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# Theme: Differential calculus of functions of many variables. Application of the gradient vector in the linear model of international trade. Integral calculus of functions of one variable

## Part 2. The indefinite integral

# Economic problems

**Economic Applications of Indefinite Integration**

Consider the following two examples as a part of your exercise to apply the tool of indefinite integration to often cited problems of economics.

a) **Investment and the Stock of Capital**

Let net investment I is the rate of change of the stock of capital K. If time is treated as a continuous variable, we can express this as

$$I(t) = \frac{dK(t)}{dt}.$$

Thus, if the rate of investment I(t) is known, the capital stock K(t) can be estimated through the formula,

$$K(t) = \int I(t)dt$$

**Example:** The rate of net investment is given by $I(t) = 12t^{\frac{1}{3}}$ and the initial stock of capital at t = 0 is 25 units. Find the equation for the stock of capital.

$$K(t) = \int 12t^{\frac{1}{3}}dt = 12\left(\frac{3}{4}\right)t^{\frac{4}{3}} + c$$

$$= 9t^{\frac{4}{3}} + c$$

As K(0) = c = 25 given,

$$K(t) = 9t^{\frac{4}{3}} + 25$$

# Economic problems

b) **Obtaining the Total from the Margin**

Integration helps us recover the total function from the marginal function if the concerned variable varies continuously. Thus, it will be possible to derive the total functions such as cost, revenue, production and saving from their marginal functions. We will examine a simple application to see the procedure involved.

**Example:** If the marginal revenue function of a firm in the production of output is $MR = 40 - 10q^2$ where q is the level of output and total revenue is 120 at 3 units of output, find the total revenue function.

Since $MR = \frac{dTR}{dq}$, we can write

$TR = \int MRdq$

$$= \int (40 - 10q^2)\, dq$$

$$= 40q - \frac{10}{3}q^3 + c$$

At q = 3, TR = 30 + c = 100 given. So c = 90. The required total revenue function is

$$TR(q) = 40q - \frac{10}{3}q^3 + 90.$$

# Economic problems

**Examples of applications of integral calculus in economics**

**Mean value of a function on the interval [*a*, *b*]:**

$$f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx.$$

**Problem 1.** Let us assume that the value of a US dollar expressed in the Polish złoty from day 1 to day 30 of a certain month is represented by a function of time:

$$f(t)=2{,}5+0{,}5\cdot\sin\left(\frac{\pi}{15}t\right).$$

Find the mean value of a US dollar in this month.

**Lecture plan**

1. Direct integration
2. Integration by substitution (change of variable)
3. Integration by parts
4. Application of the substitution method to calculate indefinite integral of rational functions with quadratic trinomial

# 1. Direct integration

The function $F(x)$ is called *the antiderivative* of the function $f(x)$ if the derivative of $F(x)$ is the function $f(x)$, i.e.

$$F'(x) = f(x)$$

The set of all antiderivatives $F(x)+C$ of the function $f(x)$ is called ***the indefinite integral*** and designated by the symbol:

$$\int f(x)dx = F(x)+C$$

where $C$ is a constant.

The symbols $\int$ and $dx$ are the symbols of an integration.

The symbol $dx$ is an integration by the variable $x$.

The integration is the finding of the function $F(x)$ with the help of its derivative $f(x)$.

# *The basic properties of an indefinite integral:*

1) $\int k \cdot f(x)dx = k \cdot \int f(x)dx$

i.e. the indefinite integral of $k$ multiplied by $f(x)$ is $k$ multiplied by the indefinite integral of $f(x)$

2) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

the indefinite integral of the sum (or difference) of two functions is the sum (or difference) of two indefinite integrals of functions

3) $\int f(kx+b)dx = \frac{1}{k}F(kx+b)+C$

$kx+b$ is a linear function

## ***The basic properties of an indefinite integral:***

4) $\left(\int f(x)dx\right)' = f(x)$

i.e. a derivative of an indefinite integral equals an integrand.

5) $d\int f(x)dx = f(x)dx$

i.e. a differential of an indefinite integral equals an integrand expression.

# The table of basic indefinite integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int dx = x + C$$

$$\int 0 \cdot dx = C$$

$$\int \frac{dx}{x} = ln|x| + C.$$

$$\int a^x dx = \frac{a^x}{ln\, a} + C \quad (a > 0, a \neq 1)$$

$$\int e^x dx = e^x + C.$$

$$\int sin\, x dx = -cos\, x + C.$$

$$\int cos\, x dx = sin\, x + C.$$

$$\int tg x dx = -ln|cos\, x| + C.$$

$$\int ctg x dx = ln|sin\, x| + C.$$

# *The table of basic indefinite integrals*

$$\int \frac{dx}{\sin^2 x} = -ctg\ x + C.$$

$$\int \frac{dx}{\cos^2 x} = tg\ x + C.$$

$$\int \frac{dx}{1+x^2} = arctg\ x + C.$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} arctg \frac{x}{a} + C$$

$$(a \neq 0)$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C.$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C.$$

$$(a \neq 0)$$

# *The table of basic indefinite integrals*

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} ln\left|\frac{x-a}{x+a}\right| + C \qquad (a \neq 0)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a}} = ln\left|x + \sqrt{x^2 \pm a}\right| + C \qquad (a \neq 0)$$

$$\int \frac{dx}{\sin x} = \ln\left|tg\,\frac{x}{2}\right| + C \qquad \int \frac{dx}{\cos x} = \ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$

