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Theme: Differential calculus of functions of many variables. Application of the gradient vector in the linear model of international trade. Integral calculus of functions of one variable

Part 1. Analysis of the function of several variables

Economic problems

Example Suppose that the production function $Q(x,y) = 4000x^{0,5}y^{0,5}$ is known. Determine the marginal productivity of labor and the marginal productivity of capital when 25 units of labor and 169 units of capital are used.

Economic problems

Example At a certain factory, the daily output is $Q = 60K^{1/2}L^{1/3}$ units, where K denotes the capital investment measured in units of $ 1000 and L the size of the labor force measured in worker-hours. The current capital investment is $ 900000 and 1000 and labor are used each day. Estimate the change in output that will result if capital investment is increased by $1000 and labor is increased by 2 worker-hours.

Economic problems

Example A firm produces two products which are sold in two separate markets with the demand schedules $P_1 = 600 - 0{,}2Q_1$, $P_2 = 500 - 0{,}2Q_2$

Production costs are related and the firm faces the total cost function

$$TC = 16 + 1{,}2Q_1 + 1{,}5Q_2 + 0{,}2Q_1Q_2$$

If the firm wishes to maximize total profits, how much of each product should it sell? What will the maximum profit level be?

Economic problems

Example A firm can sell its product in two countries, A and B, where demand in country A is given by $P_A = 100 - 2Q_A$ and in country B is $P_B = 100 - Q_B$.

It's total output is $Q_A + Q_B$, which it can produce at a cost of

$$TC = 50(Q_A+Q_B) + 0{,}5(Q_A+Q_B)^2$$

How much will it sell in the two countries assuming it maximises profits?

Economic problems

Example A firm has set different prices for its family and industrial customers. If P_1 and Q_1 are the price and demand for the household market, then the demand equation is $P_1 + Q_1 = 500$. If P_2 and Q_2 are the price and demand for the industrial market, then the demand equation is

$2P_2 + 3Q_2 = 720$. The total cost function is

$TC = 50000 + 20Q$, where $Q = Q_1 + Q_2$.

Determine the firm's pricing policy that maximizes profit, with price discrimination, and calculate the value of the maximum price.

Economic problems

Example. The company produces products of two types A and B. The cost function is $C(x,y)=x^2+2xy+2y^2$. The profit per unit output of product A is 500, the profit per unit output of product B is 600. Define an optimal plan of output if the objective is to find a maximum of a profit. Make an analysis of the obtained values in the problem.

Lecture plan

1. Function Definition Domain.
2. Partial Derivatives of Functions of Several Independent Variables
3. Gradient of the function
4. Extremum of Function of two independent variables

1. Function Definition Domain

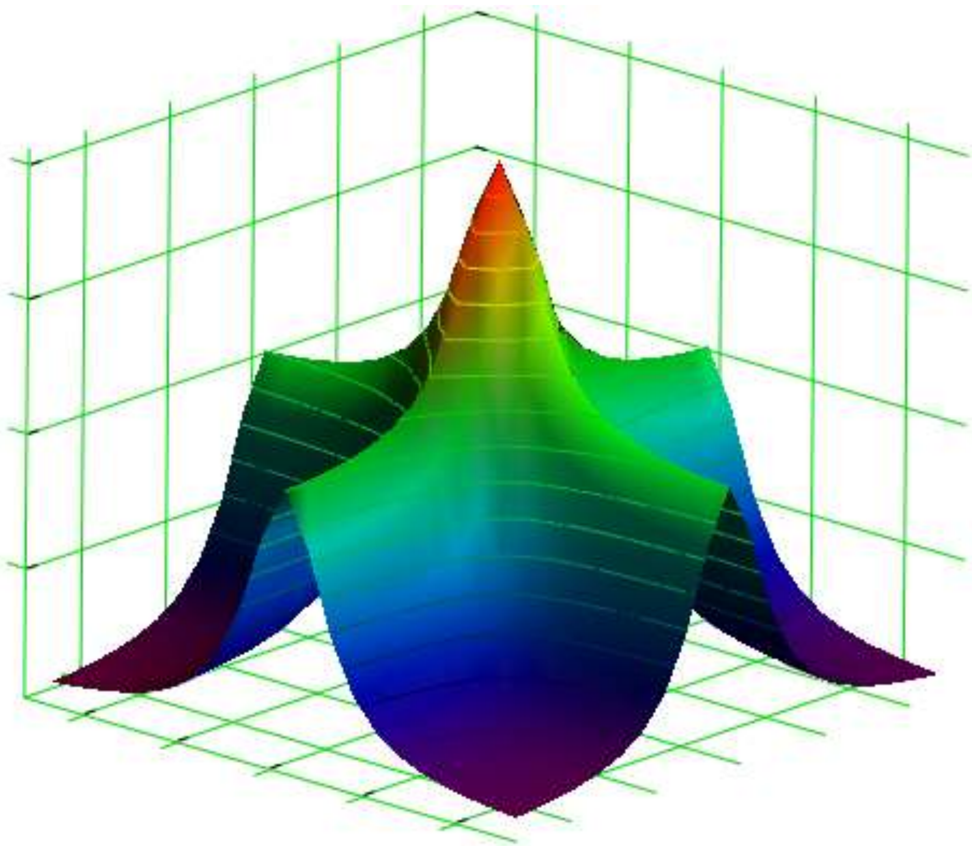
The variable z is called ***a function of independent variables*** x, y if for each set of values (x, y) of these variables from the given domain of their variations one or more values of are established according to some rule or law. The notation:

