PhD Misiura Ie.Iu. (доцент Місюра Є.Ю.)

Theme: Random variables and their economic interpretation. Basic laws of distribution

Part 2: continuous random variable

A definition of random variables and their classification

A variable is called ***random***, if it can receive real values with definite probabilities as a result of experiment.

A random variable / випадкова величина

A definition of random variables and their classification

A variable is called ***random***, if it can receive real values with definite probabilities as a result of experiment.

Definition

A random variable is a variable that takes values according to a certain probability distribution.

- Keys to know:
  - All the possible values it can take.
  - The probability distribution according to which it takes all possible values.

A definition of random variables and their classification

A variable is called ***random***, if it can receive real values with definite probabilities as a result of experiment.

In general, random variables can be **discrete** or **continuous**.

A discrete random variable /

дискретна випадкова величина

A continuous random variable /

неперервна випадкова величина

A definition of random variables and their classification

A random variable is a variable whose value is unknown, or a function that assigns values to each of an experiment's outcomes. Random variables are often designated by letters and can be classified as discrete, which are variables that have separated values, or continuous, which are variables that can have any values within a continuous range.

CONTINUOUS RANDOM VARIABLES

A ***continuous random variable*** is a random variable where the data can take **infinitely many values** on **some numerical interval** or a random variable which takes an infinite number of possible values. Continuous random variables are usually measurements.

A **continuous variable** is a variable whose value is obtained by measuring.

Examples: height of students in class

weight of students in class

time it takes to get to university

distance traveled between classes

Random Variables

According to the range of a random variable, we can define two types of random variables.

- Discrete random variables:
  - The range is finite or countably infinite.[1]
    - (1) The number of defective light bulbs in a box of ten (finite)
    - (2) The number of tails until the first heads comes up (countably infinite)
- Continuous random variables:
  - The range is uncountable (any value in an interval)
    - (1) Consider the experiment where a light bulb is tested until failure, and let X denote the time to failure.

$$\text{Range of } X = [0, \infty)$$

[1] A set of elements is countably infinite if the elements in the set can be put into one-to-one correspondence with the positive integers.

A continuous random variable is characterized by two functions:

1) **a distribution function** (the integral distribution function) $F(x)$;

2) **a density function** (the differential distribution function) $f(x)$.

a distribution function $F(x)$

/ функція розподілу

a density function / $f(x)$

/ функція щільності

The random variable X is called a ***continuous random variable***, if for any numbers $a < b$ such non-negative function $f(x)$ exists, that:

$$P(a < X < b) = \int_a^b f(x)dx$$

The random variable X is called an ***absolutely continuous random variable***, if there is a non-negative function $f(x)$ on ***R*** that:

$$P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

for every $x \in (-\infty, \infty)$

General properties of the density function $f(x)$

1. It is a non-negative function, i.e. $f(x) \geq 0$ for all *x*.

2. It is a non-decreasing function for $x \in (-\infty, \infty)$ then:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

(the condition of normalization of this function).

General properties of the density function $f(x)$

3. The relationship between the functions $f(x)$ and $F(x)$:

$$f(x) = F'(x) \qquad F(x) = \int_{-\infty}^{x} f(x)dx$$

4. The probability that a random variable X lies in the interval (x_1, x_2) is equal to the increment of its density distribution function on this interval; i.e.:

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x)dx = F(x_2) - F(x_1)$$

Example. The density function of a continuous variable ***X*** is given by

$$f(x) = \begin{cases} 0, & x \le 2 \\ c(x-2), & 2 < x \le 5 \\ 0, & x > 5 \end{cases}$$

What is the value of ***c***?

Example. The density function of a continuous variable ***X*** is given by

$$f(x) = \begin{cases} 0, & x \le 0 \\ c\left(4x - 2x^2\right), & 0 < x \le 2 \\ 0, & x > 2 \end{cases}$$

What is the value of ***c***?

Example The density function of a continuous variable X is given by

$$f(x)=\begin{cases}0, & x\le 0\\ c\left(4x-2x^2\right), & 0<x\le 2.\\ 0, & x>2\end{cases}$$

What is the value of c?

Solution. Since $f(x)$ is a probability density function, we must have the condition of normalization of this function that $\int_{-\infty}^{\infty} f(x)dx=1$, implying that:

$$\int_{-\infty}^{\infty} f(x)dx=\int_{-\infty}^{0} 0dx+\int_{0}^{2} c\left(4x-2x^2\right)dx+\int_{2}^{+\infty} 0dx=1$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} 0dx + \int_{0}^{2} c\left(4x - 2x^2\right)dx + \int_{2}^{+\infty} 0dx = 1$$

or

$$\int_{0}^{2} c\left(4x - 2x^2\right)dx = 1 \quad \text{or} \quad c\left(4\frac{x^2}{2} - 2\frac{x^3}{3}\right)\Bigg|_0^2 = c\left(2x^2 - \frac{2x^3}{3}\right)\Bigg|_0^2 = 1$$

or

$$c\left(8 - \frac{16}{3} - 0\right) = c \cdot \frac{8}{3} = 1 \quad \text{or} \quad c = \frac{3}{8}.$$

$$f(x)=\begin{cases}0, & x\le 0\\ c\left(4x-2x^2\right), & 0<x\le 2\\ 0, & x>2\end{cases}$$

So, the probability density function is:

$$f(x)=\begin{cases}0, & x\le 0\\ \frac{3}{8}\left(4x-2x^2\right), & 0<x\le 2.\\ 0, & x>2\end{cases}$$

Example. The integral distribution function of a continuous variable ***X*** is given by

$$F(x)=\begin{cases}0, & x\le 0{,}5\\ \dfrac{2x-1}{2}, & 0{,}5<x\le 1{,}5\\ 1, & x>1{,}5\end{cases}$$

Find the density function $f(x)$.

Solution. It is known that $f(x) = F'(x)$. Thus

$$F(x) = \begin{cases} 0, & x \le 0{,}5 \\ \dfrac{2x-1}{2}, & 0{,}5 < x \le 1{,}5 \\ 1, & x > 1{,}5 \end{cases}$$

$$f(x) = F'(X) = \begin{cases} (0)', & x \le 0{,}5 \\ \left(\dfrac{2x-1}{2}\right)', & 0{,}5 < x \le 1{,}5 \\ (1)', & x > 1{,}5 \end{cases} =$$

$$f(x)=F'(X)=\begin{cases}(0)', & x\le 0{,}5\\ \left(\dfrac{2x-1}{2}\right)', & 0{,}5<x\le 1{,}5\\ (1)', & x>1{,}5\end{cases}$$

$$f(x)=F'(X)=\begin{cases}0, & x\le 0{,}5\\ \dfrac{2-0}{2}, & 0{,}5<x\le 1{,}5\\ 0, & x>1{,}5\end{cases}$$

$$f(x)=F'(X)=\begin{cases}0, & x\le 0{,}5\\ \dfrac{2-0}{2}, & 0{,}5<x\le 1{,}5\\ 0, & x>1{,}5\end{cases}$$

$$f(x)=F'(X)=\begin{cases}0, & x\le 0{,}5\\ 1, & 0{,}5<x\le 1{,}5\\ 0, & x>1{,}5\end{cases}$$

Example. The differential distribution function of a continuous variable ***X*** is given by

$$f(x) = \begin{cases} 0, & x \le 1 \\ x - \dfrac{1}{2}, & 1 < x \le 2 \\ 0, & x > 2 \end{cases}$$

Find the integral distribution function $F(x)$.

