CONTINUOUS RANDOM VARIABLES

The relationships between the functions $f(x)$ and $F(x)$:	1) $F(x)=\int_{-\infty}^{x} f(x)dx$ 2) $f(x)=F'(x)$
The condition of normalization of the function $f(x)$	$\int_{-\infty}^{\infty} f(x)dx=1$
The probability that a random variable X lies in the interval (x_1,x_2)	$P(x_1<X<x_2)=F(x_2)-F(x_1)=\int_{x_1}^{x_2} f(x)dx$
Numerical characteristics	
1) *Mathematical expectation:* $M(X)=\int_{-\infty}^{\infty} x\cdot f(x)dx$ 2) *Variance*: $D(X)=\int_{-\infty}^{\infty} x^2\cdot f(x)dx-[M(X)]^2$ or $D(X)=\int_{-\infty}^{\infty}(x-M(X))^2\cdot f(x)dx$ 3) *Root-mean-square deviation*: $\sigma(X)=\sqrt{D(X)}$	

Example 1. The density function of a continuous variable X is given: $f(x)=\begin{cases}0, & x\le 0\\ c\left(4x-2x^2\right), & 0<x\le 2\\ 0, & x>2\end{cases}$. a) What is the value of c? b) Find $P(X>1)$.

Solution. a) We use the condition of normalization $\int_{-\infty}^{\infty} f(x)dx=1$:

$$\int_{-\infty}^{\infty} f(x)dx=\int_{-\infty}^{0}0dx+\int_{0}^{2}c\left(4x-2x^2\right)dx+\int_{2}^{+\infty}0dx=1 \quad\text{or}\quad \int_{0}^{2}c\left(4x-2x^2\right)dx=1 \quad\text{or}$$

$$c\left(4\frac{x^2}{2}-2\frac{x^3}{3}\right)\Bigg|_0^2=c\left(2x^2-\frac{2x^3}{3}\right)\Bigg|_0^2=1 \quad\text{or}\quad c\left(8-\frac{16}{3}-0\right)=c\cdot\frac{8}{3}=1 \quad\text{or}\quad c=\frac{3}{8}.$$

So, the probability density function is $f(x)=\begin{cases}0, & x\le 0\\ \frac{3}{8}\left(4x-2x^2\right), & 0<x\le 2\\ 0, & x>2\end{cases}$

b) Let's find $P(X>1)$:

$$P(X>1)=P(1<X<+\infty)=\int_1^{+\infty} f(x)dx=\int_1^2 \frac{3}{8}\left(4x-2x^2\right)dx+\int_2^{+\infty}0dx=$$

$$=\frac{3}{8}\left(2x^2-\frac{2x^3}{3}\right)\Bigg|_1^2=\frac{3}{8}\left(\left(2\cdot 2^2-\frac{2\cdot 2^3}{3}\right)-\left(2\cdot 1^2-\frac{2\cdot 1^3}{3}\right)\right)=\frac{3}{8}\left(\frac{8}{3}-\frac{4}{3}\right)=\frac{1}{2}.$$

Example 2. The differential distribution function of X is given:

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2.\\ 0, & x>2\end{cases}$$

Find the integral distribution function $F(x)$. Plot the graphs of the functions $f(x)$ and $F(x)$.

Solution. The integral distribution function $F(x)$ according to the formula is

$$F(x)=\int_{-\infty}^{x} f(x)dx.$$

If $x\le 1$, then $f(x)=0$ and $F(x)=\int_{-\infty}^{x} f(x)dx=\int_{-\infty}^{x}0dx=0.$

If $1<x\le 2$, then $F(x)=\int_{-\infty}^{x} f(x)dx=\int_{-\infty}^{1} f(x)dx+\int_1^x f(x)dx=$

$$=\int_{-\infty}^{1}0dx+\int_1^x\left(x-\frac{1}{2}\right)dx=\left(\frac{x^2}{2}-\frac{1}{2}x\right)\Bigg|_1^x=\frac{x^2}{2}-\frac{1}{2}x-\left(\frac{1}{2}-\frac{1}{2}\right)=\frac{x^2}{2}-\frac{x}{2}.$$