$$z = f(x, y)$$

The domain of the definition of a function can be represented as the whole plane domain *XOY* or as its part including the boundary or not.

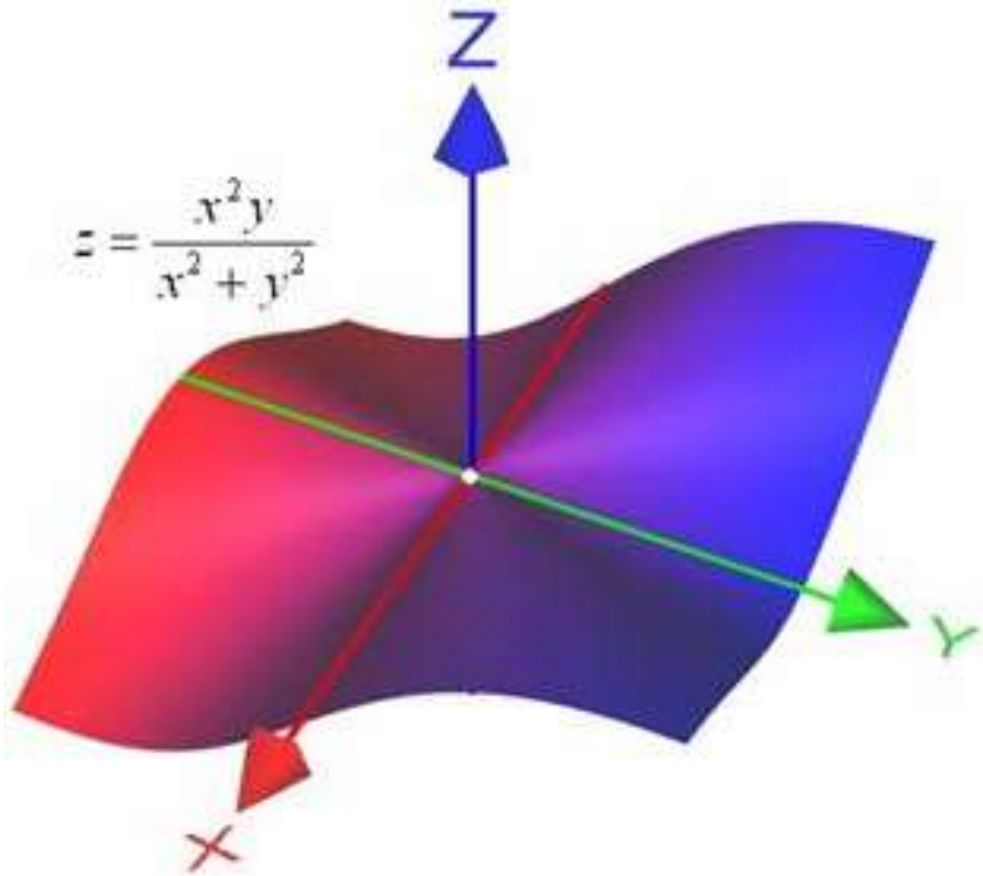
The domain of the definition of a function is defined according to the general mathematic requirements: the expression under the root in an even degree must be not negative, under the sign of logarithm it must be positive, the denominator of the fraction is not equal to zero etc.