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. It is known that $F(x)=\int_{-\infty}^{x} f(x)dx$. Thus:

If $x\le 1$, then

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. It is known that $F(x)=\int_{-\infty}^{x} f(x)dx$. Thus:

If $x\le 1$, then $f(x)=0$ and

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. It is known that $F(x)=\int\limits_{-\infty}^{x} f(x)dx$. Thus:

If $x\le 1$, then $f(x)=0$ and

$$F(x)=\int\limits_{-\infty}^{x} f(x)dx=\int\limits_{-\infty}^{x} 0dx=0$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution.

If $1<x\le 2$, then

$$F(x)=\int_{-\infty}^{x} f(x)dx=\int_{-\infty}^{1} f(x)dx+\int_{1}^{x} f(x)dx=$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution.

If $1<x\le 2$, then

$$F(x)=\int_{-\infty}^{x} f(x)dx=\int_{-\infty}^{1} f(x)dx+\int_{1}^{x} f(x)dx=$$

$$=\int_{-\infty}^{1} 0dx+\int_{1}^{x}\left(x-\frac{1}{2}\right)dx=\left(\frac{x^2}{2}-\frac{1}{2}x\right)\Bigg|_1^x=\frac{x^2}{2}-\frac{1}{2}x-\left(\frac{1}{2}-\frac{1}{2}\right)=\frac{x^2}{2}-\frac{x}{2}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution.

If $x>2$, then

$$F(x)=\int_{-\infty}^{x}f(x)dx=\int_{-\infty}^{1}f(x)dx+\int_{1}^{2}f(x)dx+\int_{2}^{x}f(x)dx=$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution.

If $x>2$, then

$$F(x)=\int_{-\infty}^{x} f(x)dx=\int_{-\infty}^{1} f(x)dx+\int_{1}^{2} f(x)dx+\int_{2}^{x} f(x)dx=$$

$$=\int_{-\infty}^{1} 0dx+\int_{1}^{2}\left(x-\frac{1}{2}\right)dx+\int_{2}^{x} 0dx=\left(\frac{x^2}{2}-\frac{1}{2}x\right)\Bigg|_1^2=\frac{2^2}{2}-\frac{1}{2}\cdot 2-\left(\frac{1^2}{2}-\frac{1}{2}\cdot 1\right)=$$

$$=2-1-0=1$$

Let’s write the formula for the integral distribution function $F(x)$:

$$F(x)=\begin{cases}0, & x\le 1\\ \dfrac{x^2}{2}-\dfrac{x}{2}, & 1<x\le 2\\ 1, & x>2\end{cases}$$

Numerical characteristics of continuous random variables

Numerical characteristics of absolutely continuous random variables

The ***mathematical expectation*** $M(X)$ of an absolutely continuous random variable is calculated by the formula:

$$M(X)=\int_{-\infty}^{\infty} x\cdot f(x)dx$$

Numerical characteristics of absolutely continuous random variables

The ***variance*** $D(X)$ of an absolutely continuous random variable is defined by the formula:

$$D(X) = \int_{-\infty}^{\infty} \left(x - M(X)\right)^2 \cdot f(x)dx$$

$$D(X) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx - \left[M(X)\right]^2$$

Numerical characteristics of absolutely continuous random variables

The ***root-mean-square deviation*** *(or* ***standard deviation****)* $\sigma(X)$ of an absolutely continuous random variable is the square root of its variance:

$$\sigma(X) = \sqrt{D(X)}$$

Numerical characteristics of absolutely continuous random variables

A ***mode*** of an absolutely continuous random variable M_o is a point of maximum of the probability density function $f(x)$.

Example 3. The density function of a continuous variable ***X*** is given by

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Calculate: a) the mathematical expectation and the variance; b) the probability that a random variable ***X*** lies in the interval $(1;3)$.

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let’s find the mathematical expectation:

$$M(X)=\int_{-\infty}^{\infty} x\cdot f(x)dx$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let’s find the mathematical expectation:

$$M(X)=\int_{-\infty}^{\infty} x\cdot f(x)dx$$

$$M(X)=\int_{-\infty}^{1} x\cdot f(x)dx+\int_{1}^{2} x\cdot f(x)dx+\int_{2}^{\infty} x\cdot f(x)dx$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let’s find the mathematical expectation:

$$M(X)=\int_{-\infty}^{1}x\cdot f(x)dx+\int_{1}^{2}x\cdot f(x)dx+\int_{2}^{\infty}x\cdot f(x)dx$$

$$M(X)=\int_{-\infty}^{1}x\cdot 0dx+\int_{1}^{2}x\cdot\left(x-\frac{1}{2}\right)dx+\int_{2}^{\infty}x\cdot 0dx$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let's find the mathematical expectation:

$$M(X)=\int_{-\infty}^{1}x\cdot 0dx+\int_{1}^{2}x\cdot\left(x-\frac{1}{2}\right)dx+\int_{2}^{\infty}x\cdot 0dx$$

$$M(X)=0+\int_{1}^{2}\left(x^2-\frac{1}{2}x\right)dx+0$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let's find the mathematical expectation:

$$M(X)=0+\int_1^2\left(x^2-\frac{1}{2}x\right)dx+0$$

$$M(X)=\left(\frac{x^3}{3}-\frac{1}{2}\frac{x^2}{2}\right)\Bigg|_1^2$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let's find the mathematical expectation:

$$M(X)=\left(\frac{x^3}{3}-\frac{1}{2}\frac{x^2}{2}\right)\Bigg|_1^2$$

$$M(X)=\left(\frac{x^3}{3}-\frac{x^2}{4}\right)\Bigg|_1^2$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let's find the mathematical expectation:

$$M(X)=\left(\frac{x^3}{3}-\frac{x^2}{4}\right)\Bigg|_1^2$$

$$M(X)=\left(\frac{2^3}{3}-\frac{2^2}{4}\right)-\left(\frac{1^3}{3}-\frac{1^2}{4}\right)$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let's find the mathematical expectation:

$$M(X)=\left(\frac{2^3}{3}-\frac{2^2}{4}\right)-\left(\frac{1^3}{3}-\frac{1^2}{4}\right)$$

$$M(X)=\left(\frac{8}{3}-\frac{4}{4}\right)-\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let's find the mathematical expectation:

$$M(X)=\left(\frac{8}{3}-\frac{4}{4}\right)-\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$M(X)=\frac{8}{3}-\frac{4}{4}-\frac{1}{3}+\frac{1}{4}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let's find the mathematical expectation:

$$M(X)=\frac{8}{3}-\frac{4}{4}-\frac{1}{3}+\frac{1}{4}$$

$$M(X)=\frac{7}{3}-\frac{3}{4}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let's find the mathematical expectation:

$$M\left(X\right)=\frac{7}{3}-\frac{3}{4}$$

$$M\left(X\right)=\frac{28-9}{12}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 1) Let's find the mathematical expectation:

$$M(X)=\frac{28-9}{12}$$

$$M(X)=\frac{19}{12}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let's find the variance:

$$D(X)=\int_{-\infty}^{\infty} x^2\cdot f(x)dx-\left[M(X)\right]^2$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let's find the variance:

$$D(X)=\int_{-\infty}^{\infty}x^2\cdot f(x)dx-\left[M(X)\right]^2$$

$$D(X)=\int_{-\infty}^{1}x^2\cdot f(x)dx+\int_{1}^{2}x^2\cdot f(x)dx+\int_{2}^{\infty}x^2\cdot f(x)dx-\left(\frac{19}{12}\right)^2$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let's find the variance:

$$D(X)=\int_{-\infty}^{1}x^2\cdot f(x)dx+\int_{1}^{2}x^2\cdot f(x)dx+\int_{2}^{\infty}x^2\cdot f(x)dx-\left(\frac{19}{12}\right)^2$$

$$D(X)=\int_{-\infty}^{1}x^2\cdot 0dx+\int_{1}^{2}x^2\cdot\left(x-\frac{1}{2}\right)dx+\int_{2}^{\infty}x^2\cdot 0dx-\left(\frac{19}{12}\right)^2$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let’s find the variance:

$$D(X)=\int_{-\infty}^{1}x^2\cdot 0dx+\int_{1}^{2}x^2\cdot\left(x-\frac{1}{2}\right)dx+\int_{2}^{\infty}x^2\cdot 0dx-\left(\frac{19}{12}\right)^2$$

$$D(X)=0+\int_{1}^{2}\left(x^3-\frac{1}{2}x^2\right)dx+0-\frac{361}{144}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let’s find the variance:

$$D(X)=0+\int_1^2\left(x^3-\frac{1}{2}x^2\right)dx+0-\frac{361}{144}$$

$$D(X)=\left.\left(\frac{x^4}{4}-\frac{1}{2}\frac{x^3}{3}\right)\right|_1^2-\frac{361}{144}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let’s find the variance:

$$D(X)=\left(\frac{x^4}{4}-\frac{1}{2}\frac{x^3}{3}\right)\Bigg|_1^2-\frac{361}{144}$$

$$D(X)=\left(\frac{x^4}{4}-\frac{x^3}{6}\right)\Bigg|_1^2-\frac{361}{144}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let's find the variance:

$$D(X)=\left(\frac{x^4}{4}-\frac{x^3}{6}\right)\Bigg|_1^2-\frac{361}{144}$$

$$D(X)=\left(\frac{2^4}{4}-\frac{2^3}{6}\right)-\left(\frac{1^4}{4}-\frac{1^3}{6}\right)-\frac{361}{144}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let's find the variance:

$$D(X)=\left(\frac{2^4}{4}-\frac{2^3}{6}\right)-\left(\frac{1^4}{4}-\frac{1^3}{6}\right)-\frac{361}{144}$$

$$D(X)=\left(\frac{16}{4}-\frac{8}{6}\right)-\left(\frac{1}{4}-\frac{1}{6}\right)-\frac{361}{144}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let's find the variance:

$$D(X)=\left(\frac{16}{4}-\frac{8}{6}\right)-\left(\frac{1}{4}-\frac{1}{6}\right)-\frac{361}{144}$$

$$D(X)=\frac{16}{4}-\frac{8}{6}-\frac{1}{4}+\frac{1}{6}-\frac{361}{144}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let's find the variance:

$$D(X)=\frac{16}{4}-\frac{8}{6}-\frac{1}{4}+\frac{1}{6}-\frac{361}{144}$$

$$D(X)=\frac{15}{4}-\frac{7}{6}-\frac{361}{144}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let's find the variance:

$$D(X)=\frac{15}{4}-\frac{7}{6}-\frac{361}{144}$$

$$D(X)=\frac{45-14}{12}-\frac{361}{144}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let’s find the variance:

$$D(X)=\frac{45-14}{12}-\frac{361}{144}$$

$$D(X)=\frac{31}{12}-\frac{361}{144}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\dfrac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let's find the variance:

$$D(X)=\frac{31}{12}-\frac{361}{144}$$

$$D(X)=\frac{372-361}{144}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 2) Let's find the variance:

$$D(X)=\frac{372-361}{144}$$

$$D(X)=\frac{11}{144}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 3) Let's find the probability that a random variable ***X*** lies in the interval $(1;3)$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 3) Let’s find the probability that a random variable ***X*** lies in the interval $(1;3)$

$$P(a<X<b)=\int_a^b f(x)dx$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 3) Let's find the probability that a random variable ***X*** lies in the interval $(1;3)$

$$P(1<X<3)=\int_1^3 f(x)dx$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 3) Let's find the probability that a random variable ***X*** lies in the interval $(1;3)$

$$P(1<X<3)=\int_1^3 f(x)dx$$

$$P(1<X<3)=\int_1^2 f(x)dx+\int_2^3 f(x)dx$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 3) Let's find the probability that a random variable ***X*** lies in the interval $(1;3)$

$$P(1<X<3)=\int_1^2 f(x)dx+\int_2^3 f(x)dx$$

$$P(1<X<3)=\int_1^2\left(x-\frac{1}{2}\right)dx+\int_2^3 0dx$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 3) Let's find the probability that a random variable ***X*** lies in the interval $(1;3)$

$$P(1<X<3)=\int_1^2\left(x-\frac{1}{2}\right)dx+\int_2^3 0dx$$

$$P(1<X<3)=\left(\frac{x^2}{2}-\frac{1}{2}x\right)\Bigg|_1^2+0$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 3) Let's find the probability that a random variable ***X*** lies in the interval $(1;3)$

$$P(1<X<3)=\left(\frac{x^2}{2}-\frac{1}{2}x\right)\Bigg|_1^2+0$$

$$P(1<X<3)=\left(\frac{2^2}{2}-\frac{1}{2}\cdot 2\right)-\left(\frac{1^2}{2}-\frac{1}{2}\cdot 1\right)$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 3) Let's find the probability that a random variable ***X*** lies in the interval $(1;3)$

$$P(1<X<3)=\left(\frac{2^2}{2}-\frac{1}{2}\cdot 2\right)-\left(\frac{1^2}{2}-\frac{1}{2}\cdot 1\right)$$

$$P(1<X<3)=\frac{4}{2}-\frac{2}{2}-\frac{1}{2}+\frac{1}{2}$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 3) Let's find the probability that a random variable ***X*** lies in the interval $(1;3)$

$$P(1<X<3)=\frac{4}{2}-\frac{2}{2}-\frac{1}{2}+\frac{1}{2}$$

$$P(1<X<3)=2-1+0$$

$$f(x)=\begin{cases}0, & x\le 1\\ x-\frac{1}{2}, & 1<x\le 2\\ 0, & x>2\end{cases}$$

Solution. 3) Let’s find the probability that a random variable ***X*** lies in the interval $(1;3)$

$$P(1<X<3)=2-1+0$$

$$P(1<X<3)=1$$

Basic distribution laws of continuous random distributions and their numerical characteristics

UNIFORM, EXPONENTIAL AND NORMAL LAWS OF PROBABILITIES DISTRIBUTION

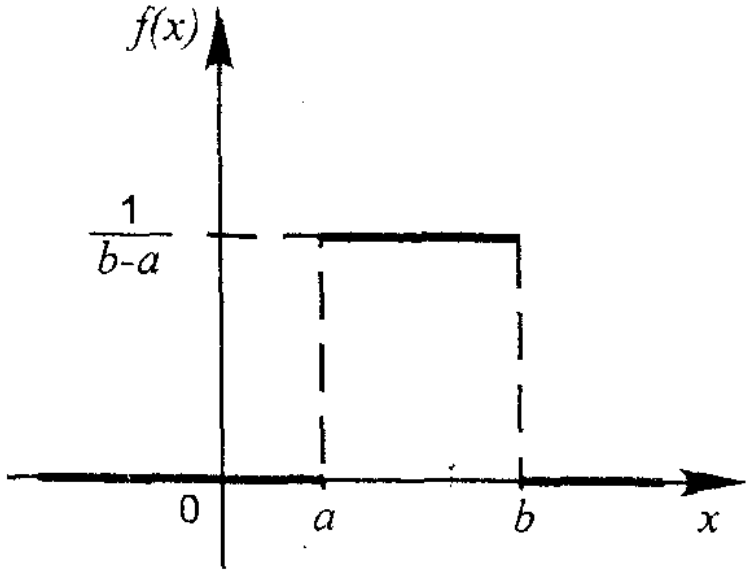
A UNIFORM LAW OF PROBABILITIES DISTRIBUTION

