Theme:

Basic concepts of game theory. Application of game theory in international trade

Game theory

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A two-person game has two players ***A*** and ***B***. Player ***A*** is called the first player (or the seller), player ***B*** is called the second player (or the customer).

Two-person game

Player ***A*** is called the first player (or the seller), player ***B*** is called the second player (or the customer).

Each player has **strategies**. What is a **strategy**?

Two-person game

Each player has strategies. What is a **strategy**?

A **strategy** is a set of rules determining the player’s behavior in each situation in the game.

Example

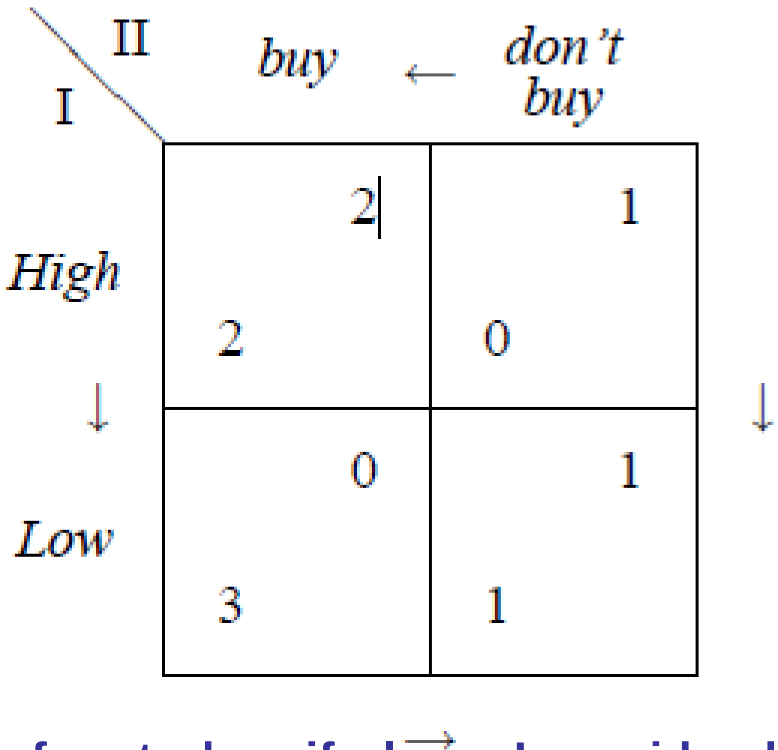
Let’s suppose player I is an internet service provider and player II is a potential customer. They consider entering into a contract of service provision for a period of time. The provider can, for himself, decide between two levels of quality of service, *High* or *Low*. High-quality service is more costly to provide, and some of the cost is independent of whether the contract is signed or not. The level of service cannot be put verifiably into the contract. High-quality service is more valuable than low-quality service to the customer, in fact so much so that the customer would prefer not to buy the service if she knew that the quality was low. Her choices are to *buy* or *not to buy* the service.

Example

I \ II	*buy*	*don't buy*
High	2 2	1 0
Low	0 3	1 1

Figure. High-low quality game between a service provider (player I) and a customer (player II)

Example



The customer prefers to buy if player I provides high-quality service, and not to buy otherwise. Regardless of whether the customer chooses to buy or not, the provider always prefers to provide the low-quality service.

Two-person game

We suppose that player ***A*** has strategies $(A_1, A_2, ..., A_m)$ and player ***B*** has strategies $(B_1, B_2, ..., B_n)$. Each strategy in the game is called the **pure strategy**.

Two-person game

Player ***A*** has strategies $\left(A_1, A_2, \ldots, A_m\right)$

Player ***B*** has strategies $\left(B_1, B_2, \ldots, B_n\right)$

Then player ***A*** can choose any of ***m*** strategies with the definite probability and player ***B*** can choose any of ***n*** strategies with the definite probability.

Two-person game

Player ***A*** has strategies $\left(A_1, A_2, ..., A_m\right)$

Player ***B*** has strategies $\left(B_1, B_2, ..., B_n\right)$

A combination of these strategies $\left(A_i, B_j\right)$

gives some **numerical result**.

Two-person game

A combination of these strategies $\left(A_i, B_j\right)$ gives some **numerical result**.

These strategies determine the **payoff matrix**:

Payoff matrix

Strategies determine the **payoff matrix**:

$$
\begin{array}{cc}
 & \begin{array}{cccc} B_1 & B_2 & \dots & B_n \end{array} \\
\Pi = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix} & \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array}
\end{array}
$$

Payoff matrix

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & \dots & B_n \end{array} \\ \Pi = & \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix} \end{array} \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array}$$

where c_{ij} is called a **payoff** (positive, negative or zero) of **player *A*** or a **loss** (positive, negative or zero) of **player *B***.

This matrix has the **size**

Payoff matrix

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\end{array}
$$

where c_{ij} is called a **payoff** (positive, negative or zero) of **player *A*** or a **loss** (positive, negative or zero) of **player *B***.