Surface z=f(x,y)



$$f(x,y) = \frac{1}{1+x^2} + \frac{1}{1+y^2}$$

Surface   z=f(x,y)



Example 1. Find the domain of the definition of the function

$$z = \ln\left(4 + 4x - y^2\right)$$

Solution. The logarithm is defined for positive values of its argument only and, therefore,

$$4 + 4x - y^2 > 0$$

$$4 + 4x > y^2$$

There are no other restrictions on the arguments x and y.

In order to show the domain D geometrically, let us, at first, find its boundary

$$4 + 4x = y^2$$

$$y^2 = 4(1 + x)$$

$$y^2 = 4(1 + x)$$

The obtained equation defines a parabola. The parabola divides the whole plane into two parts, namely internal and external parts relatively to the parabola. For points belonging to one of its parts the inequality

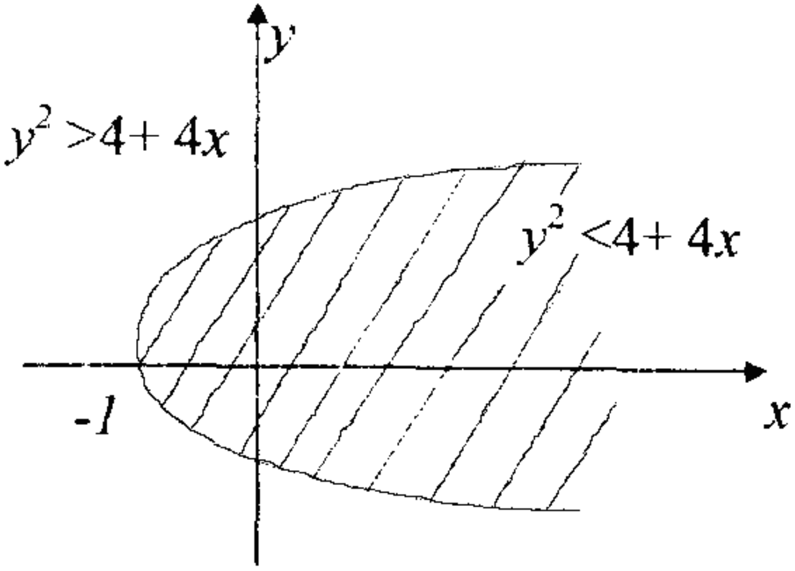
$$y^2 < 4 + 4x$$

is valid, for the other part

$$y^2 > 4 + 4x$$

on the parabola

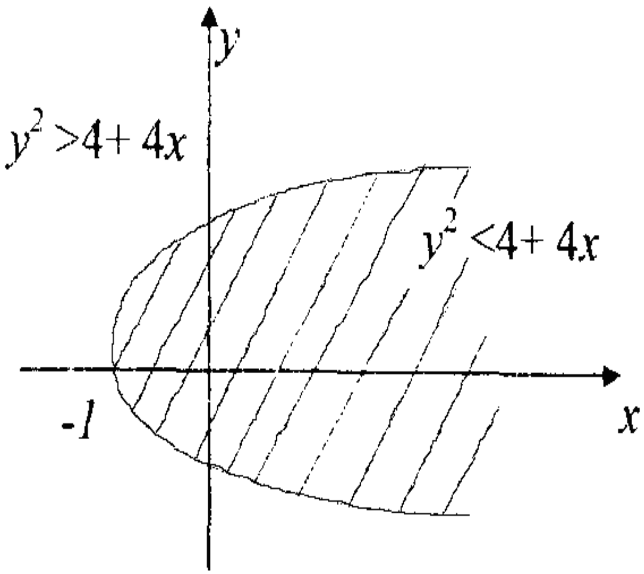
$$y^2 = 4 + 4x$$



In order to understand which of these two parts is the domain of the given function, i.e. satisfies the condition

$$y^2 < 4 + 4x$$

we should check this condition for any point not lying on the parabola. For example, the point of origin $O(0,0)$ lies inside the parabola and satisfies the necessary condition



$$0^2 < 4 + 4 \cdot 0$$

Therefore, the domain *D* consists of the points inside the parabola. As the parabola does not belong to the domain *D* we mark the boundary of the domain on the figure (the parabola) with a dotted line.

2. Partial Derivatives of Functions of Several Independent Variables

A partial derivative of the function $z = f(x, y)$ with respect to the variable ***x*** is defined corresponding to the formula

$$z_x{}' = \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

calculated at the constant ***y***.

A partial derivative of the function $z = f(x, y)$ with respect to the variable *y* calculated by the formula

$$z_y{}' = \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

calculated at the constant ***x***.