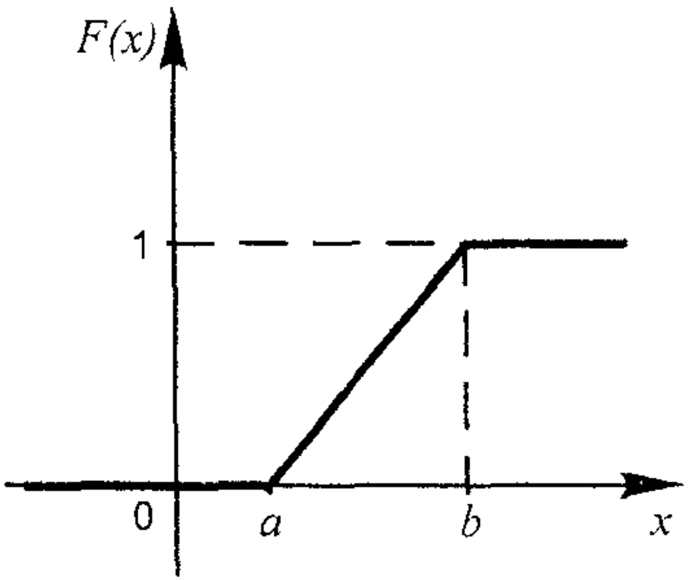
The ***uniform law*** of distribution is characterized by a **probability density function** $f(x)$ (the differential distribution function) (fig. 1) and a **cumulative distribution function** $F(x)$ (the integral distribution function) (fig. 2).

A UNIFORM LAW OF DISTRIBUTION

$$f(x) = \begin{cases} 0, & x \le a \\ \dfrac{1}{b-a}, & a < x \le b \\ 0, & x > b \end{cases}$$

$$F(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x \le b \\ 1, & x > b \end{cases}$$





A UNIFORM LAW OF DISTRIBUTION

$$\left(a,b\right)$$

$$f(x)=\begin{cases}0, & x\le a\\ \dfrac{1}{b-a}, & a<x\le b\\ 0, & x>b\end{cases}$$

$$F(x)=\begin{cases}0, & x\le a\\ \dfrac{x-a}{b-a}, & a<x\le b\\ 1, & x>b\end{cases}$$

The **probability** that a random variable ***X*** lies in the interval $\left(\alpha,\beta\right)$ is equal to the increment of its integral distribution function on this interval; i.e.:

$$P\left(\alpha<X<\beta\right)=F\left(\beta\right)-F\left(\alpha\right)$$

A UNIFORM LAW OF DISTRIBUTION

$$(a, b)$$

$$f(x) = \begin{cases} 0, & x \le a \\ \dfrac{1}{b-a}, & a < x \le b \\ 0, & x > b \end{cases}$$

$$F(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x \le b \\ 1, & x > b \end{cases}$$

Numerical characteristics are

$$M(X) = \frac{a+b}{2}$$

$$D(X) = \frac{(b-a)^2}{12}$$

$$\sigma(X) = \frac{b-a}{2\sqrt{3}}$$

TASK

The parameters a, b of the uniform law of distribution are given: $a=1$ and $b=5$. Find: a) functions $f(x)$ and $F(x)$;

b) the mathematical expectation $M(X)$, the variance $D(X)$ and the root-mean-square deviation $\sigma(X)$; c) $P(3 < X < 10)$.

HOMEWORK

TASK. The parameters a, b the uniform law of distribution are given: $a = 2$ and $b = 6$. Find: a) functions $f(x)$ and $F(x)$;

b) the mathematical expectation $M(X)$, the variance $D(X)$ and the root-mean-square deviation $\sigma(X)$; c) $P(0 < X < 4)$.

A EXPONENTIAL LAW OF DISTRIBUTION

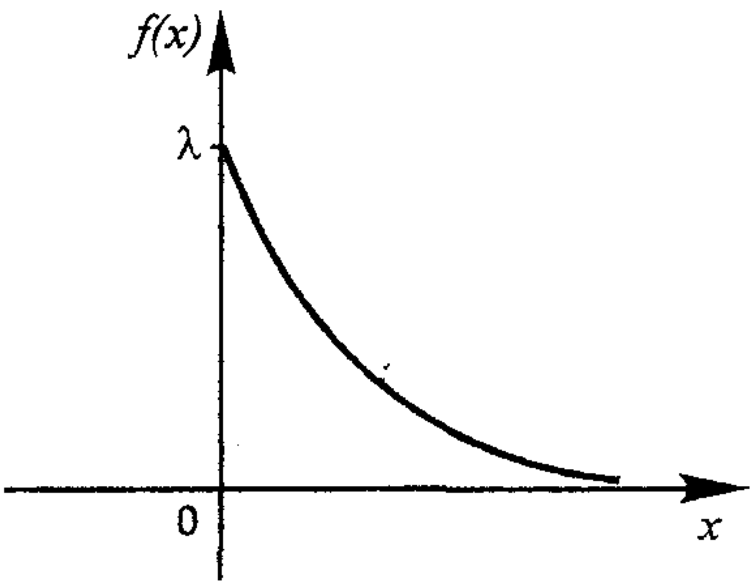
The ***exponential law*** of distribution is characterized by a **probability density function** $f(x)$ (the differential distribution function) (fig. 3) and a **cumulative distribution function** $F(x)$ (the integral distribution function) (fig. 4).

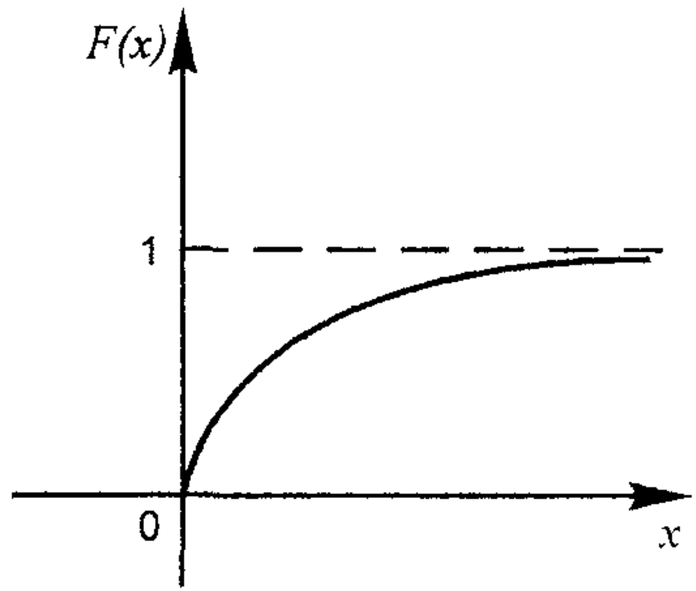
AN EXPONENTIAL LAW OF DISTRIBUTION

$$f(x) = \begin{cases} 0, & x < 0 \\ \lambda \cdot e^{-\lambda x}, & x \geq 0 \end{cases}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