This matrix has the **size** $m \times n$

Payoff matrix

$$\Pi = \begin{array}{cc} \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} & \\ \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix} & \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \end{array}$$

The rows of the payoff matrix ***C*** correspond to the strategies of player ***A***, the columns correspond to the strategies of player ***B***.

Payoff matrix

$$
\begin{array}{cc}
 & \begin{array}{cccc} B_1 & B_2 & \dots & B_n \end{array} \\
\Pi = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \dots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{pmatrix} & \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array}
\end{array}
$$

In this matrix an element c_{ij} is the payoff of player ***A*** if player ***A*** uses the pure strategy A_i and player ***B*** uses the pure strategy B_j or c_{ij} is the loss of player ***B*** if player ***B*** uses the pure strategy B_j and player ***A*** uses the pure strategy A_i.

Payoff matrix

$$
\begin{array}{cc}
 & \begin{array}{cccc} B_1 & B_2 & \dots & B_n \end{array} \\
\Pi = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \dots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{pmatrix} & \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array}
\end{array}
$$

If the payoff of player ***A*** and the loss of player ***B*** are equal, then such a game is called zero-sum game (i.e. the difference between the payoff and the loss equals 0).

Payoff matrix

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Let's consider a **two-person zero-sum game**.

Payoff matrix

Let's consider a **two-person zero-sum game**.

The number $\alpha_i = \min_j c_{ij}$ is called the guaranteed minimal payoff of player ***A***.

The number $\beta_j = \max_i c_{ij}$ is called the guaranteed maximal loss of player ***B***.

The following table corresponds to the payoff matrix of a two-person game:

B_j / A_i		**The strategies of player *B* (the customer)**				**The minimal possible payoff of player *A*** (α_i)
		B_1	B_2	$\cdots$	B_n	
The strategies of player *A* (the seller)	A_1	c_{11}	c_{12}	$\cdots$	c_{1n}	α_1
	A_2	c_{21}	c_{22}	$\cdots$	c_{2n}	α_2
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	A_m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	α_m
The maximal possible loss of player *B* (β_j)		β_1	β_2	$\cdots$	β_n	β / α

The main objective of game theory is a choice of the optimal strategies of the players.

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		B_1	B_2	…	B_n	
The strategies of player *A* (the seller)	A_1	c_{11}	c_{12}	…	c_{1n}	α_1
	A_2	c_{21}	c_{22}	…	c_{2n}	α_2
	…	…	…	…	…	…
	A_m	c_{m1}	c_{m2}	…	c_{mn}	α_m
The maximal possible loss of player *B* (β_j)		β_1	β_2	…	β_n	β / α

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The players choose the strategies according to the following **principles**:

$$\alpha = \max_i(\alpha_i) = \max_i \min_j c_{ij}$$

is called **the lower price** of the game;

$$\beta = \min_j(\beta_j) = \min_j \max_i c_{ij}$$

is called **the upper price** of the game.

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Player ***A*** chooses the optimal strategy based on the maximization of minimal payoffs (the maximin principle).

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The players choose the strategies according to the following **principles**:

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is called **the upper price** of the game.

Player ***B*** chooses the optimal strategy based on the minimization of maximal losses (the minimax principle).

GAME WITH A SADDLE POINT

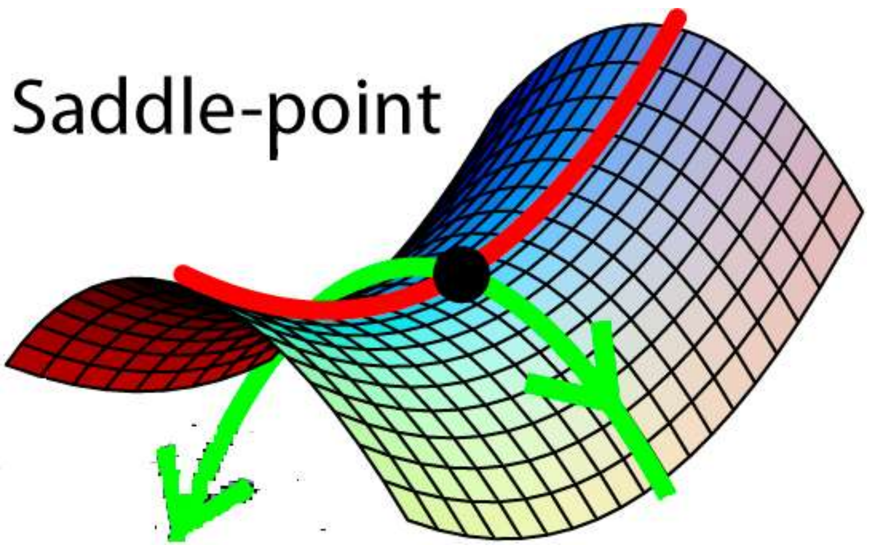
Rule:

In a two-person zero-sum game α and β satisfy the inequality:

$$\alpha \leq \beta$$

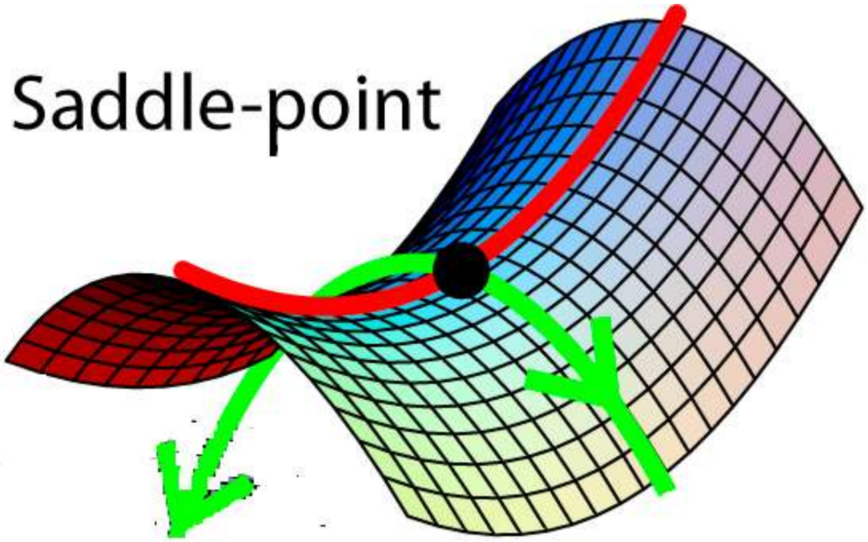
GAME WITH A SADDLE POINT

If $\alpha = \beta$ then such a game is called the game with a saddle point and of the optimal strategies is called a saddle point of the payoff matrix.



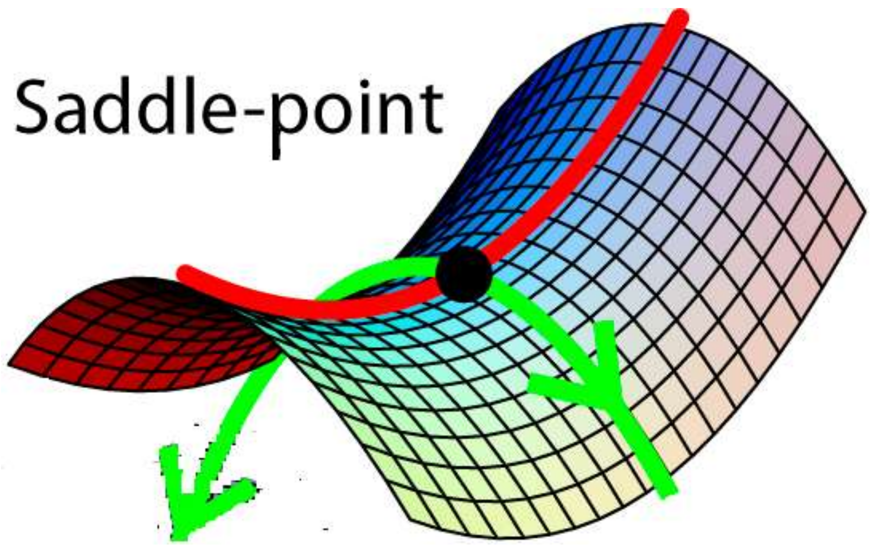
GAME WITH A SADDLE POINT

If $\alpha = \beta$ then such a game is called the game with a saddle point and $\left(A_{i\,(opt)}, B_{j\,(opt)}\right)$ of the optimal strategies is called a saddle point of the payoff matrix.



GAME WITH A SADDLE POINT $\alpha = \beta$

Then the game price ν equals α and β, i.e. $\nu = \alpha = \beta$. Such a game is called the game in the **pure strategies**.



Example 1. Determine the lower and the upper prices of the game and existence of a saddle point for given payoff matrix:

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ C = & \begin{pmatrix} 4 & 5 & 3 \\ 6 & 7 & 4 \\ 5 & 2 & 3 \end{pmatrix} \end{array} \quad \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array}$$

Solution. According to the problem statement we have:

B_j / A_i		The strategies of player ***B*** (the customer) B_1	B_2	B_3	**minimum payoff** (α_i)
The strategies of player *A* (the seller)	A_1	**4**	**5**	**3**	
	A_2	**6**	**7**	**4**	
	A_3	**5**	**2**	**3**	
maximum loss (β_j)					

Solution. According to the problem statement we have:

B_j \ A_i		The strategies of player ***B*** (the customer)			minimum payoff (α_i)
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	A_3	**5**	**2**	**3**	**2**
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The strategies of player *A* (the seller)	A_1	**4**	**5**	**3**	3
	A_2	**6**	**7**	**4**	4
	A_3	**5**	**2**	**3**	2
maximum loss (β_j)		6			

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		B_1	B_2	B_3	
The strategies of player *A* (the seller)	A_1	**4**	**5**	**3**	**3**
	A_2	**6**	**7**	**4**	**4**
	A_3	**5**	**2**	**3**	**2**
maximum loss (β_j)		**6**	**7**		

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	A_2	**6**	**7**	**4**	**4**
	A_3	**5**	**2**	**3**	**2**
maximum loss (β_j)		**6**	**7**	**4**	

$$\alpha = \max_i(\alpha_i) = \max_i \min_j c_{ij}$$

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	A_2	**6**	**7**	**4**	**4**
	A_3	**5**	**2**	**3**	**2**
maximum loss (β_j)		**6**	**7**	**4**	