Example For the function $f(x, y) = 5x^2 - 3xy + 4y^2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$?

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Solution: Treating y as a constant,

we obtain $\frac{\partial z}{\partial x} = 10x - 3y$,

Example For the function $f(x,y) = 5x^2 - 3xy + 4y^2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$?

Solution: Treating y as a constant, we obtain $\frac{\partial z}{\partial x} = 10x - 3y$,

Treating x as a constant, we obtain $\frac{\partial z}{\partial y} = -3x + 8y$.

ECONOMIC PROBLEM

Example Suppose that the production function $Q(x, y) = 4000x^{0,5}y^{0,5}$ is known. Determine the marginal productivity of labor and the marginal productivity of capital when 25 units of labor and 169 units of capital are used.

ECONOMIC PROBLEM

Example Suppose that the production function $Q(x,y)=4000x^{0,5}y^{0,5}$ is known. Determine the marginal productivity of labor and the marginal productivity of capital when 25 units of labor and 169 units of capital are used.

Solution:

$$\frac{\partial Q}{\partial x}=4000\cdot 0,5\cdot x^{-0,5}y^{0,5}=\frac{2000y^{0,5}}{x^{0,5}}\quad,\quad \frac{\partial Q}{\partial y}=4000\cdot 0,5\cdot x^{0,5}y^{-0,5}=\frac{2000x^{0,5}}{y^{0,5}}.$$

Substituting x = 25, and y = 169 , we obtain:

$$\left.\frac{\partial Q}{\partial x}\right|_{(25,169)}=\frac{2000\cdot 169^{0,5}}{25^{0,5}}=5200\ \text{units},$$

$$\left.\frac{\partial Q}{\partial y}\right|_{(25,169)}=\frac{2000\cdot 25^{0,5}}{169^{0,5}}=769{,}23\ \text{units}.$$

ECONOMIC PROBLEM

Example Suppose that the production function $Q(x,y) = 4000x^{0,5}y^{0,5}$ is known. Determine the marginal productivity of labor and the marginal productivity of capital when 25 units of labor and 169 units of capital are used.

Solution:

$$\frac{\partial Q}{\partial x} = 4000 \cdot 0{,}5 \cdot x^{-0{,}5}y^{0{,}5} = \frac{2000y^{0{,}5}}{x^{0{,}5}} \quad , \quad \frac{\partial Q}{\partial y} = 4000 \cdot 0{,}5 \cdot x^{0{,}5}y^{-0{,}5} = \frac{2000x^{0{,}5}}{y^{0{,}5}}.$$

Substituting x = 25, and y = 169 , we obtain:

$$\left.\frac{\partial Q}{\partial x}\right|_{(25,169)} = \frac{2000 \cdot 169^{0{,}5}}{25^{0{,}5}} = 5200 \text{ units },$$

$$\left.\frac{\partial Q}{\partial y}\right|_{(25,169)} = \frac{2000 \cdot 25^{0{,}5}}{169^{0{,}5}} = 769{,}23 \text{ units.}$$

Thus we see that adding one unit of labor will increase production by about 5200 units and adding one unit of capital will increase production by about 769 units.

TASK. Find the partial derivatives of the function

$$z = x^2 + 3y^2 + x - y$$

TASK. Find the partial derivatives of the function

$$z = x^2 + 2xy - 4x + 8y$$

Example 2. Find the partial derivatives of the function

$$z = \ln\left(x^2 + y\right)$$

Solution. Let's find the partial derivatives:

$$z_x{}' = \frac{\partial z}{\partial x} = \left(\ln\left(x^2 + y\right)\right)_x' = \frac{1}{x^2 + y} \cdot \left(x^2 + y\right)_x' =$$

$$= \frac{1}{x^2 + y} \cdot (2x + 0) = \frac{2x}{x^2 + y}$$

$$(y = const)$$

$$z_y{}' = \frac{\partial z}{\partial y} = \left(\ln\left(x^2 + y\right)\right)_y{}' = \frac{1}{x^2 + y} \cdot \left(x^2 + y\right)_y{}' =$$

$$= \frac{1}{x^2 + y} \cdot \left(0 + 1\right) = \frac{1}{x^2 + y}$$

$$\left(x = const\right)$$

Partial derivatives of the second order are partial derivatives obtained from the derivatives of the first order.