If $x>2$, then $F(x)=\int_{-\infty}^{x} f(x)dx=\int_{-\infty}^{1} f(x)dx+\int_1^2 f(x)dx+\int_2^x f(x)dx=$

$$=\int_{-\infty}^{1}0dx+\int_1^2\left(x-\frac{1}{2}\right)dx+\int_2^x 0dx=\left(\frac{x^2}{2}-\frac{1}{2}x\right)\Bigg|_1^2=\frac{2^2}{2}-\frac{1}{2}\cdot 2-\left(\frac{1^2}{2}-\frac{1}{2}\cdot 1\right)=2-1-0=1.$$

Let’s write the formula for the integral distribution function $F(x)$:

$$F(x)=\begin{cases}0, & x\le 1\\ \frac{x^2}{2}-\frac{x}{2}, & 1<x\le 2.\\ 1, & x>2\end{cases}$$

Example 3. The integral distribution function of X is given:

$$F(X)=\begin{cases}0, & x\le 0{,}5\\ \dfrac{2x-1}{2}, & 0{,}5<x\le 1{,}5.\\ 1, & x>1{,}5\end{cases}$$

Find the density function $f(x)$.

Solution. It is known that $f(x)=F'(x)$. Thus

$$f(x)=F'(X)=\begin{cases}(0)', & x\le 0{,}5\\ \left(\dfrac{2x-1}{2}\right)', & 0{,}5<x\le 1{,}5\\ (1)', & x>1{,}5\end{cases}=\begin{cases}0, & x\le 0{,}5\\ \dfrac{2-0}{2}, & 0{,}5<x\le 1{,}5\\ 0, & x>1{,}5\end{cases}=\begin{cases}0, & x\le 0{,}5\\ 1, & 0{,}5<x\le 1{,}5\\ 0, & x>1{,}5\end{cases}$$

Example 4. The density function of X is given: $f(x)=\begin{cases}0, & x\le 0{,}5\\ 1, & 0{,}5<x\le 1{,}5\\ 0, & x>1{,}5\end{cases}$. Calculate:

a) $M(X)$ and $D(X)$; b) $P(1<X<2.3)$; c) plot the graphs of the functions $F(X)$ and $f(x)$.

Solution. a) Let’s find the mathematical expectation:

$$M(X)=\int_{-\infty}^{\infty}x\cdot f(x)dx=\int_{-\infty}^{0.5}x\cdot 0dx+\int_{0.5}^{1.5}x\cdot 1dx+\int_{1.5}^{\infty}x\cdot 0dx=$$

$$=0+\int_{0.5}^{1.5}xdx+0=\frac{x^2}{2}\Bigg|_{0.5}^{1.5}=\frac{1}{2}\cdot\left[1.5^2-0.5^2\right]=\frac{1}{2}\cdot[2.25-0.25]=\frac{1}{2}\cdot 2=1.$$

Let’s calculate the variance: $D(X)=\int_{-\infty}^{\infty}x^2\cdot f(x)dx-[M(X)]^2=$

$$=\int_{-\infty}^{0.5}x^2\cdot 0dx+\int_{0.5}^{1.5}x^2\cdot 1dx+\int_{1.5}^{\infty}x^2\cdot 0dx-1^2=0+\int_{0.5}^{1.5}x^2dx+0-1=$$

$$=\frac{x^3}{3}\Bigg|_{0.5}^{1.5}-1=\frac{1}{3}\cdot\left[1.5^3-0.5^3\right]-1=\frac{1}{3}\cdot[3.375-0.125]-1=\frac{1}{3}\cdot 3.25-1\approx 0.0833.$$

b) Let's find the probability that a random variable X lies in the interval $(1.0, 2.3)$.