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	A_3	**5**	**2**	**3**	**2**
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maximum loss (β_j)		**6**	**7**	**4**	**4**

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	A_2	**6**	**7**	**4**	**4**
	A_3	**5**	**2**	**3**	**2**
maximum loss (β_j)		**6**	**7**	**4**	**4** \ **4**

The **lower price** of the game is

$$\alpha = \max_i (\alpha_i) = \max_i \min_j c_{ij} = \max_i (3, 4, 2) = 4$$

α is called the maximin or the lower price.

$$\Pi = \begin{pmatrix} 4 & 5 & 3 \\ 6 & 7 & 4 \\ 5 & 2 & 3 \end{pmatrix}$$

The **upper price** of the game is

$$\beta = \min_j\left(\beta_j\right) = \min_j \max_i c_{ij} = \min_j\left(6,7,4\right) = 4$$

β is called the minimax or the upper price.

$$\Pi = \begin{pmatrix} 4 & 5 & 3 \\ 6 & 7 & 4 \\ 5 & 2 & 3 \end{pmatrix}$$

$$\alpha = \max_i(\alpha_i) = \max_i \min_j c_{ij} = \max_i(3, 4, 2) = 4$$

$$\beta = \min_j(\beta_j) = \min_j \max_i c_{ij} = \min_j(6, 7, 4) = 4$$

Therefore, $\alpha = \beta = 4$, then the game has the saddle point, the game price

$$v = \alpha = \beta = 4$$

$$\alpha = \max_i(\alpha_i) = \max_i \min_j c_{ij} = \max_i(3, 4, 2) = 4$$

$$\beta = \min_j(\beta_j) = \min_j \max_i c_{ij} = \min_j(6, 7, 4) = 4$$

The optimal solution is given by using the pure strategies A_2 and B_3 , i.e.

$$\alpha = \max_i(\alpha_i) = \max_i \min_j c_{ij} = \max_i(3, 4, 2) = 4$$

$$\beta = \min_j(\beta_j) = \min_j \max_i c_{ij} = \min_j(6, 7, 4) = 4$$

The optimal solution is given by using the pure strategies A_2 and B_3, i.e.

A_2 is the most profitable strategy of player ***A***;

B_3 is the most profitable strategy of player ***B***.

Probabilities of strategies for player ***A*** are denoted as x_1, x_2, x_3 $\left(A_1, A_2, A_3\right)$ and for player ***B*** are denoted as y_1, y_2, y_3 $\left(B_1, B_2, B_3\right)$

A_2 is the most profitable strategy of player ***A***;

B_3 is the most profitable strategy of player ***B***.

In this case we have:

$$X_{opt} = (0,1,0) \qquad Y_{opt} = (0,0,1)$$

Answer: $X_{opt} = (0,1,0)$ $Y_{opt} = (0,0,1)$

i.e. we take only the strategy A_2 with probability 1

and the strategy B_3 with probability 1

and the game price is $\nu = 4$

TASK 1.

Solve the payoff matrix and find the game price:

$$\Pi = \begin{pmatrix} 6 & 3 & 6 & 5 \\ 5 & 2 & 4 & 9 \\ 7 & 1 & 3 & 7 \end{pmatrix}$$

Unprofitable and profitable strategies

For a payoff matrix with size $m \times n$ ($m \neq 2, n \neq 2$), we decrease its size with the help of exclusion of unprofitable strategies.

Example 7.2.

Investigate and simplify the given payoff matrix:

$$\Pi = \begin{pmatrix} 7 & 6 & 5 & 4 & 2 \\ 5 & 4 & 3 & 2 & 3 \\ 5 & 6 & 6 & 3 & 5 \\ 2 & 3 & 3 & 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 7 & 6 & 5 & 4 & 2 \\ 5 & 4 & 3 & 2 & 3 \\ 5 & 6 & 6 & 3 & 5 \\ 2 & 3 & 3 & 4 & 1 \end{pmatrix}$$

Solution.

Let A_1, A_2, A_3, A_4 be strategies of player A

B_1, B_2, B_3, B_4, B_5 be strategies of player B

Let A_1, A_2, A_3, A_4 be strategies of player A

Let's denote: x_1 is the probability that the first player uses the 1-st strategy; x_2, x_3, x_4 are probabilities that the first player uses the 2-nd, 3-rd and 4-th strategies; since events A_1, A_2, A_3, A_4 form the complete group, then

$$x_1 + x_2 + x_3 + x_4 = 1$$

Let B_1, B_2, B_3, B_4, B_5 be strategies of player B

Denote: y_1 is the probability that the second player uses the 1-st strategy; y_2, y_3, y_4, y_5 are the probability that the second player uses the 2-nd, 3-rd, 4-th and 5-th strategies, then

$$y_1 + y_2 + y_3 + y_4 + y_5 = 1$$

Let’s find the upper and the lower prices:

$$\alpha = \max\left(2,2,3,1\right) = 3, \quad \beta = \min\left(7,6,6,4,5\right) = 4, \quad \alpha \neq \beta$$