**Example 1.** Find integrals:

$$\int\left(2x^2+\frac{7}{2x^3}-e^x\right)dx=\int\left(2x^2+\frac{7}{2}\cdot x^{-3}-e^x\right)dx=\left|\text{apply tabular integrals}\right|$$

$$=2\frac{x^3}{3}+\frac{7}{2}\cdot\frac{x^{-3+1}}{-3+1}-e^x+C=\frac{2}{3}x^3-\frac{7}{4}x^{-2}-e^x+C$$

$$\int\frac{1}{\sin^2(5x-3)}=\left|\text{apply the }3-\text{rd property}\right|=-\frac{1}{5}ctg(5x-3)+C$$

$$\int\frac{dx}{5-9x}=-\frac{1}{9}\ln\left|5-9x\right|+C$$

$$\int e^{4x+3}dx=\frac{1}{4}e^{4x+3}+C$$

# TASKS

Find:

$$\int \frac{dx}{\sqrt{9-4x^2}}$$

$$\int (3x-5)^6 \, dx$$

$$\int \frac{dx}{9x^2+4}$$

$$\int \sqrt{2x+3} \, dx$$

# TASKS

Find:

$$\int (3x-5)^6\,dx$$

# TASKS

Find:

$$\int (3x-5)^6\, dx$$

$$\int f(kx+b)dx = \frac{1}{k}F(kx+b)+C$$

# TASKS

Find:

$$\int (3x-5)^6 \, dx$$

$$\int f(kx+b)dx = \frac{1}{k}F(kx+b)+C$$

$$k=3$$

$$\int (3x-5)^6 dx = \frac{1}{3} \cdot \frac{(3x-5)^7}{7} + C = \frac{(3x-5)^7}{21} + C$$

# TASKS

Find:

$$\int \sqrt{2x+3}\, dx$$

# TASKS

Find:

$$\int \sqrt{2x+3}\, dx$$

$$\int f(kx+b)dx = \frac{1}{k}F(kx+b)+C$$

$$k=2$$

$$\int \sqrt{2x+3}\, dx = \int (2x+3)^{1/2} dx = \frac{1}{2} \cdot \frac{(2x+3)^{3/2}}{3/2} + C =$$

# TASKS

Find:

$$\int \sqrt{2x+3}\,dx \qquad \boxed{k=2}$$

$$\int \sqrt{2x+3}\,dx = \int (2x+3)^{1/2}\,dx = \frac{1}{2}\cdot\frac{(2x+3)^{3/2}}{3/2}+C=$$

$$=\frac{(2x+3)^{3/2}}{3}+C=\frac{\sqrt{(2x+3)^3}}{3}+C$$

# TASKS

Find:

$$\int \frac{dx}{9x^2+4}$$

# TASKS

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} arctg \frac{x}{a} + C$$

Find:

$$\int \frac{dx}{x^2 + 4}$$

$$\int \frac{dx}{x^2 + 4} = \int \frac{dx}{2^2 + x^2} = \left| a = 2 \right| = \frac{1}{2} arctg \frac{x}{2} + C$$

# TASKS

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} arctg \frac{x}{a} + C$$

$$\int f(kx+b)dx = \frac{1}{k} F(kx+b) + C$$

$$\int \frac{dx}{x^2+4} = \int \frac{dx}{2^2+x^2} = \left| a=2 \right| = \frac{1}{2} arctg \frac{x}{2} + C$$

$$\int \frac{dx}{9x^2+4} = \int \frac{dx}{2^2+(3x)^2} = \begin{vmatrix} a=2 \\ k=3 \end{vmatrix} = \frac{1}{3} \cdot \frac{1}{2} arctg \frac{3x}{2} + C =$$

$$= \frac{1}{6} arctg \frac{3x}{2} + C$$

# TASKS

Find: $\int \frac{dx}{\sqrt{9-4x^2}}$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = arcsin\frac{x}{a} + C.$$

$$\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{\sqrt{3^2-(2x)^2}} = \left| \begin{array}{l} a=3 \\ k=2 \end{array} \right| =$$

$$= \frac{1}{2} \cdot \arcsin \frac{2x}{3} + C$$

# TASKS

Find: $$\int \frac{dx}{4x^2-9}$$

$$\int \frac{dx}{4x^2-9} = \int \frac{dx}{(2x)^2-3^2} = \begin{vmatrix} a=3 \\ k=2 \end{vmatrix} =$$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot 3}\ln\left|\frac{2x-3}{2x+3}\right| + C = \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C$$

# 2. Integration by substitution (change of variable)

This method is one of the main methods to calculate indefinite integrals. If the integrand has the function $f\left(g\left(x\right)\right)$ and the derivative $g'\left(x\right)$, i.e.

$$\int f\left(g\left(x\right)\right)g'\left(x\right)dx$$

then this integral is transformed to the integral $\int f\left(t\right)dt$

by the substitution $t = g\left(x\right)$. This method reduces such integrals to the tabular ones.