Thus, $P(1 < X < 2.3) = F(2.3) - F(1) = 1 - \frac{2 \cdot 1 - 1}{2} = 1 - \frac{1}{2} = \frac{1}{2} = 0.5$.

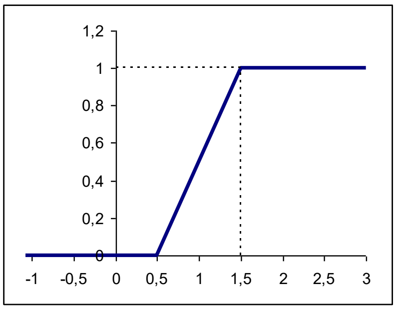


Fig. 1. The graph of $F(X)$

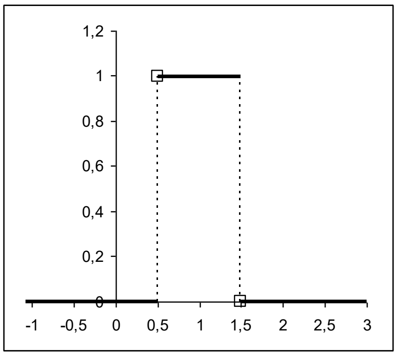
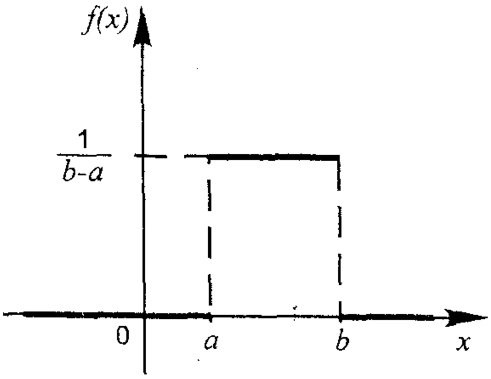


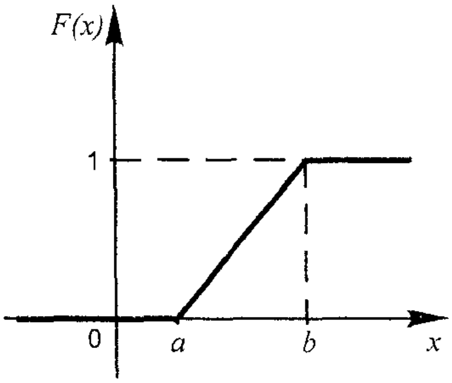
Fig. 2. The graph of $f(x)$

TOPIC: MAIN CONTINUOUS DISTRIBUTIONS

1. UNIFORM DISTRIBUTION LAW

$$f(x) = \begin{cases} 0, & x \le a \\ \frac{1}{b-a}, & a < x \le b, \\ 0, & x > b \end{cases} \qquad F(x) = \begin{cases} 0, & x \le a \\ \frac{x-a}{b-a}, & a < x \le b. \\ 1, & x > b \end{cases}$$





The probability that a random variable X lies in the interval:

$$P(\alpha < X < \beta) = F(\beta) - F(\alpha) = \frac{\beta - \alpha}{b - a}.$$

Numerical characteristics of uniform random variables are

1) *mathematical expectation*: $M(X) = \frac{a+b}{2}$; 2) *variance*: $D(X) = \frac{(b-a)^2}{12}$;

3) *root-mean-square deviation*: $\sigma(X) = \frac{b-a}{2\sqrt{3}}$.

Example 5. The parameters a, b of the uniform law of distribution are given: $a = 2$ and $b = 6$. Find: a) functions $f(x)$ and $F(x)$; b) the mathematical expectation $M(X)$, the variance $D(X)$ and the root-mean-square deviation $\sigma(X)$; c) $P(0 < X < 3)$.

