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DISCRETE RANDOM VARIABLES

Numerical characteristics	
Mathematical expectation	$M(X)=\sum_{i=1}^{n} x_i \cdot p_i = x_1 \cdot p_1 + x_2 \cdot p_2 + ... + x_n \cdot p_n$
Variance	$D(X)=M\left[(X-M(X))^2\right]=\sum_{i=1}^{n}(x_i - M(X))^2 \cdot p_i$ or $D(X)=M(X^2)-[M(X)]^2$, where $M(X^2)=\sum_{i=1}^{n} x_i^2 \cdot p_i$
Root-mean-square deviation	$\sigma(X)=\sqrt{D(X)}$

Example. Distribution law of a discrete random variable X

x_i	-2	2	6	10	14
p_i	0.05	0.16	0.35	0.31	0.13

is given. Let's find the numerical characteristics of a random variable:

a) a mathematical expectation is defined by the following formula:

$$M(X)=\sum_{i=1}^{5} x_i \cdot p_i = x_1 \cdot p_1 + x_2 \cdot p_2 + x_3 \cdot p_3 + x_4 \cdot p_4 + x_5 \cdot p_5 =$$
$$= -2 \cdot 0.05 + 2 \cdot 0.16 + 6 \cdot 0.35 + 10 \cdot 0.31 + 14 \cdot 0.13 =$$
$$= -0.1 + 0.32 + 2.1 + 3.1 + 1.82 = 7.24;$$

b) a variance is defined as:

1) $D(X)=\sum_{i=1}^{5}(x_i - M(X))^2 \cdot p_i = (-2-7.24)^2 \cdot 0.05 + (2-7.24)^2 \cdot 0.16 +$

$+(6-7.24)^2 \cdot 0.35 + (10-7.24)^2 \cdot 0.31 + (14-7.24)^2 \cdot 0.13 = 17.5024;$

2) $D(X)=M(X^2)-[M(X)]^2$, where $M(X^2)=\sum_{i=1}^{5} x_i^2 \cdot p_i$.

$$M(X^2)=x_1^2 \cdot p_1 + x_2^2 \cdot p_2 + x_3^2 \cdot p_3 + x_4^2 \cdot p_4 + x_5^2 \cdot p_5 =$$
$$=(-2)^2 \cdot 0.05 + 2^2 \cdot 0.16 + 6^2 \cdot 0.3 + 10^2 \cdot 0.31 + 14^2 \cdot 0.13 = 69.92,$$

$D(X) = 69.92 - 7.24^2 = 17.5024;$

c) a root-mean-square deviation is calculated as: $\sigma(X) = \sqrt{D(X)} = \sqrt{17.5024} = 4.1836$.

A distribution function	$F(x) = P(X < x)$
The probability that a random variable X ***lies in the interval*** (x_1, x_2)	$P(x_1 < X < x_2) = F(x_2) - F(x_1)$

$$F(x) = \begin{cases} 0, & x < x_1, \\ p_1, & x_1 \le x < x_2, \\ p_1 + p_2, & x_2 \le x < x_3, \\ \ldots, & \\ p_1 + p_2 + \ldots + p_{n-1}, & x_{n-1} \le x < x_n, \\ 1, & x \ge x_n. \end{cases}$$

Binomial distribution law	$P_n(x = k) = C_n^k p^k q^{n-k}, \quad k = 0,1,\ldots,n$ $0 < p < 1,\ q = 1 - p,\ n \ge 1$
The numerical characteristics	$M(X) = np,\ D(X) = npq,\ \sigma(X) = \sqrt{npq}$

x_i	0	1	…	$n-1$	n
p_i	q^n	$C_n^1 pq^{n-1}$	…	$C_n^{n-1} p^{n-1} q$	p^n

Example. The probability of passing an exam excellently for each of three students equals 0.4. Make up a distribution law of a number of excellent marks which are got by the students at the exam. Find a mathematical expectation, a variance and a root-mean square deviation of a discrete random variable.

Solution. Let a discrete random variable X be a number of students with the mark "5". It has such possible values: $x_1 = 0$ (no student passed the exam with the mark "5");

$x_2 = 1$ (one student passed the exam with the mark "5");

$x_3 = 2$ (two students passed the exam with the mark "5");

$x_4 = 3$ (three students passed the exam with the mark “5”).