# Please, define the substitution

## Task 1

$$\int xe^{x^2}dx$$

# Please, define the substitution

## Task 1

$$\int xe^{x^2}dx$$

$$\int f\left(x^2\right)xdx$$

## Substitution

$$x^2 = t$$

# Please, define the substitution

## Task 2

$$\int \frac{x^2 dx}{\sqrt{9 - x^6}}$$

# Please, define the substitution

## Task 2

$$\int \frac{x^2 dx}{\sqrt{9 - x^6}}$$

$$\int f\left(x^3\right) x^2 dx$$

## Substitution

$$x^3 = t$$

# Please, define the substitution

## Task 3

$$\int \frac{ln^3 x}{x} dx$$

# Please, define the substitution

## Task 3

$$\int \frac{ln^3 x}{x} dx$$

$$\int f\left(ln\, x\right) \frac{dx}{x}$$

## Substitution

$$ln\, x = t$$

# Please, define the substitution

## Task 4

$$\int \frac{e^x dx}{e^{2x}+9}$$

## Substitution

# Please, define the substitution

## Task 4

$$\int \frac{e^x dx}{e^{2x}+9}$$

$$\int f\left(e^x\right)e^x dx$$

## Substitution

$$e^x = t$$

**Example 2.** Calculate the following integral: $\int \frac{xdx}{16+5x^2}$

Solution. The given integral is not the tabular integral, because the numerator has the expression $x$

Let's consider the derivative of the denominator:

$$\left(16+5x^2\right)' = 10x$$

Thus, the numerator is the derivative of the denominator.

Let's apply the method of integration by substitution and make the substitution of the variable:

$$t = 16+5x^2$$

then the derivative of $t$ is

$$dt = \left(16 + 5x^2\right)' dx = 10x dx$$

The numerator of the integrand does not contain the factor 10, therefore let's divide both parts of the derivative $dt = 10x dx$ by 10

$$\frac{dt}{10} = x dx$$

Let's get back to the initial integral. Its numerator is $\frac{dt}{10}$

the denominator is $t$.

$$= \int \frac{\frac{dt}{10}}{t} =$$

Let's apply one of the basic properties of indefinite integral to the given integral, i.e. the factor $\frac{1}{10}$ can be taken out of the sign of the integral:

$$= \frac{1}{10}\int \frac{dt}{t} =$$

This integral is the tabular integral, and it equals

$$= \frac{1}{10}\ln|t| + C =$$

Let's get back to the previous variable and obtain:

$$= \frac{1}{10}\ln\left|16 + 5x^2\right| + C$$

**Example 3.** Calculate following integrals:

$$\int \frac{dx}{x\sqrt{4-\ln^2 x}} = \left| \begin{array}{l} t = \ln x \\ dt = (\ln x)' dx = \frac{1}{x} dx \end{array} \right| = \int \frac{dt}{\sqrt{4-t^2}} =$$

$$= \arcsin \frac{t}{2} + C = \arcsin \frac{\ln x}{2} + C$$

$$\int \frac{\sin x dx}{\cos^5 x} = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x dx \\ -dt = \sin x dx \end{array} \right| = \int \frac{-dt}{t^5} = -\int \frac{dt}{t^5} =$$

$$= -\int t^{-5} dt = -\frac{t^{-5+1}}{-5+1} + C = \frac{1}{4t^4} + C = \frac{1}{4\cos^4 x} + C$$

# TASKS

Find:

$$\int e^{\sin x} \cos x dx$$

$$\int \frac{e^x dx}{\sqrt{e^x + 2}}$$

$$\int \frac{x dx}{9 - 4x^4}$$

$$\int x e^{-5x^2} dx$$

# 3. Integration by parts

This method is based on the famous formula of the derivative of the product of two functions:

$$(u \cdot v)' = u' \cdot v + v' \cdot u$$

where $u = u(x)$ and $v = v(x)$ are some functions of ***x***

In the differential form we have:

$$d(u \cdot v) = u \cdot dv + v \cdot du$$

We integrate this formula and obtain

$$\int d(uv) = \int u dv + \int v du$$

then we apply the property

$$\int dF(x) = F(x) + C$$

and find:

$$uv = \int u dv + \int v du$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

Let the functions $u = u(x)$ and $v = v(x)$ have continuous derivatives, the following formula of integration by parts is valid:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

This method transforms the given integral $\int u \cdot dv$ to calculation of the integral $\int v \cdot du$ . The last one may be reduced to the tabular integrals or the similar ones.

**Remark.** The name “integration by parts” can be explained as follows: the formula does not produce a final result, but only transforms

the problem from calculation of the integral $\int u \cdot dv$ to calculation of the integral $\int v \cdot du$, which is simpler at the successful choice of

$u = u(x)$ and $v = v(x)$.

*This method is applied to two types of integrals.*

| The first type | | | |
|---|---|---|---|
| | Kind of integral | The factor $u$ | The factor $dv$ |
| | $\int P_n(x) a^x dx$ | $P_n(x)$ | $a^x dx$ |
| | $\int P_n(x) \cdot e^x dx$ | $P_n(x)$ | $e^x dx$ |
| | $\int P_n(x) \cdot \sin x dx$ | $P_n(x)$ | $\sin x dx$ |
| | $\int P_n(x) \cdot \cos x dx$ | $P_n(x)$ | $\cos x dx$ |

| The second type | | | |
|---|---|---|---|
| | Kind of integral | The factor $u$ | The factor $dv$ |
| | $\int P_n(x)\cdot \arccos x dx$ | $\arccos x$ | $P_n(x)dx$ |
| | $\int P_n(x)\cdot \arcsin x dx$ | $\arcsin x$ | $P_n(x)dx$ |
| | $\int P_n(x)\cdot \mathrm{arctg} x dx$ | $\mathrm{arctg} x$ | $P_n(x)dx$ |
| | $\int P_n(x)\cdot \ln x dx$ | $\ln x$ | $P_n(x)dx$ |

**Example 4.** Find the integral: $\int x \cdot \cos 3x dx$

Solution. It is the integral of the first type:

$$u = x \qquad\qquad dv = cos\,3xdx$$

Let’s find

$$du = dx \qquad\qquad v = \int cos\,3xdx = \frac{1}{3} sin\,3x$$

(suppose that $C = 0$ ).