The given matrix game has no a saddle point.

$$\begin{pmatrix} 7 & 6 & 5 & 4 & 2 \\ 5 & 4 & 3 & 2 & 3 \\ 5 & 6 & 6 & 3 & 5 \\ 2 & 3 & 3 & 4 & 1 \end{pmatrix}$$

$$\alpha = \max_i \min_j c_{ij}$$

We need to exclude rows with the smallest elements.

$$\begin{pmatrix} 7 & 6 & 5 & 4 & 2 \\ 5 & 4 & 3 & 2 & 3 \\ 5 & 6 & 6 & 3 & 5 \\ 2 & 3 & 3 & 4 & 1 \end{pmatrix}$$

$$\alpha = \max_i \min_j c_{ij}$$

Let's find unprofitable strategies.

$$\begin{pmatrix} 7 & 6 & 5 & 4 & 2 \\ 5 & 4 & 3 & 2 & 3 \\ 5 & 6 & 6 & 3 & 5 \\ 2 & 3 & 3 & 4 & 1 \end{pmatrix}$$

$$\alpha = \max_i \min_j c_{ij}$$

Let’s find unprofitable strategies. All elements A_2 are less than A_3 , i.e. A_2 is more unprofitable for the first player, and A_2 can be excluded. All elements A_4 are less than A_3 then A_4 can be excluded.

$$\begin{pmatrix} 7 & 6 & 5 & 4 & 2 \\ 5 & 4 & 3 & 2 & 3 \\ 5 & 6 & 6 & 3 & 5 \\ 2 & 3 & 3 & 4 & 1 \end{pmatrix}$$

$$\alpha = \max_i \min_j c_{ij}$$

Let’s find unprofitable strategies. We compare columns.

$$\begin{pmatrix} 7 & 6 & 5 & 4 & 2 \\ 5 & 6 & 6 & 3 & 5 \end{pmatrix}$$

$$\beta = \min_{j} \max_{i} c_{ij}$$

We need to exclude columns with the greatest elements.

$$\begin{pmatrix} 7 & 6 & 5 & 4 & 2 \\ 5 & 6 & 6 & 3 & 5 \end{pmatrix} \qquad \beta = \min_j \max_i c_{ij}$$

For the second player: we compare B_1 and B_4, and exclude B_1; we compare B_2 and B_4, and exclude B_2; we compare B_3 and B_4, and exclude B_3.

In the result of transformations we obtain the new matrix:

$$\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}_{2\times 2}.$$

The size of the payoff matrix is reduced with the help of exclusion of unprofitable strategies, i.e. A_2, A_4 and B_1, B_2, B_3. Therefore $x_2 = x_4 = y_1 = y_2 = y_3 = 0$

and the matrix has the following form:

$$\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}_{2\times 2}.$$

Then for player ***A***:

$$\begin{cases} 4x_1 + 3x_3 = v \\ 2x_1 + 5x_3 = v \\ x_1 + x_3 = 1 \end{cases}$$

For player ***B***:

$$\begin{cases} 4y_4 + 2y_5 = v \\ 3y_4 + 5y_5 = v \\ y_4 + y_5 = 1 \end{cases}$$

Solve systems and get:

$$x_1 = x_3 = 1/2, \nu = 7/2;$$

$$y_4 = 3/4, y_5 = 1/4, \nu = 7/2.$$

The optimal strategy of the first player:

$$X_{onm} = (1/2, 0, 1/2, 0),$$

The game price: $v = 7/2.$

The optimal strategy of the second player:

$$Y_{onm} = (0, 0, 0, \frac{3}{4}, \frac{1}{4})$$

The game price: $v = 7/2.$

TASK 4.

Simplify the payoff matrix and define unprofitable strategies:

$$\begin{pmatrix} 8 & 6 & 4 & 7 & 7 \\ 5 & 4 & 3 & 4 & 6 \\ 4 & 3 & 2 & 3 & 4 \\ 7 & 2 & 6 & 5 & 9 \end{pmatrix}$$

Answer: $\begin{pmatrix} 6 & 4 \\ 2 & 6 \end{pmatrix}$

GRAPHICAL METHOD

If a game has size

$$\boxed{2 \times n},$$

$$\boxed{m \times 2}$$

or

$$\boxed{2 \times 2}$$

then we can use the graphical method.