Students’ passing an exam with the mark “5” are independent events. The probabilities of passing an exam of each student are equal, then we use Bernoulli’s formula: $P_n(x = k) = C_n^k p^k q^{n-k}$.

According to the condition we have: $n = 3$, $p = 0.4$, $q = 1 - p = 0.6$.

Let’s find $x_1 = 0$, $P_3(0) = C_3^0 p^0 q^{3-0} = 1 \cdot 1 \cdot q^3 = 0.6^3 = 0.216$;

$x_2 = 1$, $P_3(1) = C_3^1 p^1 q^{3-1} = 3 \cdot p \cdot q^2 = 3 \cdot 0.4 \cdot 0.6^2 = 0.432$;

$x_3 = 2$, $P_3(2) = C_3^2 p^2 q^{3-2} = 3 \cdot p^2 \cdot q = 3 \cdot 0.4^2 \cdot 0.6 = 0.288$;

$x_4 = 3$, $P_3(3) = C_3^3 p^3 q^{3-3} = 1 \cdot p^3 \cdot q^0 = 1 \cdot 0.4^3 \cdot 1 = 0.064$.

A distribution law of a discrete random variable X is defined by the table:

x_i	0	1	2	3
p_i	0.216	0.432	0.288	0.064

According to the formulas for a mathematical expectation, a variance and a root-mean square deviation we obtain: $M(X) = np = 3 \cdot 0.4 = 1.2$; $D(X) = npq = 3 \cdot 0.4 \cdot 0.6 = 0.72$;

$$\sigma(X) = \sqrt{npq} = \sqrt{0.72} \approx 0.85.$$

Poisson distribution law	$P_n(x = k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$, $k = 0,1,2,\ldots$, $e \approx 2{,}718$, where $\lambda = np$, $k! = 1 \cdot 2 \cdot \ldots \cdot k$
The numerical characteristics	$M(X) = np = \lambda$; $D(X) = npq \approx \lambda$; $\sigma(X) = \sqrt{npq} = \sqrt{\lambda}$.

x_i	0	1	2	…	n

p_i	$e^{-\lambda}$	$\lambda e^{-\lambda}$	$\frac{\lambda^2 e^{-\lambda}}{2!}$	…	$\frac{\lambda^n e^{-\lambda}}{n!}$

Example. The probability of finding a mistake on a book page is equal to 0.004. 500 pages are checked. Make up a distribution law of a number of finding a mistake on a book page. Find a mathematical expectation, a variance and a root-mean square deviation of a discrete random variable.

Solution. Let X be a number of finding a mistake on a book page, then the possible values of X are 0, 1, 2, 3, …, 500. Here $p = 0.004, n = 500$, then $\lambda = 500 \cdot 0.004 = 2$. Let's make up a distribution law of X according to Poisson formula $P_n(x=k) \approx \frac{\lambda^k e^{-\lambda}}{k!} = \frac{2^k e^{-2}}{k!}$:

$P_{500}(x=0) \approx \frac{2^0 \cdot e^{-2}}{0!} \approx 0.13534,$ $P_{500}(x=1) \approx \frac{2^1 \cdot e^{-2}}{1!} \approx 0.27067,$

$P_{500}(x=2) \approx \frac{2^2 \cdot e^{-2}}{2!} \approx 0.27067,$ $P_{500}(x=3) \approx \frac{2^3 \cdot e^{-2}}{3!} \approx 0.18045,$

$P_{500}(x=4) \approx \frac{2^4 \cdot e^{-2}}{4!} \approx 0.09022,$ $P_{500}(x=5) \approx \frac{2^5 \cdot e^{-2}}{5!} \approx 0.03609,$

$P_{500}(x=6) \approx \frac{2^6 \cdot e^{-2}}{6!} \approx 0.01203,$ $P_{500}(x=7) \approx \frac{2^7 \cdot e^{-2}}{7!} \approx 0.00344,$

$P_{500}(x=8) \approx \frac{2^8 \cdot e^{-2}}{8!} \approx 0.00086,$ $P_{500}(x=9) \approx \frac{2^9 \cdot e^{-2}}{9!} \approx 0.00019,$

$P_{500}(x=10) \approx \frac{2^{10} \cdot e^{-2}}{10!} \approx 0.00004$ and so on. At $k \geq 11$ we have that $P_{500}(x \geq 11) \approx 0$.