λ





AN EXPONENTIAL LAW OF DISTRIBUTION

$$f(x)=\begin{cases}0, & x<0\\ \lambda\cdot e^{-\lambda x}, & x\geq 0\end{cases}$$

λ

$$F(x)=\begin{cases}0, & x<0\\ 1-e^{-\lambda x}, & x\geq 0\end{cases}$$

The **probability** that a random variable ***X*** lies in the interval (α,β) is equal to the increment of its integral distribution function on this interval; i.e.:

$$P(\alpha<X<\beta)=F(\beta)-F(\alpha)$$

AN EXPONENTIAL LAW OF DISTRIBUTION

$$f(x) = \begin{cases} 0, & x < 0 \\ \lambda \cdot e^{-\lambda x}, & x \ge 0 \end{cases}$$

λ

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \ge 0 \end{cases}$$

Numerical characteristics are

$$M(X) = \frac{1}{\lambda}$$

$$D(X) = \frac{1}{\lambda^2}$$

$$\sigma(X) = \frac{1}{\lambda}$$

TASK

TASK. The parameter λ of the exponential law of distribution is given: $\lambda = 0.01$. Find: a) functions $f(x)$ and $F(x)$;
b) the mathematical expectation $M(X)$, the variance $D(X)$ the root-mean-square deviation $\sigma(X)$; c) $P(3 < X < 5)$.

HOMEWORK

TASK 2. The parameter λ of the exponential law of distribution is given: $\lambda = 0.05$.
Find: a) functions $f(x)$ and $F(x)$;
b) the mathematical expectation $M(X)$, the variance $D(X)$ the root-mean-square deviation $\sigma(X)$; c) $P(2 < X < 10)$.

A NORMAL LAW OF DISTRIBUTION

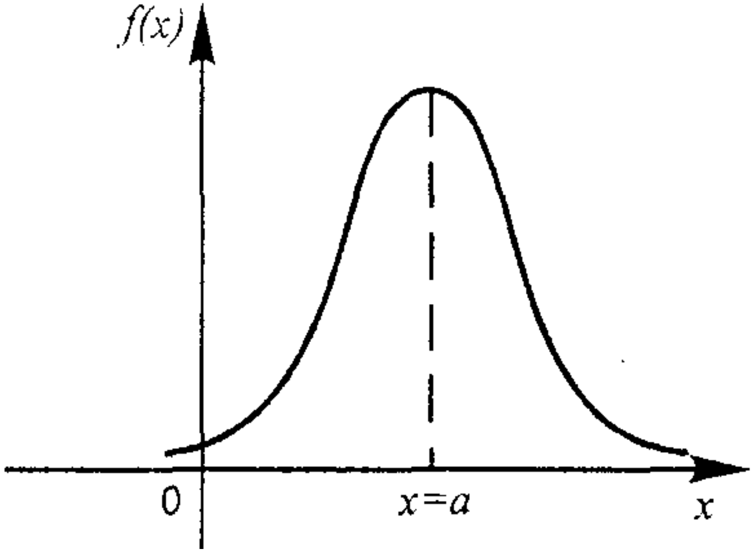
The ***normal law*** of distribution is characterized by a **probability density function** $f(x)$ (the differential distribution function) (fig. 5) and a **cumulative distribution function** $F(x)$ (the integral distribution function) (fig. 6).

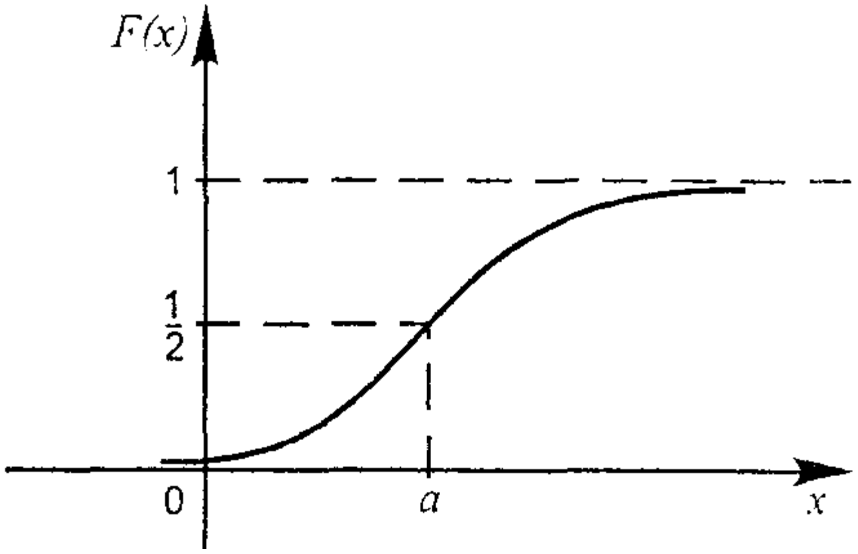
A NORMAL LAW OF DISTRIBUTION

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2\sigma^2}}$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2\sigma^2}} dx$$

$$\left(a, \sigma^2\right)$$





A NORMAL LAW OF DISTRIBUTION

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2\sigma^2}}$$

$$\left(a, \sigma^2\right)$$

$$f(x) = \frac{\varphi(t)}{\sigma}$$

Laplace differential function

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}}$$

$$t = \frac{x-a}{\sigma}$$

A NORMAL LAW OF DISTRIBUTION

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2\sigma^2}} dx$$

$$\left(a, \sigma^2\right)$$

$$F(x) = \frac{1}{2} + \Phi(t)$$

Laplace integral function

$$t = \frac{x-a}{\sigma}$$

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_0^t e^{-\frac{t^2}{2}} dt$$

A NORMAL LAW OF DISTRIBUTION (a, σ^2)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2\sigma^2}}$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2\sigma^2}} dx$$

The **probability** that a random variable ***X*** lies in the interval (α, β) is equal to the increment of its integral distribution function on this interval; i.e.:

$$P(\alpha < X < \beta) = F(\beta) - F(\alpha) = \Phi\left(\frac{\beta - a}{\sigma}\right) - \Phi\left(\frac{\alpha - a}{\sigma}\right)$$

A NORMAL LAW OF DISTRIBUTION $\left(a, \sigma^2\right)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2\sigma^2}}$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2\sigma^2}} dx$$

***Numerical characteristics* are**

$$M(X) = a$$

$$D(X) = \sigma^2$$

$$\sigma(X) = \sigma$$

TASK

The parameters $\left(a,\sigma^2\right)$ of the normal law of distribution are given: $a=8$ and $\sigma=4$. Find: a) functions $F\left(x\right)$ and $f\left(x\right)$ b) the mathematical expectation $M(X)$, the variance $D(X)$ and the root-mean-square deviation $\sigma(X)$; c) $P(5<X<10)$.

HOMEWORK

TASK 3. The parameters $\left(a,\sigma^2\right)$ of the normal law of distribution are given: $a=3$ and $\sigma=2$. Find: a) functions $F\left(x\right)$ and $f\left(x\right)$ b) the mathematical expectation $M(X)$, the variance $D(X)$ and the root-mean-square deviation $\sigma(X)$; c) $P(1<X<7)$.