Let's consider the payoff matrix with the size $\left[2\times n\right]$

Probabilities		The second player				Strategies of the first player
		y_1	y_2	...	y_n	
The first player	x_1	a_{11}	a_{12}	...	a_{1n}	A_1
	x_2	a_{21}	a_{22}	...	a_{2n}	A_2
Strategies of the second player		B_1	B_2	...	B_n	

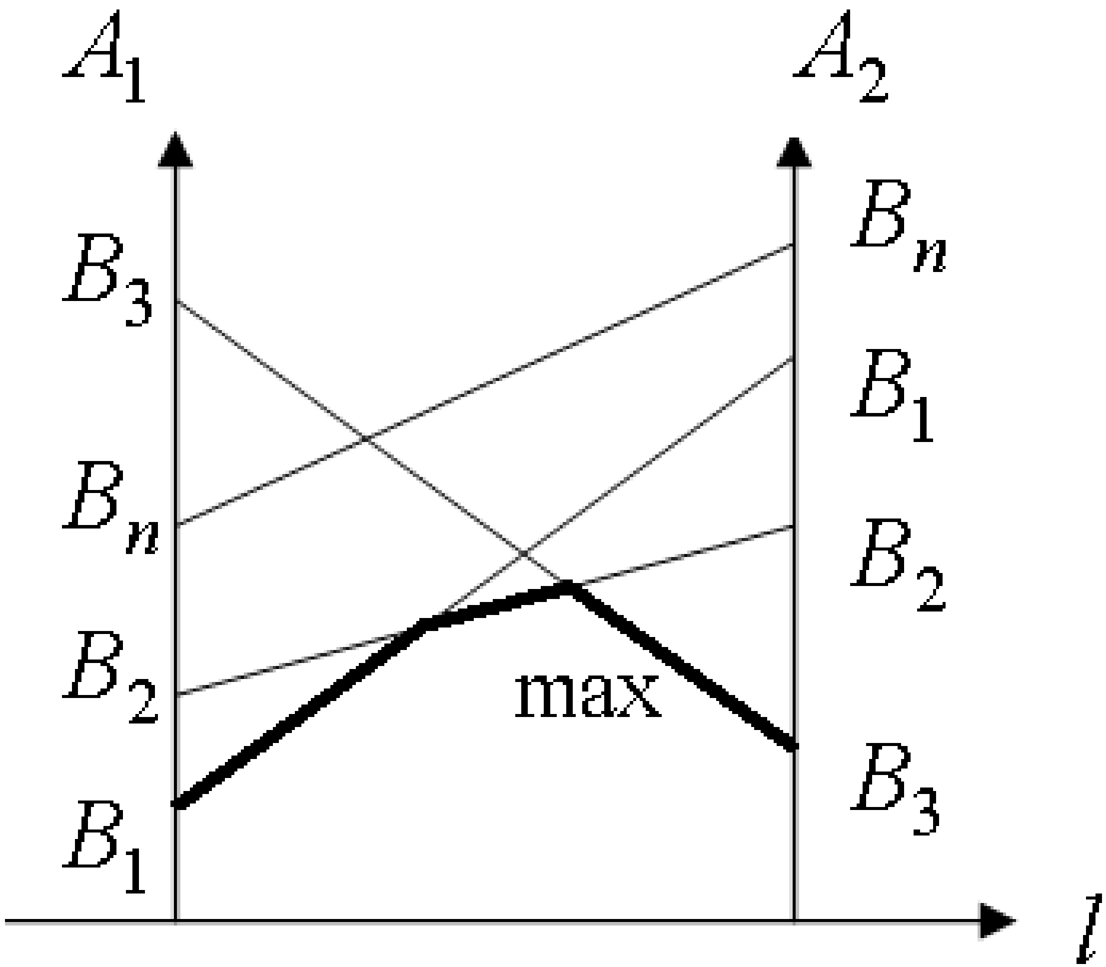
The expected payoff of the first player if the second player uses the first strategy equals the mathematical expectation of the payoff:

$$M(v) = a_{11}x_1 + a_{21}x_2.$$

Like this for the rest of strategies we have:.

Pure strategies of the second player	**Expected payoffs for the first player**
1	$a_{11}x_1 + a_{21}x_2$
2	$a_{12}x_1 + a_{22}x_2$
$\cdots$	$\cdots$
n	$a_{1n}x_1 + a_{2n}x_2$

$$\alpha = \max_{j} \min_{i} a_{ij}$$



Player ***A*** chooses the optimal strategy based on the maximization of minimal payoffs (the maximin principle).

It means **we choose *the lower polyline and the upper point on this polyline (the point with the maximal ordinate)*.**

For example, we have the intersection: B_2, B_3

$$\Pi = \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$$

For player ***A***:

$$\begin{cases} a_{12}x_1 + a_{22}x_2 = v, \\ a_{13}x_1 + a_{23}x_2 = v, \\ x_1 + x_2 = 1. \end{cases}$$