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial z}{\partial x}\right)'_x=\frac{\partial^2 z}{\partial x^2}=z''_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial z}{\partial y}\right)'_y=\frac{\partial^2 z}{\partial y^2}=z''_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial z}{\partial x}\right)'_y=\frac{\partial^2 z}{\partial x\partial y}=z''_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial z}{\partial y}\right)'_x=\frac{\partial^2 z}{\partial y\partial x}=z''_{yx}$$

If mixed partial derivatives are $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ continuous then they are equal.

$$z''_{xy} = z''_{yx}$$

TASK. Find the partial derivatives of the function

$$z = x^2 + 2xy - 4x + 8y$$

If mixed partial variables are $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ continuous then they are equal.

Example 3. Find the partial derivatives of the second order of the function

$$z = \ln\left(x^2 + y\right)$$

Solution. Let's the second order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial z}{\partial x}\right)'_x = \frac{2\left(x^2 + y\right) - 2x \cdot 2x}{\left(x^2 + y\right)^2} = \frac{2y - 2x^2}{\left(x^2 + y\right)^2}$$

$$(y = const)$$

$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial z}{\partial y}\right)'_y = -\frac{1}{\left(x^2 + y\right)^2}$$

$$\left(x = const\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{\partial z}{\partial x}\right)'_y = 2x\left(-\frac{1}{\left(x^2 + y\right)^2}\right) = -\frac{2x}{\left(x^2 + y\right)^2}$$

$$\left(x = const\right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \left(\frac{\partial z}{\partial y} \right)'_x = -\frac{1}{\left(x^2 + y \right)^2} \cdot 2x$$

$$\left(y = const \right)$$

3. Extremum of Function of two independent variables

The *necessary condition in investigation of an extremum* of the function at the point $M_0(x_0, y_0)$ is an equality to zero of the first partial derivatives

$$z'_x\Big|_{(x_0;y_0)} = 0 \qquad z'_y\Big|_{(x_0;y_0)} = 0$$

Such points are called stationary. Not any stationary point is a point of extremum, thus let's formulate sufficient conditions.

4. Extremum of Function of two independent variables

The *necessary condition in investigation of an extremum* of the function at the point $M_0(x_0, y_0)$ is an equality to zero of the first partial derivatives

$$\frac{\partial f\left(x_0; y_0\right)}{\partial x} = 0 \qquad \frac{\partial f\left(x_0; y_0\right)}{\partial y} = 0$$

Such points are called stationary. Not any stationary point is a point of extremum, thus let's formulate sufficient conditions.

Let $M_0(x_0, y_0)$ be the stationary point. Let’s compose

$$\Delta = AC - B^2$$

where

$$A = \frac{\partial^2 f\left(x_0; y_0\right)}{\partial x^2} \qquad B = \frac{\partial^2 f\left(x_0; y_0\right)}{\partial x \partial y}$$

$$C = \frac{\partial^2 f\left(x_0; y_0\right)}{\partial y^2}$$

Let $M_0(x_0, y_0)$ be the stationary point. Let’s compose

$$\Delta = AC - B^2$$

where

$$A = z''_{xx}\Big|_{(x_0;y_0)} \qquad B = z''_{xy}\Big|_{(x_0;y_0)}$$

$$C = z''_{yy}\Big|_{(x_0;y_0)}$$

Then if

$$\Delta\begin{cases} >0, \textit{there is extremum} \begin{cases} \max, \ A<0, C<0 \\ \min, \ A>0, C>0 \end{cases} \\ <0, \textit{there is not extremum} \\ =0, \textit{to futher function investigation is required} \end{cases}$$

Example. Investigate the extremum of the function

$$z = x^2 + xy + y^2 - 3x - 6y.$$

Task. Investigate the extremum of the function

$$z = x^2 + xy + y^2 - 3x - 6y.$$

$$z_{min} = z(0;3) = -9$$

Task. Investigate the extremum of the function

$$z = x^3 - 7x^2 + xy - y^2 + 9x + 3y + 12$$

$$z_{max} = z(1;2) = 19.$$

Example. Investigate the extremum of the function

$$z = x^3 + y^3 - 6xy$$

Solution. Let’s the first order partial derivatives:

$$\frac{\partial z}{\partial x} = 3x^2 - 6y \qquad \frac{\partial z}{\partial y} = 3y^2 - 6x.$$

Let’s define the coordinates of stationary points from the necessary condition:

$$\begin{cases} \dfrac{\partial z}{\partial x} = 0, \\ \dfrac{\partial z}{\partial y} = 0; \end{cases} \qquad \begin{cases} 3x^2 - 6y = 0, \\ 3y^2 - 6x = 0 \end{cases} \qquad \begin{cases} x^2 - 2y = 0, \\ 2x - y^2 = 0. \end{cases}$$