Probability that a module of the deviation of the normal distributed random variable from its mathematical expectation

Let’s find ***probability that a module of the deviation of the normal distributed random variable from its mathematical expectation*** is less than any nonnegative ε , i.e.

$$P\left(\left|X-a\right|<\varepsilon\right)$$

Let's find ***probability that a module of the deviation of the normal distributed random variable from its mathematical expectation*** is less than any nonnegative ε , i.e.

$$P(|X-a|<\varepsilon)=P(-\varepsilon<X-a<\varepsilon)=P(a-\varepsilon<X<\varepsilon+a)=$$

Let's find ***probability that a module of the deviation of the normal distributed random variable from its mathematical expectation*** is less than any nonnegative ε , i.e.

$$P(|X-a|<\varepsilon)=P(-\varepsilon<X-a<\varepsilon)=P(a-\varepsilon<X<\varepsilon+a)=$$

$$=\Phi\left(\frac{a+\varepsilon-a}{\sigma}\right)-\Phi\left(\frac{a-\varepsilon-a}{\sigma}\right)=\Phi\left(\frac{\varepsilon}{\sigma}\right)-\Phi\left(-\frac{\varepsilon}{\sigma}\right)=2\Phi\left(\frac{\varepsilon}{\sigma}\right)$$

Let's find ***probability that a module of the deviation of the normal distributed random variable from its mathematical expectation*** is less than any nonnegative ε , i.e.

$$P\left(\left|X-a\right|<\varepsilon\right)=2\Phi\left(\frac{\varepsilon}{\sigma}\right)$$

HOMEWORK

$$a = 40; \sigma = 0{,}4; P = 0{,}8$$

$$\varepsilon = ?$$

Three sigma rule

Let’s transform the formula

$$P(|X-a|<\varepsilon)=2\Phi\left(\frac{\varepsilon}{\sigma}\right)$$

Let $\varepsilon=\sigma\cdot t$, then

$$P(|X-a|<\sigma t)=2\Phi(t)$$

Three sigma rule

Let's transform the formula

$$P(|X-a|<\varepsilon)=2\Phi\left(\frac{\varepsilon}{\sigma}\right)$$

Let $\varepsilon=\sigma\cdot t$, then

$$P(|X-a|<\sigma t)=2\Phi(t)$$

$t=1$

$$P(|X-a|<\sigma)=2\Phi(1)=0.6826$$

$\varepsilon=\sigma$

Three sigma rule

It means that 68 % of values of a random variable ***X*** is located on the interval $(a \pm \sigma)$.

$t = 1$

$\varepsilon = \sigma$

$$P(|X - a| < \sigma) = 2\Phi(1) = 0.6826$$

Three sigma rule

Let's transform the formula

$$P\left(\left|X - a\right| < \varepsilon\right) = 2\Phi\left(\frac{\varepsilon}{\sigma}\right)$$

Let $\varepsilon = \sigma \cdot t$, then

$$P\left(\left|X - a\right| < \sigma t\right) = 2\Phi\left(t\right)$$

$t = 2$

$$P\left(\left|X - a\right| < 2\sigma\right) = 2\Phi\left(2\right) = 0.9544$$

$\varepsilon = 2\sigma$

Three sigma rule

It means that 95 % of values of a random variable ***X*** is located on the interval $(a \pm 2\sigma)$.

$t = 2$

$\varepsilon = 2\sigma$

$$P(|X - a| < 2\sigma) = 2\Phi(2) = 0.9544$$

Three sigma rule

Let's transform the formula

$$P(|X-a|<\varepsilon)=2\Phi\left(\frac{\varepsilon}{\sigma}\right)$$

Let $\varepsilon=\sigma\cdot t$, then

$$P(|X-a|<\sigma t)=2\Phi(t)$$

$t=3$

$$P(|X-a|<3\sigma)=2\Phi(3)=0.9973$$

$\varepsilon=3\sigma$

Three sigma rule

Hence *three sigma rule* means the normal distributed random variable ***X*** possesses all its values on the interval $(a \pm 3\sigma)$ with the probability 100 %.

$t = 3$

$$P(|X - a| < 3\sigma) = 2\Phi(3) = 0.9973$$

$\varepsilon = 3\sigma$

EXAMPLE

X is a random variable, distributed by a normal law with the mathematical expectation $a = 2$

and the root-mean-square deviation $\sigma = 0{,}1$

Find limits for values of ***X*** with probability $99{,}73\,\%$

SOLUTION

According to the condition of the task:

$P = 0{,}9973$ $\sigma = 0{,}1$ $a = 2$

Let's substitute:

$3\sigma = 0{,}3$

$a - 3\sigma < X < a + 3\sigma$

SOLUTION

According to the condition of the task:

$P = 0{,}9973$ $\sigma = 0{,}1$ $a = 2$

Let’s substitute:

$3\sigma = 0{,}3$

$$2 - 0{,}3 < X < 2 + 0{,}3$$

SOLUTION

According to the condition of the task:

$P = 0{,}9973$ $\sigma = 0{,}1$ $a = 2$

Let’s substitute:

$3\sigma = 0{,}3$

$1{,}7 < X < 2{,}3$

Let's consider a *deviation* from a *normal law*

Asymmetry

Asymmetry of a distribution shows a deviation of a value from its central position on the left or on the right:

$$A_S = \frac{\mu_3}{\sigma^3}$$

where

μ_3 – the central moment of the 3-rd order;

σ – the root-mean square deviation.

Asymmetry for normal distribution

Asymmetry of a distribution shows a deviation of a value from its central position on the left or on the right:

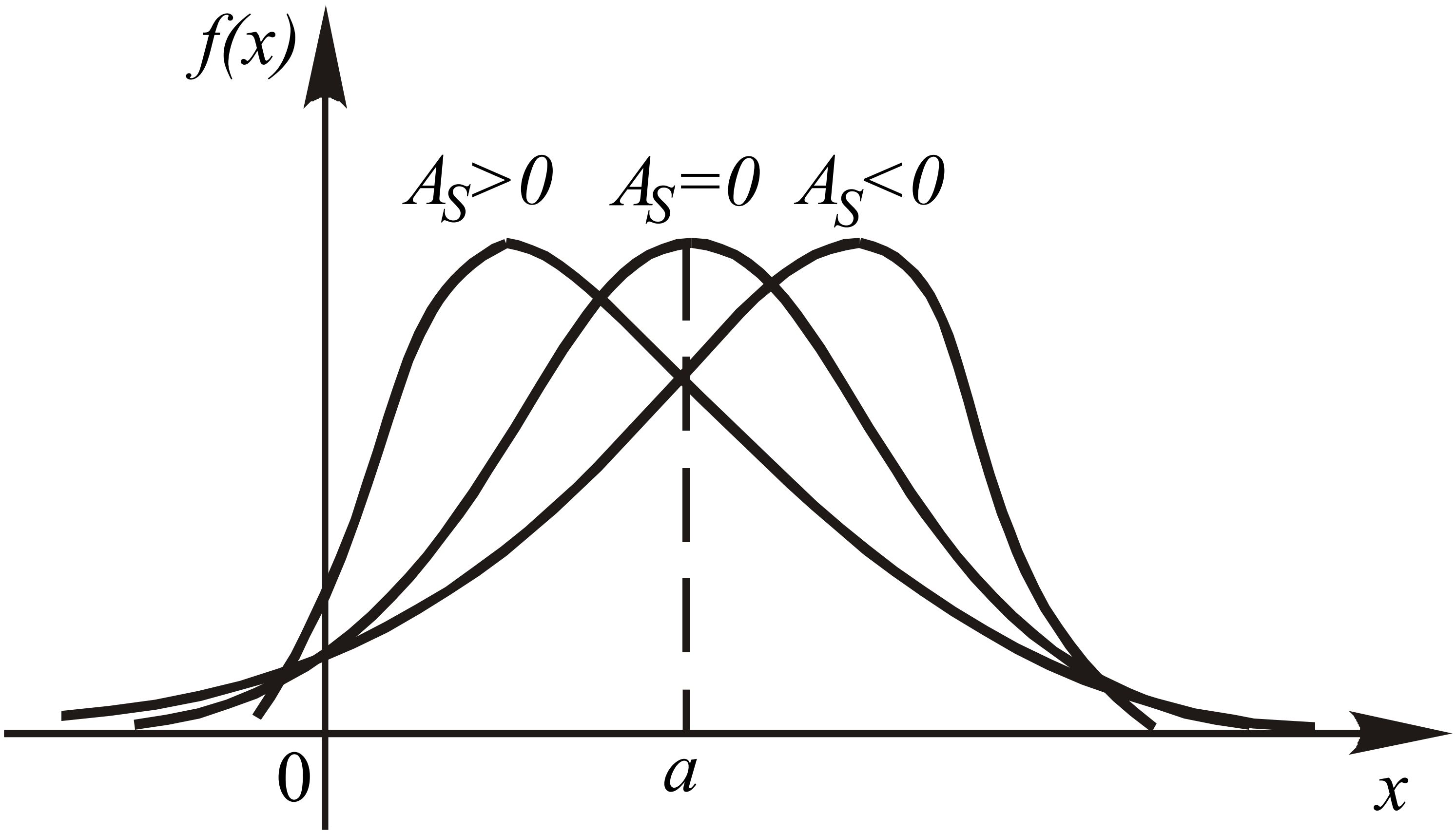
$$A_S = \frac{\mu_3}{\sigma^3} = 0$$

where

μ_3 – the central moment of the 3-rd order;

σ – the root-mean square deviation.

Asymmetry



Excess

Excess characterizes a deviation of a value from its central position down or up:

$$E_S = \frac{\mu_4}{\sigma^4} - 3$$

where

μ_4 – the central moment of the 4-th order;

σ – the root-mean square deviation.

Excess for normal distribution

Excess of a distribution is a value:

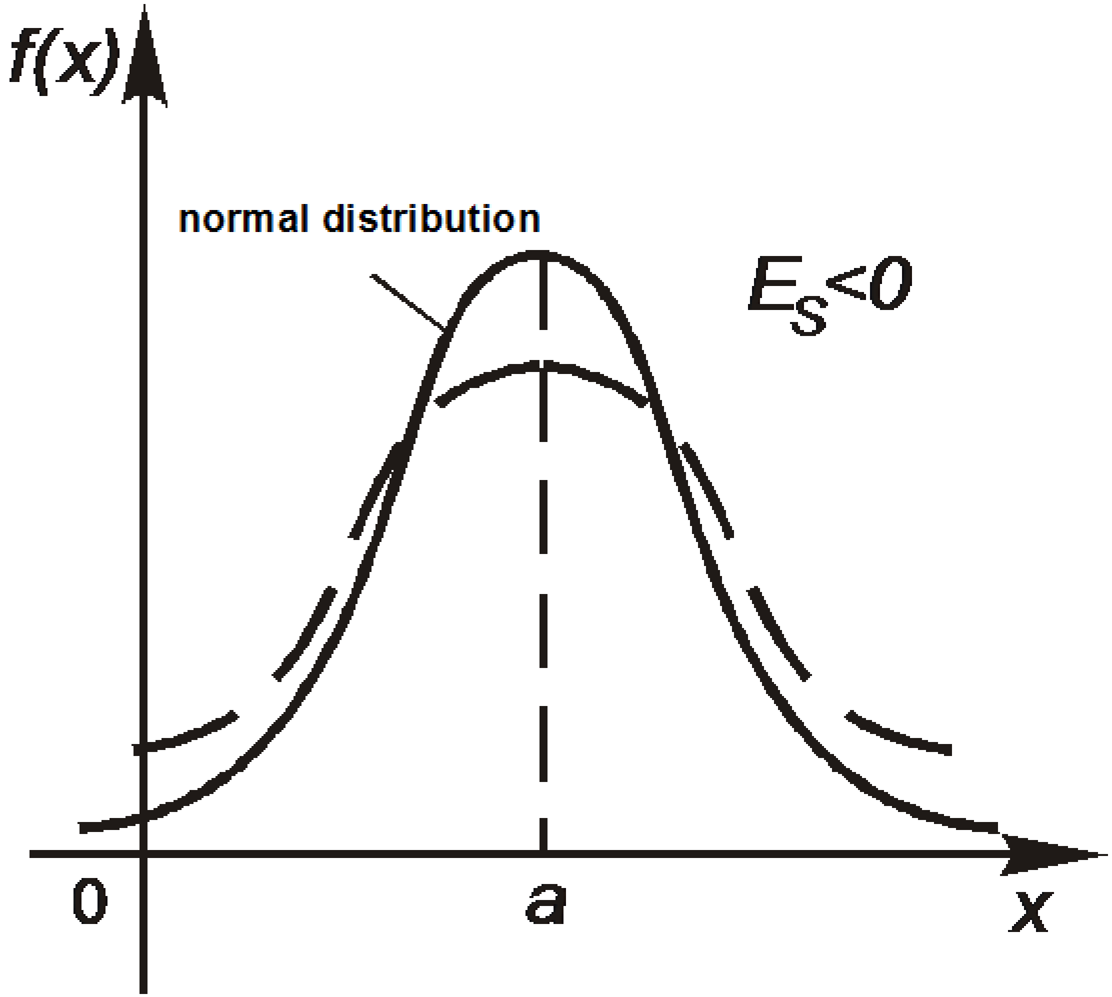
$$E_S = \frac{\mu_4}{\sigma^4} - 3 = 0$$

where

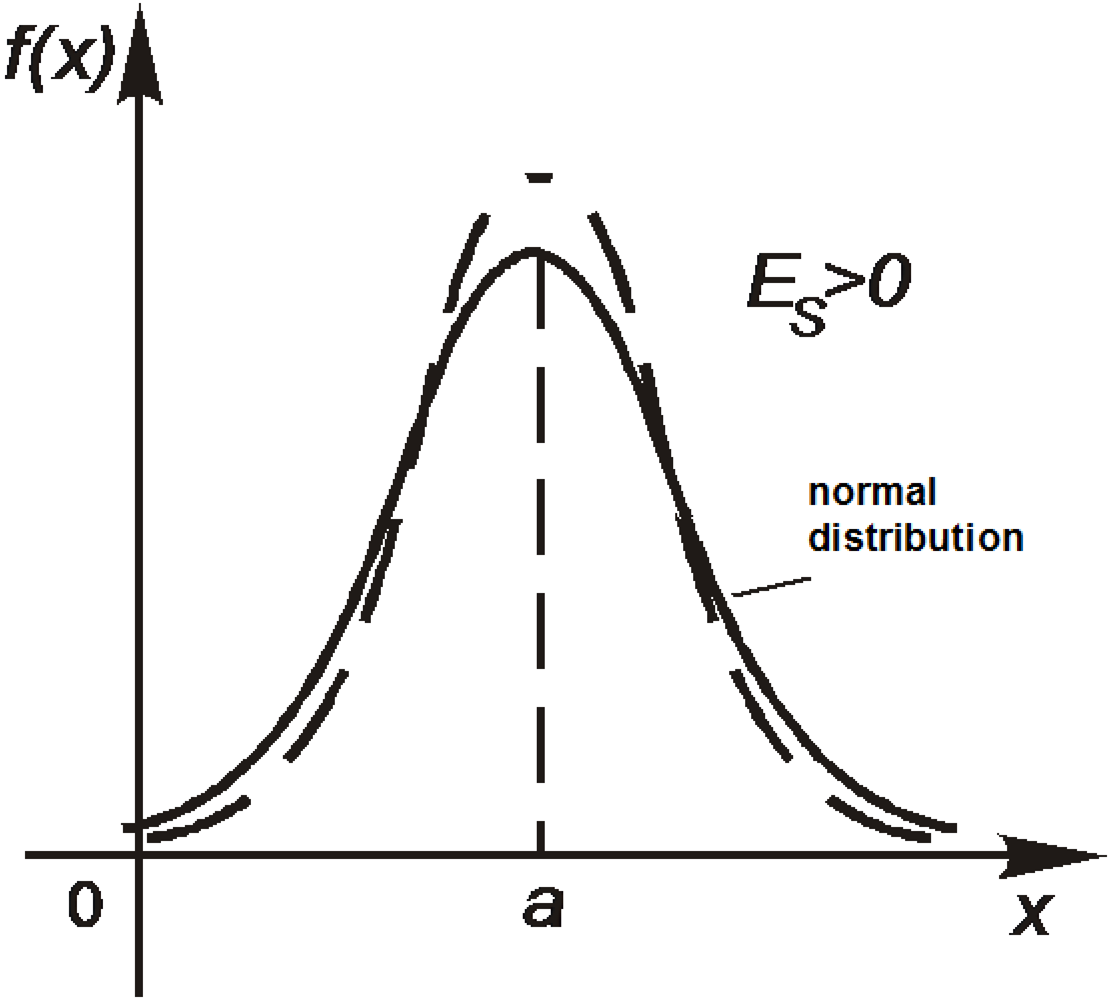
μ_4 – the central moment of the 4-th order;