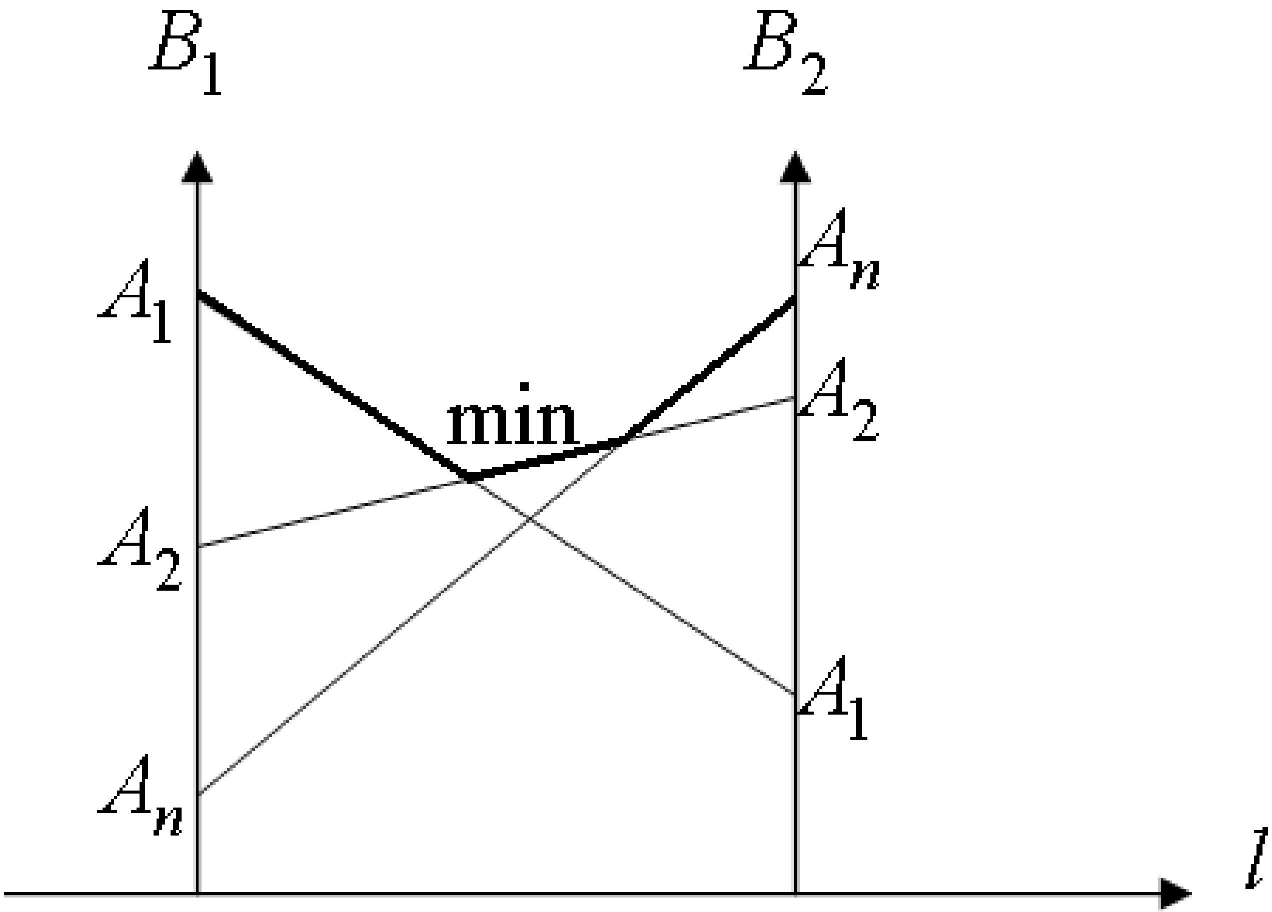
For player ***B***:

$$\begin{cases} a_{12}y_2 + a_{13}y_3 = \nu, \\ a_{22}y_2 + a_{23}y_3 = \nu, \\ y_2 + y_3 = 1. \end{cases}$$

Let’s consider the payoff matrix with the size $\boxed{\left[m \times 2\right]}$

Probabilities		The second player		Strategies for the first player
		y_1	y_2	
The first player	x_1	a_{11}	a_{12}	A_1
	x_2	a_{21}	a_{22}	A_2
	…	…	…	…
	x_m	a_{m1}	a_{m2}	A_m
Strategies for the second player		B_1	B_2	

$$\beta = \min_i \max_j a_{ij}$$



For example, we have the intersection: A_1, A_2

$$x_3 = x_4 = \ldots = x_m = 0$$

$$\Pi = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

For player ***A***:

$$\begin{cases} a_{11}x_1 + a_{21}x_2 = v, \\ a_{12}x_1 + a_{22}x_2 = v, \\ x_1 + x_2 = 1. \end{cases}$$

For player ***B***:

$$\begin{cases} a_{11}y_1 + a_{12}y_2 = \nu, \\ a_{21}y_1 + a_{22}y_2 = \nu, \\ y_1 + y_2 = 1. \end{cases}$$

GRAPHICAL METHOD

Let’s consider a general scheme of graphical solving the game in mixed strategies:

1. To plot a system of strategies considering the behavior of player ***A*** or player ***B***, which has 2 strategies, because a system of coordinates has 2 axes.

GRAPHICAL METHOD

Let’s consider a general scheme of graphical solving the game in mixed strategies:

2. To mark the values of y_1, …, y_n and x_1, …, x_m from 0 to 1, because a probability is changed from 0 to 1.

GRAPHICAL METHOD

Let’s consider a general scheme of graphical solving the game in mixed strategies:

3. To plot the straight lines which correspond to another player.

GRAPHICAL METHOD

Let's consider a general scheme of graphical solving the game in mixed strategies:

4. To find two strategies for corresponding player which are intersected at the point of maximin (player ***A***) or minimax (player ***B***).