So, a distribution law has a form

x_i	0	1	2	3	4	5
p_i	0.13534	0.27067	0.27067	0.18045	0.09022	0.03609
x_i	6	7	8	9	10	…

p_i	0.01203	0.00344	0.00086	0.00019	0.00004	…

Let's calculate the total sum of probabilities: $\sum_{i=1}^{n} p_i = \sum_{i=1}^{500} p_i \approx 0.99999 \approx 1.$

Let's find the numerical characteristics: $M(X) = np = \lambda = 2;$ $D(X) = npq \approx \lambda = 2;$

$$\sigma(X) = \sqrt{npq} = \sqrt{\lambda} = \sqrt{2} \approx 0.41421.$$

TASKS

Task 1. Statisticians use sampling plans to either accept or reject batches or lots of material. Suppose one of these sampling plans involves sampling independently 10 items from a lot of 100 items in which 12 are defective. Make up a distribution law of the number of items found defective in the sample of 10 (in this case, the random variable takes on the values 0, 1, 2, ..., 9, 10).

Task 2. The factory sent 5 000 products of high quality to the warehouse. The probability of damaging the products on the way is equal to 0.0004. Make up a distribution law of a number of damaged products received at the warehouse. Find the probability that a) from 2 to 3 damaged products will be received at the warehouse; b) at least one damaged product will be received at the warehouse. Calculate the numerical characteristics $M(X)$, $D(X)$ and $\sigma(X)$. Find a distribution function of a random variable X.

Task 3. There are 500 details. The probability of producing defective details is equal to 0.01. Make up a distribution law of a number of defective details. Find the probability that there will be: a) 3 defective details; b) from 2 to 4 defective ones. Calculate $M(X)$, $D(X)$ and $\sigma(X)$.

Task 4. A coin is tossed 5 times. Make up a distribution law of a number of occurrences of heads. Find the probability that the coin will land with a head at a least 2 times. Calculate $M(X)$, $D(X)$ and $\sigma(X)$. Find a distribution function of a random variable X.

Task 5. The probability of train arrival at the station on time is equal to 0.8. Make up a distribution law of a number of train arrival at the station on time out of 4 expecting trains. Find the probability that no less than 2 and no more than 3 trains will arrive on time. Find the probability that at least one train will arrive on time. Calculate $M(X)$, $D(X)$ and $\sigma(X)$. Find a distribution function of a random variable X.

Task 6. The probability of the birth of a boy is equal to 0.51. Make up a distribution law of a number of newborn boys out of 10 newborns. Find the probability that among 10 newborns there will be from 3 to 7 boys. Calculate $M(X)$, $D(X)$ and $\sigma(X)$. Find a distribution function of a random variable X.

Task 7. The probability that the student will pass a test from the very first time is equal to 0.9. Make up a distribution law of a number of the students who will pass a test from the very first among 7 students of the same knowledge level. Find the probability that from 4 to 6 students will pass a test.

Task 8. There is a random variable X

a)

x_i	1	2	3	4	5
p_i	0,2	0,4	0,1	0,2	0,1

b)

x_i	1	4	5	7
p_i	0,4	0,1	0,3	0,2

c)

x_i	5	10	15	20
p_i	0,2	0,3	0,4	0,1

d)

x_i	0	1	2	3	4
p_i	0,1	0,25	0,35	0,2	?

e)

x_i	-1	1	3	5	7
p_i	0,10	0,19	0,31	0,25	0,15

f)

x_i	1	2	3	4	5
p_i	0,2	?	0,1	0,3	0,1

g)

x_i	1	2	5
p_i	0,3	0,5	0,2

h)

x_i	1	2	3
p_i	1/6	1/2	1/3

i)

x_i	1	2	4
p_i	0,1	0,3	0,6

j)

x_i	2	4	5	8
p_i	0,1	0,1	0,5	0,3

Draw a distribution polygon. Calculate $M(X)$, $D(X)$ and $\sigma(X)$. Find a distribution function of a random variable X . Calculate the a mode $Mo(X)$, a median $Me(X)$.

Task 9. There is a random variable X

x_i	2	4
p_i	p_1	p_2

Find unknowns if $M(X)$=3,4 and $D(X)$=0,84.

Task 10. There is a random variable X

x_i	x_1	x_2
p_i	0,8	p_2

Find unknowns if $M(X)$=3,2 and $D(X)$=0,16.