Let’s substitute into the formula $\int u \cdot dv = u \cdot v - \int v \cdot du$

$$\int x\,cos\,3xdx = x \cdot \frac{1}{3} sin\,3x - \int \frac{1}{3} sin\,3xdx$$

Let’s apply the table of the basic integrals:

$$\int x\cos 3x\,dx = x\cdot\frac{1}{3}\sin 3x - \int\frac{1}{3}\sin 3x\,dx$$

Let’s apply the table of the basic integrals:

$$\int x\cos 3x\,dx = \frac{x}{3}\sin 3x - \frac{1}{3}\left(-\frac{1}{3}\cos 3x\right) + C =$$

$$= \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + C.$$

**Example 5.** Find the integral: $\int \frac{\ln x}{\sqrt[3]{x}} dx$

Solution. It is the integral of the second type:

$$\int P_n(x) \cdot \ln x dx = \int u \cdot dv$$

$$u = ln\, x \qquad\qquad dv = \frac{dx}{\sqrt[3]{x}}$$

Then

$$du = (\ln x)' dx = \frac{1}{x} dx$$

$$v = \int dv = \int \frac{dx}{\sqrt[3]{x}} = \int x^{-\frac{1}{3}} dx = \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} = \frac{3}{2} x^{\frac{2}{3}}$$

Let's substitute into the formula

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int \frac{\ln x}{\sqrt[3]{x}} dx = \frac{3}{2} x^{\frac{2}{3}} \ln x - \int \frac{3}{2} x^{\frac{2}{3}} \frac{dx}{x} = \frac{3}{2} x^{\frac{2}{3}} \ln x - \frac{3}{2} \int x^{-\frac{1}{3}} dx =$$

$$= \frac{3}{2} x^{\frac{2}{3}} \ln x - \frac{3}{2} \cdot \frac{3}{2} x^{\frac{2}{3}} + C = \frac{3}{2} x^{\frac{2}{3}} \ln x - \frac{9}{4} x^{\frac{2}{3}} + C.$$

# PLEASE, DEFINE TYPES

Find:

$$\int (x+2)\cos x dx$$

$$\int \ln(x+1) dx$$

$$\int \arcsin x \, dx$$

$$\int x e^{-5x} dx$$

# PLEASE, DEFINE TYPES

Find:

$$\int (x+2)\cos x dx$$

$$\int \ln(x+1) dx$$

$$\int \arcsin x \, dx$$

$$\int x e^{-5x} dx$$

# THE 1st TYPE

Find:

$$\int (x+2)\cos x dx$$

$$\int \ln(x+1) dx$$

$$\int \arcsin x \, dx$$

$$\int xe^{-5x} dx$$

# THE 2nd TYPE

Find:

$$\int (x+2)\cos x dx$$

$$\int \ln\left(x+1\right)dx$$

$$\int \arcsin x \, dx$$

$$\int xe^{-5x}dx$$

# 4. Application of the substitution method to calculate indefinite integral of rational functions with quadratic trinomial

A perfect square of a quadratic trinomial is

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

where $ax^2 + bx + c$ is the quadratic trinomial.

Rational functions with quadratic trinomial are functions of kinds:

$$\int \frac{A}{ax^2+bx+c}dx \qquad \int \frac{Ax+B}{ax^2+bx+c}dx$$

$$\int \frac{A}{\sqrt{ax^2+bx+c}}dx \qquad \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$$

where $ax^2+bx+c$ is the quadratic trinomial.

Let’s consider the first and third kinds:

$$\int \frac{A}{ax^2+bx+c}\,dx \qquad \int \frac{A}{\sqrt{ax^2+bx+c}}\,dx$$

They are reduced to tabular integrals if you allocate the perfect square in the denominator with the help of formulas:

$$(y \pm z)^2 = y^2 \pm 2yz + z^2$$

(the square of the sum or the square of the difference).

# TASK

Let's find this integral:

$$\int \frac{dx}{x^2 - 2x + 10}$$

# TASK

Let's find this integral:

$$\int \frac{dx}{\sqrt{5+4x-x^2}}$$

**Example 6.** Find $$\int \frac{2}{\sqrt{x^2+8x+25}}\,dx$$

Solution. Let's allocate the perfect square in the denominator with the help of the formula:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$x^2+8x+25 = x^2+2\cdot 4\cdot x+16+9 = \underbrace{x^2+2\cdot 4\cdot x+4^2}+9 = (x+4)^2+9$$

Let's substitute

$$\int \frac{2}{\sqrt{x^2+8x+25}}\,dx = \int \frac{2}{\sqrt{(x+4)^2+9}}\,dx = 2\cdot\int \frac{dx}{\sqrt{(x+4)^2+9}}$$

Let's substitute

$$\int \frac{2}{\sqrt{x^2+8x+25}}dx = \int \frac{2}{\sqrt{(x+4)^2+9}}dx = 2\cdot\int \frac{dx}{\sqrt{(x+4)^2+9}}$$