From the first equation $y = \dfrac{x^2}{2}$ we find and substitute it into the second one:

$$2x - \left(\frac{x^2}{2}\right)^2 = 0$$

Then

$$8x - x^4 = 0 \qquad\qquad x\left(8 - x^3\right) = 0.$$

The equation solution is:

$$x_1 = 0 \qquad\qquad x_2 = \sqrt[3]{8} = 2$$

Then $y_1 = 0$, $y_2 = 2$. We have two critical points $M_1(0;0)$, $M_2(2;2)$. Let's find the second order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2} = 6x \qquad \frac{\partial^2 z}{\partial x \partial y} = -6 \qquad \frac{\partial^2 z}{\partial y^2} = 6y.$$

For the point $M_1(0;0)$

$$\mathrm{A} = 6 \cdot 0 = 0 \qquad \mathrm{B} = -6 \qquad C = 6 \cdot 0 = 0.$$

We calculate $\Delta = AC - B^2$

$$\Delta = 0 \cdot 0 - (-6)^2 = -36 < 0.$$

Thus there is not extremum at the point M_1

For the point $M_2(2;2)$

$$\mathrm{A} = 6 \cdot 2 = 12 \qquad \mathrm{B} = -6 \qquad C = 6 \cdot 2 = 12$$

We calculate $\Delta = AC - B^2$

$$\Delta = 12 \cdot 12 - (-6)^2 = 108 > 0$$

We have $\Delta > 0$ and $A > 0$, the function has the minimum at the point M_2

$$z_{min} = z(2;2) = 2^3 + 2^3 - 6 \cdot 2 \cdot 2 = -8$$

Economic problem 1

Example. The company produces products of two types A and B. The cost function is $C(x, y) = x^2 + 2xy + 2y^2$. The profit per unit output of product A is 500, the profit per unit output of product B is 600. Define an optimal plan of output if the objective is to find a maximum of a profit. Make an analysis of the obtained values in the problem.

Economic problem 1

Example. The company produces products of two types A and B. The cost function is $C(x,y)=x^2+2xy+2y^2$. The profit per unit output of product A is 500, the profit per unit output of product B is 600. Define an optimal plan of output if the objective is to find a maximum of a profit. Make an analysis of the obtained values in the problem.

Solution. Let x, y be quantities of product A and product B. Then the profit function is:

$$P(x,y)=p_1q_1+p_2q_2=500x+600y.$$

Economic problem 1

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Solution. Let x, y be quantities of product A and product B. Then the profit function is:

$$P(x,y)=p_1q_1+p_2q_2=500x+600y.$$

The profit for this company is the difference between the profit function and the cost function, i.e.

$$z(x,y)=P(x,y)-C(x,y)=500x+600y-x^2-2xy-2y^2.$$

Economic problem 1

The profit for this company is the difference between the profit function and the cost function, i.e.

$$z(x,y) = P(x,y) - C(x,y) = 500x + 600y - x^2 - 2xy - 2y^2.$$

Let's find the extremum of this function.

Step 1. The partial derivatives of the first order:

$$z'_x = 500 - 2x - 2y,\ z'_y = 600 - 2x - 4y.$$

Step 2. The extremum is found as: $\begin{cases} z'_x = 0 \\ z'_y = 0 \end{cases} \Rightarrow \begin{cases} 500 - 2x - 2y = 0 \\ 600 - 2x - 4y = 0 \end{cases} \Rightarrow \begin{cases} x_0 = 200 \\ y_0 = 50 \end{cases}.$

Economic problem 1

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Step 3. We calculate the value for finding of extremum: $\Delta = AC - B^2$.

$$A = z''_{xx}\big|_{M(200;50)} = -2\big|_{M(200;50)} = -2,\ B = z''_{xy}\big|_{M(200;50)} = -2\big|_{M(200;50)} = -2,$$

$$C = z''_{yy}\big|_{M(200;50)} = -4\big|_{M(200;50)} = -4.$$

Economic problem 1

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Economic problem 1

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$$C = z''_{yy}\big|_{M(200;50)} = -4\big|_{M(200;50)} = -4.$$

Then $\Delta = A \cdot C - B^2 = (-2)\cdot(-4) - (-2)^2 = 8 - 4 = 4 > 0$ and $A = -2 < 0$,

thus, the point $M(200;50)$ is maximum.