Solution. Let's find functions $f(x)$ and $F(x)$ substituting $a = 2$ and $b = 6$ into formulas for functions:

$$f(x)=\begin{cases}0, & x\le 2\\ \dfrac{1}{6-2}, & 2<x\le 6\\ 0, & x>6\end{cases}=\begin{cases}0, & x\le 2\\ \dfrac{1}{4}, & 2<x\le 6,\\ 0, & x>6\end{cases}$$

$$F(x)=\begin{cases}0, & x\le 2\\ \dfrac{x-2}{6-2}, & 2<x\le 6\\ 1, & x>6\end{cases}=\begin{cases}0, & x\le 2\\ \dfrac{x-2}{4}, & 2<x\le 6.\\ 1, & x>6\end{cases}$$

Let's calculate the numerical characteristics $M(X)$, $D(X)$ and $\sigma(X)$:

$$M(X)=\frac{a+b}{2}=\frac{2+6}{2}=4, \qquad D(X)=\frac{(6-2)^2}{12}=\frac{4^2}{12}=\frac{4}{3},$$

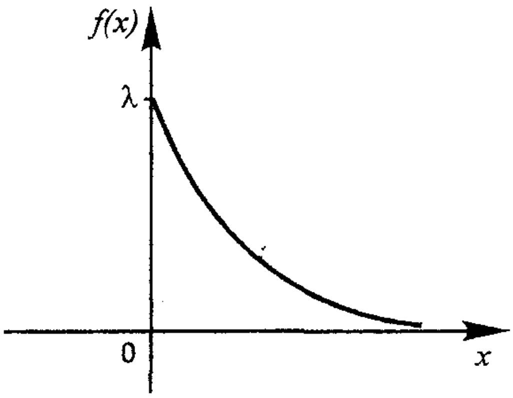
$$\sigma(X)=\frac{b-a}{2\sqrt{3}}=\frac{6-2}{2\sqrt{3}}=\frac{4}{2\sqrt{3}}=\frac{2}{\sqrt{3}}.$$

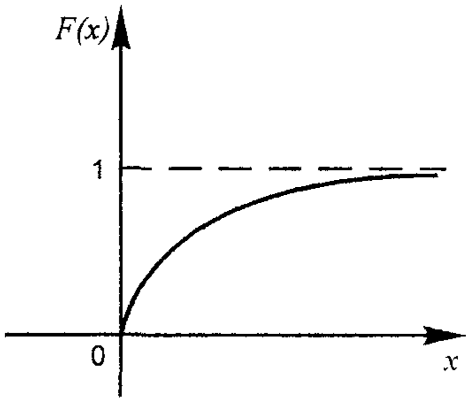
Let's calculate the probability that a uniform random variable X lies in the interval $(0,3)$:

$$P(\alpha<X<\beta)=P(0<X<3)=F(3)-F(0)=\frac{3-0}{6-2}=\frac{3}{4}=0.75.$$

2. EXPONENTIAL DISTRIBUTION LAW

$$f(x)=\begin{cases}0, & x<0\\ \lambda\cdot e^{-\lambda x}, & x\ge 0\end{cases}, \qquad F(x)=\begin{cases}0, & x<0\\ 1-e^{-\lambda x}, & x\ge 0\end{cases}.$$





The probability that a random variable X lies in the interval (α,β):

$$P(\alpha<X<\beta)=F(\beta)-F(\alpha)=e^{-\lambda\alpha}-e^{-\lambda\beta}.$$

Numerical characteristics of exponential random variables are

1) *mathematical expectation*: $M(X)=\dfrac{1}{\lambda}$.

2) *variance* $D(X)$: $D(X)=\dfrac{1}{\lambda^2}$,

3) *root-mean-square deviation*: $\sigma(X)=\dfrac{1}{\lambda}$.