Let's use the tabular integral:

$$\int \frac{dx}{\sqrt{x^2 \pm a}} = ln\left|x+\sqrt{x^2 \pm a}\right| + C$$

We have

$$2\cdot\int \frac{dx}{\sqrt{(x+4)^2+9}} = 2\cdot\ln\left|\left(x+4+\sqrt{(x+4)^2+9}\right)\right| + C =$$

$$= 2\cdot\ln\left|\left(x+4+\sqrt{(x+4)^2+9}\right)\right| + C = 2\cdot\ln\left|\left(x+4+\sqrt{x^2+8x+25}\right)\right| + C$$

Let’s consider the second and fourth kinds:

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx \qquad\qquad \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$$

To integrate these functions we should use the following rules:

1) to allocate the perfect square in the trinomial with the help of formulas $(y \pm z)^2 = y^2 \pm 2yz + z^2$ (the square of the sum or the square of the difference) and obtain a new denominator

$$(x \pm p)^2 \pm q$$

2) to apply the substitution:

$$t = x + p \qquad\qquad t = x - p$$

$$dt = dx \qquad\qquad dt = dx$$

$$x = t - p \qquad\qquad x = t + p$$

3) to present the initial integral as a sum of two integrals, the first one is the tabular integral and the second one may be integrated by substitution.

Let's consider four kinds of such integrals (see the table of basic integrals, 23-26):

$$\int \frac{xdx}{x^2 \pm a} = \frac{1}{2}\ln\left|x^2 \pm a\right| + C$$

$$\int \frac{xdx}{\sqrt{a - x^2}} = -\sqrt{a - x^2} + C$$

$$\int \frac{xdx}{a - x^2} = -\frac{1}{2}\ln\left|x^2 - a\right| + C$$

$$\int \frac{xdx}{\sqrt{x^2 \pm a}} = \sqrt{x^2 \pm a} + C$$

4) to get back to the previous variable by substitution:

$$t = x + p \qquad t = x - p$$

**Example 7.** Let's find this integral

$$\int \frac{(7-8x)dx}{x^2-6x+2} = \int \frac{(7-8x)dx}{(x-3)^2-7} = \left| \begin{matrix} t = x-3 \\ dt = dx \\ x = t+3 \end{matrix} \right| = \int \frac{(7-8(t+3))dt}{t^2-7} =$$

$$= \int \frac{(10-8t)dt}{t^2-7} = 10\int \frac{dt}{t^2-7} - 8\int \frac{tdt}{t^2-7} = 10\int \frac{dt}{t^2-(\sqrt{7})^2} - 8\int \frac{tdt}{t^2-7} =$$

$$= \frac{10}{2\sqrt{7}} \ln\left|\frac{t-\sqrt{7}}{t+\sqrt{7}}\right| - \frac{8}{2}\ln\left|t^2-7\right| + C =$$

$$= \frac{5}{\sqrt{7}} \ln\left|\frac{x-3-\sqrt{7}}{x-3+\sqrt{7}}\right| - \frac{8}{2}\ln\left|x^2-6x+2\right| + C$$

# TASKS

Find:

$$\int \frac{(5x-2)dx}{x^2+6x+7}$$

$$\int \frac{6x+1}{\sqrt{5-4x-x^2}}dx$$

# 5. Integration of rational fractions

The rational fraction is called the following relation of the polynomials:

$$\frac{P_n(x)}{Q_m(x)} = \frac{a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n}{b_0 x^m + b_1 x^{m-1} + ... + b_{m-1} x + b_m}$$

where $P_n(x), Q_m(x)$ are polynomials

The rational fraction is called a improper fraction, if $n \geq m$ i.e. if the degree of the numerator is greater than or equal to the degree of the denominator.

The rational fraction is called a proper fraction, if $n < m$ i.e. if the degree of the numerator is less than the degree of the denominator.

$$\frac{x^3+1}{x^3-x^2} \qquad \frac{x^2+2x}{x^4+5x^2}$$

# Proper fractions

$$\frac{x+1}{x^2-2}$$

# Improper fractions

$$\frac{x^2 + 1}{x^2 - 2}$$

$$\frac{x^3 - 3x + 1}{x^2 - 2}$$

# Improper fractions

$$\frac{x^2 + 1}{x^2 - 2}$$

$$\frac{x^3 - 3x + 1}{x^2 - 2}$$

$$\begin{array}{rl|l}
- & x^2 + x - 7 & x + 1 \\ \cline{3-3}
 & x^2 + x & x \\ \cline{2-2}
 & \quad\quad - 7 &
\end{array}$$

$$\frac{x^2 + x - 7}{x + 1} = x + \frac{-7}{x + 1} = x - \frac{7}{x + 1}$$

# Improper fractions

$$
\begin{array}{r|l}
 x^2 - 2x + 2 & x - 1 \\ \cline{2-2}
 - \quad x^2 - x \phantom{{}+2} & x - 1 \\ \cline{1-1}
 -x + 2 & \\
 - \quad -x + 1 & \\ \cline{1-1}
 1 &
\end{array}
$$

$$\frac{x^2 - 2x + 2}{x - 1} = x - 1 + \frac{1}{x - 1}$$

# Improper fractions

$$\begin{array}{rl|l} - & x^2-2x+2 & x-1 \\ \cline{3-3} & x^2-x & x-1 \\ \cline{2-2} & -x+2 & \\ - & -x+1 & \\ \cline{2-2} & 1 & \end{array}$$

$$\frac{x^2-2x+2}{x-1}=x-1+\frac{1}{x-1}$$

$$\int\frac{x^2-2x+2}{x-1}dx=\int\left(x-1+\frac{1}{x-1}\right)dx=$$

$$=\int x\,dx-\int dx+\int\frac{dx}{x-1}=\frac{x^2}{2}-x+\ln|x-1|+C$$

Any rational fraction can be presented as a sum of a polynomial and partial fractions. The partial fractions are fractions of three following types:

$$\frac{A}{x-a} \qquad \frac{A}{(x-a)^k} \qquad \int \frac{Ax+B}{x^2+px+q}\,dx$$

$$D = p^2 - 4q < 0$$

Finding integrals of rational fractions should be carried out according to the following scheme:

1) If $n \geq m$ (an improper fraction), we should secure (выделять) an integer part, representing the integrand as a sum of an integer part (a polynomial) and a proper rational fraction.