Economic problem 1

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Economic problem 1

Then $\Delta = A \cdot C - B^2 = (-2)\cdot(-4) - (-2)^2 = 8 - 4 = 4 > 0$ and $A = -2 < 0$, thus, the point $M(200;50)$ is maximum.

The maximum of the profit is $z_{max} = 500 \cdot 200 + 600 \cdot 50 - 200^2 - 2 \cdot 200 \cdot 50 - 2 \cdot 50^2 = 65000$ (units of money) at 200 units of product A and 50 units of product B.

Economic problem 2

Example A firm produces two products which are sold in two separate markets with the demand schedules $P_1 = 600 – 0{,}2Q_1$, $P_2 = 500 – 0{,}2Q_2$

Production costs are related and the firm faces the total cost function

$$TC = 16 + 1{,}2Q_1 + 1{,}5Q_2 + 0{,}2Q_1Q_2$$

If the firm wishes to maximize total profits, how much of each product should it sell? What will the maximum profit level be?

Economic problem 2

Example A firm produces two products which are sold in two separate markets with the demand schedules $P_1 = 600 – 0,2Q_1$, $P_2 = 500 – 0,2Q_2$

Production costs are related and the firm faces the total cost function

$$TC = 16 + 1,2Q_1 + 1,5Q_2 + 0,2Q_1Q_2$$

If the firm wishes to maximize total profits, how much of each product should it sell? What will the maximum profit level be?

Solution. The total revenue is

$$TR = TR_1 + TR_2 = P_1Q_1 + P_2Q_2 = (600 – 0,3Q_1)Q_1 + (500 – 0,2Q_2)Q_2 =$$

$$=600Q_1 – 0,3\,Q_1^2 + 500Q_2 – 0,2\,Q_2^2$$

Economic problem 2

Example A firm produces two products which are sold in two separate markets with the demand schedules $P_1 = 600 - 0{,}2Q_1$, $P_2 = 500 - 0{,}2Q_2$

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Solution. The total revenue is

$$TR = TR_1 + TR_2 = P_1Q_1 + P_2Q_2 = (600 - 0{,}3Q_1)Q_1 + (500 - 0{,}2Q_2)Q_2 =$$

$$=600Q_1 - 0{,}3\,Q_1^2 + 500Q_2 - 0{,}2\,Q_2^2$$

Therefore profit is

$$\pi = TR - TC = 600Q_1 - 0{,}3\,Q_1^2 + 500Q_2 - 0{,}2\,Q_2^2 - (16 + 1{,}2Q_1 + 1{,}5Q_2 + 0{,}2Q_1Q_2) =$$

$$=600Q_1 - 0{,}3\,Q_1^2 + 500Q_2 - 0{,}2\,Q_2^2 - 16 - 1{,}2Q_1 - 1{,}5Q_2 - 0{,}2Q_1Q_2 =$$

$$= -16 + 598{,}8Q_1 - 0{,}3\,Q_1^2 + 498{,}5Q_2 - 0{,}2\,Q_2^2 - 0{,}2Q_1Q_2$$

Economic problem 2

Therefore profit is

$$\pi = TR - TC = 600Q_1 - 0{,}3\,Q_1^2 + 500Q_2 - 0{,}2\,Q_2^2 - (16 + 1{,}2Q_1 + 1{,}5Q_2 + 0{,}2Q_1Q_2) =$$

$$=600Q_1 - 0{,}3\,Q_1^2 + 500Q_2 - 0{,}2\,Q_2^2 - 16 - 1{,}2Q_1 - 1{,}5Q_2 - 0{,}2Q_1Q_2 =$$

$$= -16 + 598{,}8Q_1 - 0{,}3\,Q_1^2 + 498{,}5Q_2 - 0{,}2\,Q_2^2 - 0{,}2Q_1Q_2$$

First-order conditions for maximization of this profit function are

$$\partial\pi / \partial Q_1 = 598{,}8 - 0{,}6Q_1 - 0{,}2Q_2 = 0 \qquad (18)$$

And

$$\partial\pi / \partial Q_2 = 498{,}5 - 0{,}4Q_2 - 0{,}2Q_1 = 0 \qquad (19)$$

Economic problem 2

First-order conditions for maximization of this profit function are

$$\partial\pi / \partial Q_1 = 598{,}8 - 0{,}6Q_1 - 0{,}2Q_2 = 0 \qquad (1)$$

And

$$\partial\pi / \partial Q_2 = 498{,}5 - 0{,}4Q_2 - 0{,}2Q_1 = 0 \qquad (2)$$

Simultaneous equations (1) and (2) can now be solved to find the optimal values of q_1 and q_2. Multiplying (2) by 3 and Rearranging (1)

$$1495{,}5 - 1{,}2Q_2 - 0{,}6Q_1 = 0$$

$$598{,}8 - 0{,}2Q_2 - 0{,}6Q_1 = 0.$$

Subtracting gives Q_2=896,7 . Substituting this value for Q_2 into (1)