Example 6. The probability density function of the exponential law of distribution $f(x)=\begin{cases}0, & x<0\\ 0.05\cdot e^{-0.05x}, & x\geq 0\end{cases}$ is given. Find: a) the function $F(x)$; b) the mathematical expectation $M(X)$, the variance $D(X)$ and the root-mean-square deviation $\sigma(X)$; c) $P(2<X<10)$.

Solution. We have that $\lambda=0.05$ from the formula of $f(x)$. Let's substitute this parameter into the formula for $F(x)$ and find the numerical characteristics:

$$F(x)=\begin{cases}0, & x<0\\ 1-e^{-\lambda x}, & x\geq 0\end{cases}=\begin{cases}0, & x<0\\ 1-e^{-0.05x}, & x\geq 0\end{cases},$$

$$M(X)=\frac{1}{\lambda}=\frac{1}{0.05}=20,\qquad D(X)=\frac{1}{\lambda^2}=\frac{1}{0.05^2}=400,\qquad \sigma(X)=\frac{1}{\lambda}=\frac{1}{0.05}=20.$$

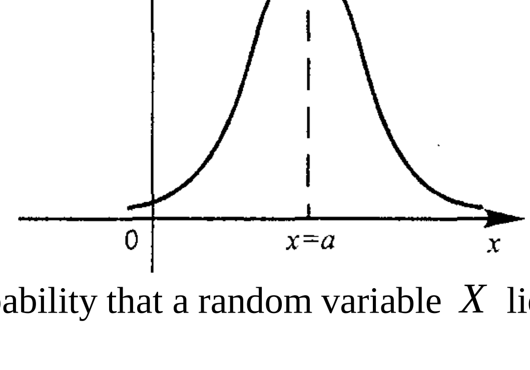
Let's calculate the probability $P(2<X<10)$ that a random variable X lies in the interval $(2,10)$:

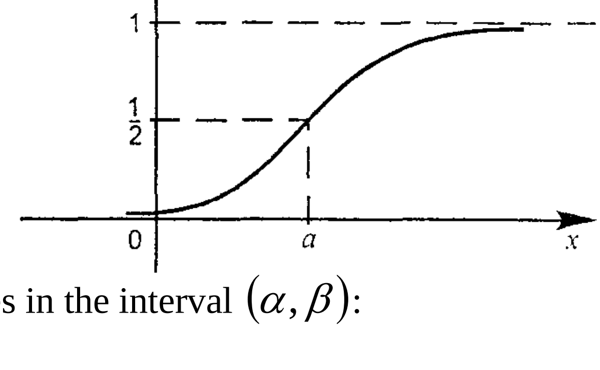
$$P(2<X<10)=e^{-\lambda\alpha}-e^{-\lambda\beta}=e^{-0.05\cdot 2}-e^{-0.05\cdot 10}=0.90484-0.60653=0.29831.$$

3. NORMAL DISTRIBUTION LAW

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{(x-a)^2}{2\sigma^2}}=\frac{\varphi(t)}{\sigma}\qquad\qquad F(x)=\int_{-\infty}^{x}\frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{(x-a)^2}{2\sigma^2}}dx=\frac{1}{2}+\Phi(t),$$

where $\varphi(t)=\dfrac{1}{\sqrt{2\pi}}\cdot e^{-\frac{t^2}{2}}$ and $\Phi(t)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^t\cdot e^{-\frac{t^2}{2}}dt$ are Laplace functions (Tables 1,2)





The probability that a random variable X lies in the interval (α,β):

$$P(\alpha < X < \beta) = F(\beta) - F(\alpha) = \Phi\left(\frac{\beta - a}{\sigma}\right) - \Phi\left(\frac{\alpha - a}{\sigma}\right).$$

Numerical characteristics of normal random variables are

1) *mathematical expectation*: $M(X) = a$, 2) *variance*: $D(X) = \sigma^2$,

3) *root-mean-square deviation*: $\sigma(X) = \sigma$.