2) Decompose the denominator of the proper rational fraction into factors like $(x-a)^k$ and $\left(x^2+px+q\right)$, where $D = p^2 - 4q < 0$

3) Decompose the proper rational fraction into partial ones according to the following theorem:

**Theorem.** If $Q_m(x) = b_0(x-a)^\alpha \cdot (x-b)^\beta \cdot \ldots \cdot \left(x^2 + px + q\right)^\mu$ then the proper irreducible (несократимая) rational fraction represented by the following way:

$$\frac{P_n(x)}{Q_m(x)} = \frac{A_0}{(x-a)^\alpha} + \frac{A_1}{(x-a)^{\alpha-1}} + \ldots + \frac{A_{\alpha-1}}{(x-a)} + \frac{B_0}{(x-b)^\beta} + \frac{B_1}{(x-b)^{\beta-1}} + \ldots + \frac{B_{\beta-1}}{(x-b)} +$$

$$+ \frac{M_0 x + N_0}{\left(x^2 + px + q\right)^\mu} + \frac{M_1 x + N_1}{\left(x^2 + px + q\right)^{\mu-1}} + \ldots + \frac{M_{\mu-1} x + N_{\mu-1}}{\left(x^2 + px + q\right)}$$

The coefficients $A_0, A_1, \ldots, B_0, B_1, \ldots$ can be defined according to the following. The obtained equation is an identity, therefore, reducing the fractions to a common denominator, we obtain identical polynomials in the numerators on the right and the left. Equating the coefficients of the same power of *x*, we obtain a system of equations for defining undetermined coefficients

$$A_0, A_1, \ldots, B_0, B_1, \ldots$$

Integrals of partial rational fractions are calculated according to the following formulas:

$$\int \frac{A}{x-a}dx = A\ln|x-a| + C$$

$$\int \frac{A}{(x-a)^k}dx = \int A(x-a)^{-k}dx = A\frac{(x-a)^{-k+1}}{-k+1} + C$$

In this case $\int \frac{Ax+B}{x^2+px+q}dx$ we apply the rule of integration of

a quadrate trinomial.

# EXAMPLES

$$\frac{P_1(x)}{(x-x_1)(x-x_2)} = \frac{A}{x-x_1} + \frac{B}{x-x_2}$$

$$\frac{x+1}{x^2+3x-4} = \frac{x+1}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$$

# EXAMPLES

$$\frac{P_2(x)}{(x-x_1)(x-x_2)(x-x_3)} = \frac{A}{x-x_1} + \frac{B}{x-x_2} + \frac{C}{x-x_3}$$

# EXAMPLES

$$\frac{2x^2-x-7}{x^3-5x^2+6x} = \frac{2x^2-x-7}{x(x-3)(x-2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-2}$$

# EXAMPLES

$$\frac{P_2(x)}{(x-x_1)(x-x_2)^2} = \frac{A}{x-x_1} + \frac{B}{x-x_2} + \frac{C}{(x-x_2)^2}$$

# EXAMPLES

$$\frac{x^2+x}{(x-1)^3(x-2)(x-3)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1} + \frac{D}{x-2} + \frac{E}{x-3}$$

$A, B, C, D, E$

# EXAMPLES

$$\frac{P_2(x)}{(x-x_1)(x^2+px+q)} = \frac{A}{x-x_1} + \frac{Bx+C}{x^2+px+q}$$

# EXAMPLES

$$\frac{P_2(x)}{(x-x_1)(x^2+px+q)} = \frac{A}{x-x_1} + \frac{Bx+C}{x^2+px+q}$$

$$\frac{4x^2+x+22}{(x-2)\cdot(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

# EXAMPLES

$$\frac{x^4+3x^3+2x-11}{(x-1)(x+1)(x-3)^3} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C_3}{(x-3)^3} + \frac{C_2}{(x-3)^2} + \frac{C_1}{x-3}$$

# EXAMPLES

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x+2}+\frac{E}{x+3}+\frac{F}{(x+3)^2}+\frac{Gx+H}{x^2+2x+13}=\frac{x^2-19x+6}{x^3(x+2)(x+3)^2(x^2+2x+13)}$$

**Example 8.** Let's find $\int \frac{x^3+1}{x^3-x^2}dx$

This fraction $\frac{x^3+1}{x^3-x^2}$ is improper, because the degree of the numerator is equal to the degree of the denominator. Let us separate the integer part by the division of the polynomials:

$$\begin{array}{r|l} x^3+1 & x^3-x^2 \\ \cline{2-2} -\underline{x^3-x^2} & 1 \\ x^2+1 & \end{array}$$

and obtain

$$\frac{x^3+1}{x^3-x^2}=1+\frac{x^2+1}{x^3-x^2}$$

Then the initial integral will be transformed to the following two integrals:

$$\int \frac{x^3+1}{x^3-x^2}dx = \int \left( 1 + \frac{x^2+1}{x^3-x^2} \right) dx = \int dx + \int \frac{x^2+1}{x^3-x^2}dx$$

2) The denominator of the second integral has real multiple roots and can be represented as the product

$$x^2(x-1)$$

Let us decompose the integrand into partial fractions:

$$\frac{x^2+1}{x^2(x-1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1}$$

where $A, B, C$ are undetermined coefficients.