And we get Q_1=699,1

Economic problem 2

Checking second-order conditions by differentiating (1) and (2') again:

$$\frac{\partial^2 \pi}{\partial Q_1^2} = -0{,}6 < 0\,, \quad \frac{\partial^2 \pi}{\partial Q_2^2} = -0{,}4 < 0$$

This satisfies one set of second-order conditions for a maximum. The cross partial derivative will be

$$\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = -0{,}2$$

Therefore

$$\frac{\partial^2 \pi}{\partial Q_1^2} \cdot \frac{\partial^2 \pi}{\partial Q_2^2} = (-0{,}6) \cdot (-0{,}4) = 0{,}24 > (-0{,}2)^2 = \left(\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} \right)^2$$

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and so the remaining second-order condition for a maximum is satisfied. The actual profit is found by substituting the optimum values Q_1 = 699,1 and Q_2 = 896,7. into the profit function. Thus

$$\pi = -16 + 598{,}8Q_1 - 0{,}3\,Q_1^2 + 498{,}5Q_2 - 0{,}2\,Q_2^2 - 0{,}2Q_1Q_2 = -16 + 598{,}8(699{,}1) - 0{,}3(699{,}1)\ 2 +$$
$$+498{,}5(896{,}7) - 0{,}2(896{,}7)\ 2 - 0{,}2(699{,}1)(896{,}7) = 432\ 797{,}02$$

Economic tasks 3 and 4 (pages 8 – 11)

Gradient of the function

The gradient of the function $z = f\left(x, y\right)$ at the point $M\left(x, y\right)$ is called vector from the point $M\left(x, y\right)$:

$$\overrightarrow{grad}z = \frac{\partial z}{\partial x}\bar{i} + \frac{\partial z}{\partial y}\bar{j}$$

The gradient shows the direction of the fastest function increasing at the point $M\left(x, y\right)$

Example 3. Find the gradient of the function $z = \sqrt{x^2 - y^2}$ at the point $M\left(5;3\right)$

Solution. We find the values of the partial derivatives at the point $M\left(5;3\right)$

$$z_x{}' = \frac{\partial z}{\partial x} = \left(\sqrt{x^2 - y^2}\right)_x' = \frac{1}{2\sqrt{x^2 - y^2}} \cdot \left(x^2 - y^2\right)_x' =$$

$$z_x{}' = \frac{\partial z}{\partial x} = \left(\sqrt{x^2 - y^2}\right)_x{}' = \frac{1}{2\sqrt{x^2 - y^2}} \cdot \left(x^2 - y^2\right)_x{}' =$$

$$= \frac{1}{2\sqrt{x^2 - y^2}} \cdot (2x - 0) = \frac{2x}{2\sqrt{x^2 - y^2}} = \frac{x}{\sqrt{x^2 - y^2}}$$

$$\left(\frac{\partial z}{\partial x}\right)_M = \frac{5}{\sqrt{25 - 9}} = \frac{5}{4}$$

$$z_y{}' = \frac{\partial z}{\partial y} = \left(\sqrt{x^2 - y^2}\right)_y' = \frac{1}{2\sqrt{x^2 - y^2}} \cdot \left(x^2 - y^2\right)_y' =$$

$$= \frac{1}{2\sqrt{x^2 - y^2}} \cdot \left(0 - 2y\right) = \frac{-2y}{2\sqrt{x^2 - y^2}} = -\frac{y}{\sqrt{x^2 - y^2}}$$

$$\left(\frac{\partial z}{\partial y}\right) = \frac{-3}{\sqrt{25 - 9}} = -\frac{3}{4}$$

Let’s substitute these values into the equation

$$\overline{gradz} = \frac{\partial z}{\partial x}\bar{i} + \frac{\partial z}{\partial y}\bar{j}$$

and obtain:

$$\overline{gradz} = \frac{5}{4}\bar{i} - \frac{3}{4}\bar{j}$$