi(x)}} \quad \left(\frac{0}{0}\right) \qquad\qquad \frac{\varphi(x)}{\frac{1}{f(x)}} \quad \left(\frac{\infty}{\infty}\right)$$

Indetermination $\infty - \infty$ is transformed to $\dfrac{0}{0}$

A product $f(x) - \varphi(x)$ is transformed to

$$f(x)\left(1 - \frac{\varphi(x)}{f(x)}\right)$$ $\left(\dfrac{0}{0}\right)$

Indeterminations $0^0, 1^\infty, \infty^0$ are transformed to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ with the help of the natural logarithm