Let us reduce partial fractions to the common denominator:

$$\frac{x^2+1}{x^2(x-1)}=\frac{A(x-1)+Bx(x-1)+Cx^2}{x^2(x-1)}$$

Since the denominators on the left and right are equal and the fractions are identically equal, then the numerators are equal too.

$$x^2+1=A(x-1)+Bx(x-1)+Cx^2$$

Let us collect (приводить подобные):

$$x^2 + 1 = x^2(B + C) + x(A - B) + (-A)$$

*Consider the first method of finding of undetermined coefficients.*

To find the coefficients, we equate the coefficients at the equal powers of *x* on the left and on the right:

at $x^2$: $1 = B + C$

at $x^0$: $1 = -A$

at $x^1$: $0 = A - B$

We find the **undetermined coefficients:**

from the 3-rd equation: $A = -1$

from the 2-nd equation: $A = B = -1$

from the 1-st equation: $C = 1 - B = 1 - \left(-1\right) = 2$

So,

$$\int \frac{x^3+1}{x^3-x^2}\,dx = \int dx + \int \frac{x^2+1}{x^3-x^2}\,dx = \int dx + \int \left( \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1} \right) dx =$$

$$= \int dx + \int \left( \frac{-1}{x^2} + \frac{-1}{x} + \frac{2}{x-1} \right) dx = \int dx - \int \frac{1}{x^2}\,dx - \int \frac{1}{x}\,dx + 2\int \frac{1}{x-1}\,dx =$$

$$= x + \frac{1}{x} - \ln|x| + 2\ln|x-1| + C$$

**Example.** Find the indefinite integral: $\int \frac{15x^2-4x-81}{(x-3)(x+4)(x-1)}dx$

The integrand is a proper fraction, let us decompose it into partial fractions:

$$\frac{15x^2-4x-81}{(x-3)(x+4)(x-1)} = \frac{A}{x-3} + \frac{B}{x+4} + \frac{C}{x-1}$$

Since the denominators on the left and right are equal and the fractions are identically equal, then the numerators are equal too.

Let us reduce to the common denominator and equate the numerators:

$$15x^2 - 4x - 81 = A(x+4)(x-1) + B(x-3)(x-1) + C(x-3)(x+4)$$

**Consider the second method of finding of undetermined coefficients.**

Let us substitute the first root $x = 3$ of the denominator into this expression:

$$15x^2 - 4x - 81 = A(x+4)(x-1) + B(x-3)(x-1) + C(x-3)(x+4)$$

$$15 \cdot 9 - 4 \cdot 3 - 81 = A(3+4)(3-1) + B \cdot 0 + C \cdot 0$$

$$42 = 14A$$

$$A = 3$$

**Consider the second method of finding of undetermined coefficients.**

Let us substitute the second root $x = -4$ of the deno-minator into this expression:

$$15x^2 - 4x - 81 = A(x+4)(x-1) + B(x-3)(x-1) + C(x-3)(x+4)$$

$$15 \cdot 16 + 4 \cdot 4 - 81 = A \cdot 0 + B(-4-3)(-4-1) + C \cdot 0$$

$$175 = 35B$$

$$B = 5$$

**Consider the second method of finding of undetermined coefficients.**

Let us substitute the third root $x = 1$ of the denominator into this expression:

$$15x^2 - 4x - 81 = A(x+4)(x-1) + B(x-3)(x-1) + C(x-3)(x+4)$$

$$15 \cdot 1 - 4 \cdot 1 - 81 = A \cdot 0 + B \cdot 0 + C(1-3)(1+4)$$

$$-70 = -10B$$

$$B = 7$$

So,

$$\int \frac{15x^2 - 4x - 81}{(x-3)(x+4)(x-1)}dx = \int\left(\frac{A}{x-3} + \frac{B}{x+4} + \frac{C}{x-1}\right)dx =$$

$$= \int\left(\frac{3}{x-3} + \frac{5}{x+4} + \frac{7}{x-1}\right)dx = 3\int \frac{1}{x-3}dx + 5\int \frac{1}{x+4}dx + 7\int \frac{1}{x-1}dx =$$

$$= 3\ln|x-3| + 5\ln|x+4| + 7\ln|x-1| + C$$

**Example.** Find the indefinite integral: $\int \frac{dx}{x^3 - 1}$

Let us decompose the fraction into partial fractions:

$$\frac{1}{x^3 - 1} = \frac{1}{(x-1)\left(x^2 + x + 1\right)} = \frac{A}{x-1} + \frac{Bx + C}{\left(x^2 + x + 1\right)}$$

# Integration of expressions containing trigonometric functions

Let's consider the integrals of kind:

$$\int \sin mx \cdot \cos nx dx \qquad \int \cos mx \cdot \cos nx dx$$

$$\int \sin mx \cdot \sin nx dx$$

These products of trigonometric functions are transformed in sums by formulas:

$$\sin\alpha \cdot \cos\beta = \frac{1}{2}\left(\sin\left(\alpha+\beta\right)+\sin\left(\alpha-\beta\right)\right)$$