Example 7. The probability density function of the normal law of distribution $f(x) = \frac{1}{2\sqrt{2\pi}} \cdot e^{-\frac{(x-3)^2}{8}}$ is given. Find $F(x)$, calculate $M(X)$, $D(X)$ and $P(1 < X < 7)$.

Solution. We have that $a = 3$ and $\sigma = 2$ from the formula of $f(x)$. Let's substitute these parameters into the formula for $F(x)$ and find the numerical characteristics:

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2\sigma^2}} dx = \int_{-\infty}^{x} \frac{1}{2\sqrt{2\pi}} \cdot e^{-\frac{(x-3)^2}{2 \cdot 2^2}} dx = \int_{-\infty}^{x} \frac{1}{2\sqrt{2\pi}} \cdot e^{-\frac{(x-3)^2}{8}} dx,$$

$$M(X) = a = 3, \quad D(X) = \sigma^2 = 2^2 = 4 \quad \text{and} \quad \sigma(X) = \sigma = 2.$$

Let's calculate the probability $P(1 < X < 7)$ that a random variable X lies in the interval $(1,7)$:

$$P(1 < X < 7) = \Phi\left(\frac{\beta - a}{\sigma}\right) - \Phi\left(\frac{\alpha - a}{\sigma}\right) = \Phi\left(\frac{7-3}{2}\right) - \Phi\left(\frac{1-3}{2}\right) = \Phi(2) - \Phi(-1) =$$

$$= \left\| \begin{matrix} \text{use the property} \\ \Phi(-x) = -\Phi(x) \end{matrix} \right\| = \Phi(2) + \Phi(1) = \left\| \begin{matrix} \text{use table 2} \\ \Phi(2) = 0.4772 \\ \Phi(1) = 0.3413 \end{matrix} \right\| = 0.4772 + 0.3413 = 0.8185.$$

CONTROL TASKS

Task 1. The integral distribution function of a continuous variable X is given. Find: a) the density function $f(x)$; b) the numerical characteristics $M(X)$, $D(X)$ and $\sigma(X)$; c) the probability $P(\alpha < X < \beta)$; d) plot the graphs of functions $F(X)$ and $f(x)$.

1) $$F(X) = \begin{cases} 0, & x \le -2 \\ \frac{x+2}{4}, & -2 < x \le 2 \\ 1, & x > 2 \end{cases}$$
$\alpha = 0{,}5$, $\beta = 4{,}5$;

2) $$F(X) = \begin{cases} 0, & x \le 0{,}5 \\ \frac{2x-1}{2}, & 0{,}5 < x \le 1{,}5 \\ 1, & x > 1{,}5 \end{cases}$$
$\alpha = 1{,}0$, $\beta = 2{,}3$;

3) $F(X)=\begin{cases}0, & x\le 1\\ \frac{1}{4}(x-1)^2, & 1<x\le 3\\ 1, & x>3\end{cases}$ $\alpha=0{,}2,\ \beta=2{,}6;$

4) $F(X)=\begin{cases}0, & x\le 5\\ \frac{1}{4}(x-5)^2, & 5<x\le 7\\ 1, & x>7\end{cases}$ $\alpha=3{,}4,\ \beta=6{,}6.$

Task 2. The density function of a continuous variable X is given. Find: a) the probability function $F(x)$; b) the numerical characteristics $M(X)$, $D(X)$, $\sigma(X)$; c) the probability $P(\alpha<X<\beta)$; d) plot graphs of $F(X)$ and $f(x)$.