σ – the root-mean square deviation.

Excess



Excess



EXAMPLE

We have:

$\mu_3 = 0{,}4$ $\mu_4 = 6{,}17$ $\sigma^2 = 1{,}66$

Find A_S, E_S

EXAMPLE

We have: $\mu_3 = 0{,}4$ $\mu_4 = 6{,}17$ $\sigma^2 = 1{,}66$

Find: A_S, E_S

Solution.

Asymmetry is:

$$A_S = \frac{0{,}4}{1{,}66\sqrt{1{,}66}} \approx 0{,}19$$

Excess:

$$E_S = \frac{6{,}17}{1{,}66^2} - 3 = -0{,}76$$

EXAMPLE

Asymmetry :

$$A_S = \frac{0,4}{1,66\sqrt{1,66}} \approx 0,19$$

Excess :

$$E_S = \frac{6,17}{1,66^2} - 3 = -0,76$$

We have:

$$A_S > 0 \qquad E_S < 0$$

INDEPENDENT WORK

TRANSFORMATION OF SEQUENCES OF NORMAL DISTRIBUTED RANDOM VARIABLE

Gamma-distribution

A ***random variable x*** that is ***gamma-distributed*** with shape k and scale θ is denoted by $\Gamma(k, \theta)$

.

Gamma-distribution

The probability density function and the cumulative distribution function of the gamma distribution can be expressed in terms of the gamma function parameterized in terms of a shape parameter k and scale parameter θ and the lower incomplete gamma function, i.e.

$$f(x)=\begin{cases}\dfrac{1}{\theta^k\cdot\Gamma(k)}x^{k-1}e^{-\frac{x}{\theta}}, & \text{if } x\geq 0\\ 0, & \text{if } x<0\end{cases}$$

Gamma-distribution

$$f(x)=\begin{cases}\dfrac{1}{\theta^k\cdot\Gamma(k)}x^{k-1}e^{-\frac{x}{\theta}}, & \text{if } x\geq 0\\ 0, & \text{if } x<0\end{cases}$$

$$F(x)=\int_0^x f(x)dx$$

where $\Gamma(k)=\int_0^{+\infty} t^{k-1}e^{-t}dt$ is the gamma function,

both k and θ are positive values.

Gamma-distribution

Numerical characteristics of gamma distribution law are

1) *mathematical expectation:* $M(X) = k\theta$

2) *variance:* $D(X) = k\theta^2$

3) *root-mean-square deviation (or standard deviation)*: $\sigma(X) = \theta\sqrt{k}$

Chi-square (χ^2) distribution

A random variable $X = \chi^2(n)$ has the ***chi-square distribution*** with ***n*** degrees of freedom if its probability density function and the cumulative distribution function have the forms:

Chi-square (χ^2) distribution

A random variable $X=\chi^2(n)$ has the ***chi-square distribution*** with ***n*** degrees of freedom if its probability density function and the cumulative distribution function have the forms:

$$f(x)=\begin{cases}\dfrac{1}{2^{\frac{n}{2}}\cdot\Gamma\left(\dfrac{\alpha}{2}\right)}x^{\frac{n}{2}-1}e^{-\frac{x}{2}}, \text{if } x>0\\ 0, \qquad\qquad\qquad\qquad\qquad , \text{if } x\le 0\end{cases}$$

$$F(x)=\frac{1}{2^{\frac{n}{2}}\cdot\Gamma\left(\dfrac{\alpha}{2}\right)}\int_0^x t^{\frac{n}{2}-1}e^{-\frac{t}{2}}dt$$

Chi-square (χ^2) distribution

$$f(x)=\begin{cases}\dfrac{1}{2^{\frac{n}{2}}\cdot\Gamma\left(\dfrac{\alpha}{2}\right)}x^{\frac{n}{2}-1}e^{-\frac{x}{2}}, \text{ if } x>0\\ 0, \qquad\qquad\qquad\qquad\quad , \text{ if } x\le 0\end{cases} \qquad (*)$$

$$F(x)=\frac{1}{2^{\frac{n}{2}}\cdot\Gamma\left(\dfrac{\alpha}{2}\right)}\int_0^x t^{\frac{n}{2}-1}e^{-\frac{t}{2}}dt$$

where $\Gamma\left(\dfrac{\alpha}{2}\right)$ is the gamma function.

Chi-square (χ^2) distribution

Numerical characteristics of gamma distribution law are:

1) *mathematical expectation:*

$$M(X) = M\left(\chi^2(n)\right) = n$$

2) *variance:* $D(X) = D\left(\chi^2(n)\right) = 2n$

3) *root-mean-square deviation*:

$$\sigma(X) = \sigma\left(\chi^2(n)\right) = \sqrt{2n}$$

Chi-square (χ^2) distribution

Main property of chi-square distribution. For an arbitrary ***n*** the sum:

$$X = \sum_{k=1}^{n} X_k^2$$

of squares of independent random variables obeying the standard normal distribution has the chi-square distribution with ***n*** degrees of freedom.

Chi-square (χ^2) distribution

The values are tabulated.

Relationship with other distributions:

1. For ***n=1,*** this formula (*) gives the probability density function of the square X^2 of a random variable with the standard normal distribution.

Chi-square (χ^2) distribution

2. For $n = 2$, the formula (*) gives the exponential distribution with parameter $\lambda = \frac{1}{2}$

3. As $n \to \infty$ the random variable $X = \chi^2(n)$ has an asymptotically normal distribution with parameters $(n, 2n)$.

HOMEWORK

To read about

1) ***Student's distribution** (t**-distribution**) **with n degrees of freedom***
2) ***F-distribution*** (or ***Fisher–Snedecor distribution***)

(see Lecture)

Thank You for your time and attention! Hope this presentation was educational and helpful to you ☺