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}\left(\cos\left(\alpha+\beta\right)+\cos\left(\alpha-\beta\right)\right)$$

$$\sin\alpha \cdot \sin\beta = \frac{1}{2}\left(\cos\left(\alpha-\beta\right)-\cos\left(\alpha+\beta\right)\right)$$

**Example.** Find the integral: $\int \sin 5x \cos 2x dx$

Solution. Let's use the formula

$$sin\,\alpha \cdot cos\,\beta = \frac{1}{2}\left(sin\left(\alpha+\beta\right)+sin\left(\alpha-\beta\right)\right)$$

We have

$$\int sin\,5x\,cos\,2xdx = \int \frac{1}{2}\left(sin\,7x + sin\,3x\right)dx =$$

$$= \frac{1}{2}\int sin\,7xdx + \frac{1}{2}\int sin\,3xdx = \frac{1}{2}\left(-\frac{cos\,7x}{7}\right) +$$

$$+ \frac{1}{2}\left(-\frac{cos\,3x}{3}\right) + C = -\frac{cos\,7x}{14} - \frac{cos\,3x}{6} + C.$$

The integral of kind $\int R\left(\sin x, \cos x\right) dx$ , where $R\left(\sin x, \cos x\right)$ is a rational function of $\sin x$ and $\cos x$ is reduced to a rational function using the general trigonometric substitution $tg\frac{x}{2} = t\left(-\pi < x < \pi\right)$

then

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$$tg\frac{x}{2} = t \qquad \frac{x}{2} = arctgt \qquad x = 2arctgt$$

We have $\qquad dx = \frac{2}{1+t^2}dt$

We have

$$\int R(\sin x, \cos x)\,dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2t\,dt}{1+t^2}$$

**Example.** Find the integral: $$\int \frac{dx}{3+5\sin x+3\cos x}$$

Solution. Let’s use the substitution

$$tg\frac{x}{2} = t \qquad \frac{x}{2} = arctgt$$

$$x = 2arctgt \qquad dx = 2\frac{dt}{1+t^2}$$

Then

$$\int \frac{dx}{3+5\sin x+3\cos x}=\int \frac{2}{3+5\cdot\dfrac{2t}{1+t^2}+3\cdot\dfrac{1-t^2}{1+t^2}}\cdot\frac{dt}{1+t^2}=$$

$$=\int \frac{2\left(1+t^2\right)}{3\left(1+t^2\right)+5\cdot 2t+3\left(1-t^2\right)}\cdot\frac{dt}{1+t^2}=$$

$$=\int \frac{2dt}{3+3t^2+10t+3-3t^2}=\int \frac{2dt}{10t+6}=\int \frac{dt}{5t+3}=$$

$$=\frac{1}{5}ln\left|5t+3\right|+C=\frac{1}{5}ln\left|5tg\frac{x}{2}+3\right|+C.$$

Let’s consider other three cases this integral $\int R\left(\sin x, \cos x\right)dx$

We have the substitutions:

a) $t = \sin x$ , if $R\left(\sin x, -\cos x\right) = -R\left(\sin x, \cos x\right)$

b) $t = \cos x$ , if $R\left(-\sin x, \cos x\right) = -R\left(\sin x, \cos x\right)$

c) $t = tgx$ , if $R\left(-\sin x, -\cos x\right) = R\left(\sin x, \cos x\right)$

a) if we have $\int R\left(\sin x\right)\cos x dx$

then the substitution $\sin x = t$ reduces to the integral $\int R\left(t\right)dt$

b) if we have $\int R\left(\cos x\right)\sin x dx$

then the substitution $\cos x = t$ reduces to the integral $\int R\left(t\right)dt$

# Integration of irrational functions

Integrals of the form: $$\int R\left(x, x^{\frac{m}{n}}, \dots, x^{\frac{r}{s}}\right) dx$$

where *R* is a rational function of its arguments are calculated by the substitution $x = t^k$ (*k* is a common denominator of the fractions $\frac{m}{n}, \dots, \frac{r}{s}$) allowing to get rid of irrationality

**Example.** Find the integral: $$\int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x\left(1 + \sqrt[3]{x}\right)} dx.$$

**Example.** Find the integral: $$\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx.$$

Solution. Let's find the common denominator of fractions

$$\frac{2}{3},\frac{1}{6},\frac{1}{3}$$

It's 6, then the substitution is $x=t^6$ and $dx=6t^5dt$

We have

$$\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx=\int \frac{t^6+t^4+t}{t^6\left(1+t^2\right)}\cdot 6t^5dt=$$

$$=6\int \frac{t\left(t^5+t^3+1\right)t^5}{t^6\left(1+t^2\right)}dt=6\int \frac{t^5+t^3+1}{t^2+1}dt.$$

Let’s consider integrals of the form:

a) $\int R\left(x, \sqrt{a^2 - x^2}\right) dx$

b) $\int R\left(x, \sqrt{a^2 + x^2}\right) dx,$

c) $\int R\left(x, \sqrt{x^2 - a^2}\right) dx.$

Such integrals are found by the substitution:

a) $x = a \sin t$ or $x = a \cos t$

b) $x = a tg t$ or $x = a ctg t$

c) $x = \dfrac{a}{\cos t}$ or $x = \dfrac{a}{\sin t}$