1) $f(x)=\begin{cases}0, & x\le 1\\ \frac{x}{4}, & 1<x\le 3\\ 0, & x>3\end{cases}$ $\alpha=0,\ \beta=2;$

2) $f(x)=\begin{cases}0, & x\le 1\\ \frac{1}{2}(x-1), & 1<x\le 3\\ 0, & x>3\end{cases}$ $\alpha=1.5,\ \beta=5;$

3) $f(x)=\begin{cases}0, & x\le 0,\\ \frac{3x^2}{8}, & 0<x\le 2,\\ 0, & x>2.\end{cases}$ $\alpha=-1,\ \beta=1.5;$

4) $f(x)=\begin{cases}0, & x\le 3,\\ \frac{1}{25}(x-3)^2, & 3<x\le 8,\\ 0, & x>8.\end{cases}$ $\alpha=4,\ \beta=6.$

Task 3. The density function of a continuous variable X is given:

1) $f(x)=\begin{cases}0, & x\le 0\\ cx^2, & 0<x\le 1,\\ 0, & x>1\end{cases}$

2) $f(x)=\begin{cases}0, & x\le 2\\ c(x-2), & 2<x\le 5.\\ 0, & x>5\end{cases}$

Find: a) the parameter c; b) the integral distribution function $F(x)$; c) the probability $P(0.5<X<3)$; d) the numerical characteristics $M(X)$, $D(X)$ and $\sigma(X)$. Plot the graphs of the functions $f(x)$ and $F(x)$.

Task 4. Measurements of scientific systems are always subject to variation, some more than others. There are many structures for measurement error and statisticians spend a great deal of time modeling these errors. Suppose the measurement error X of a certain physical quantity is decided by the density function $f(x)=\begin{cases}0, & x\le -1\\ c(3-x^2), & -1<x\le 1.\\ 0, & x>1\end{cases}$ Determine c that renders $f(x)$ a valid density function. Find the probability that a random error in measurement is less than 1/2. For this particular measurement, it is undesirable if the magnitude of the error, exceeds 0.8. What is the probability that this occurs?

Task 5. The proportion of people who respond to a certain mail-order solicitation is a continuous random variable X that has the density function $f(x)=\begin{cases}0, & x\le 0\\ \dfrac{2(x+2)}{4}, & 0<x\le 1\\ 0, & x>1\end{cases}$. Show that $P(0<X<1)=1$. Find the probability that more than 1/4 but fewer than 1/2 of the people contacted will respond to this type of solicitation.

Task 6. The density function of the exponential distribution $f(x)=\begin{cases}0, & x<0\\ 0.04\cdot e^{-0.04x}, & x\ge 0\end{cases}$ is given. Find: a) the function $F(x)$; b) $M(X)$, $D(X)$ and $\sigma(X)$; c) $P(1<X<2)$.

Task 7. The waiting time, in hours, between successive speeders spotted by a radar unit is an exponential continuous random variable with cumulative distribution function $F(x)=\begin{cases}0, & x<0\\ 1-e^{-8x}, & x\ge 0\end{cases}$. Find: a) the density function $f(x)$; b) $M(X)$, $D(X)$ and $\sigma(X)$; c) the probability of waiting less than 12 minutes between successive speeders.

Task 8. The mathematical expectation of a normal random variable X equals 20, the root-mean-square deviation equals 5. Find: a) functions $f(x)$ and $F(x)$; b) the numerical characteristics $M(X)$, $D(X)$ and $\sigma(X)$; c) the probability that a random variable X lies in the interval $(15,25)$.

Task 9. The mathematical expectation of a normal random variable X equals 25. It is known that $P(35<X<40)=0,2$. Find the probability $P(10<X<15)$.

Task 10. The parameters a,b of the uniform law of distribution are given: $a=1$ and $b=4$. Find: a) the functions $f(x)$ and $F(x)$; b) $M(X)$, $D(X)$ and $\sigma(X)$; c) $P(0<X<3)$, $P(2<X<5)$.

Task 11. A man's height is distributed by the normal law. The mathematical expectation of a normal random variable X equals 170 centimeters, the root-mean-square deviation equals 5 centimeters. Find the probability that the men' height there will be a) less than 160 centimeters; b) greater than 180 centimeters; c) from 160 to 175 centimeters.