GRAPHICAL METHOD

Let's consider a general scheme of graphical solving the game in mixed strategies:

5. To calculate the probabilities of the optimal strategies and the game price.

GRAPHICAL METHOD

If a payoff matrix doesn't have a saddle point, i.e. $\alpha < \beta$, then search of an optimal solution gives using mixed strategies which consist of using two or more strategies with definite probabilities. In a game with a payoff matrix $m \times n$, probabilities of strategies of the first player are denoted by $X = (x_1, x_2, \ldots, x_m)$ with the condition:

$$\sum_{i=1}^{m} x_i = 1, \quad x_i \geq 0, \quad i = \overline{1, m}.$$

GRAPHICAL METHOD

For the second player probabilities of his strategies are denoted $Y = (y_1, y_2, ..., y_n)$ with the condition:

$$\sum_{j=1}^{n} y_j = 1, \; y_j \geq 0, \; j = \overline{1, n}.$$

GRAPHICAL METHOD

For the second player probabilities of his strategies are denoted $Y=(y_1, y_2, ..., y_n)$ with the condition:

$$\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = \overline{1,n}.$$

Using an optimal strategy allows obtaining the payoff (or the loss) which equals the game price:

$$\alpha \leq v \leq \beta.$$

GRAPHICAL METHOD

For player ***A*** using an optimal strategy must be equal to the game price at any actions of player ***B***. It gives the following relation:

$$\sum_{i=1}^{m} c_{ij} x_{i\,opt} = v$$

GRAPHICAL METHOD

For player ***B*** using an optimal strategy must be equal to the game price at any actions of player ***A***. It gives the following relation:

$$\sum_{j=1}^{n} c_{ij} y_{j\,opt} = v$$

GRAPHICAL METHOD

For a payoff matrix with size $m \times n$ ($m \neq 2, n \neq 2$), we decrease its size with the help of exclusion of unprofitable strategies.

EXAMPLE

Example 2. Solve the matrix game and define the game price, if the payoff matrix is given by:

$$
\begin{array}{cc}
 & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\
C = & \begin{pmatrix} 2 & 1 & 5 & 3 \\ 1 & 3 & 4 & 0{,}5 \end{pmatrix}
\end{array}
\begin{array}{c} A_1 \\ A_2 \end{array}
$$

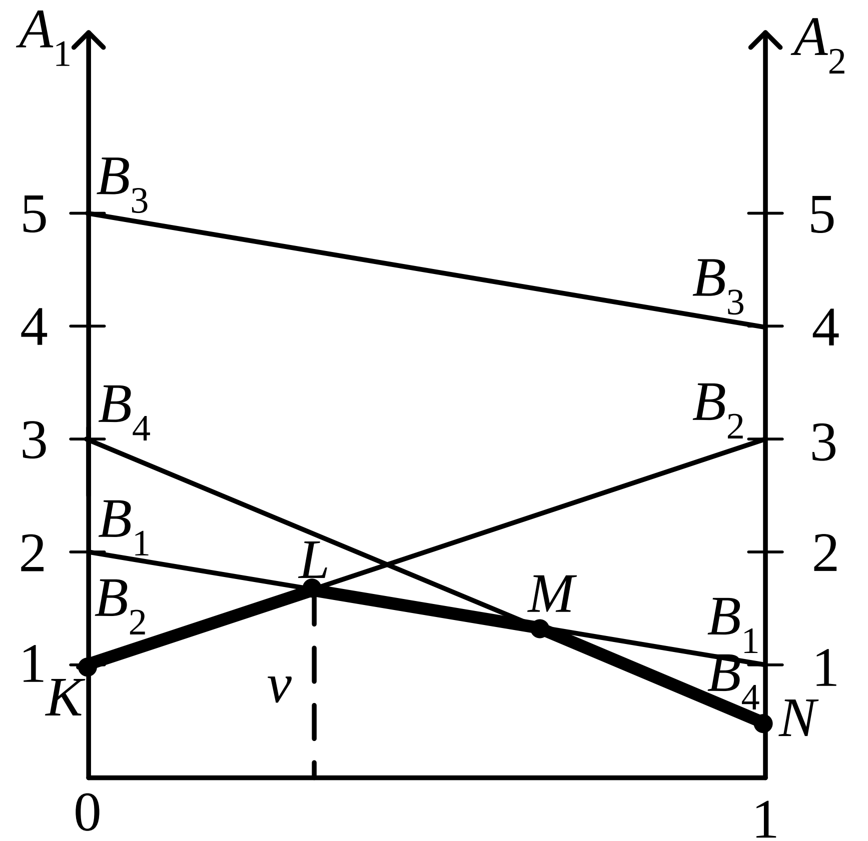
EXAMPLE

Solution. Since player ***A*** has 2 strategies, then we can use the graphical method for this problem with the given payoff matrix.

$$
\begin{array}{cc}
 & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\
C = & \begin{pmatrix} 2 & 1 & 5 & 3 \\ 1 & 3 & 4 & 0{,}5 \end{pmatrix} \begin{array}{c} A_1 \\ A_2 \end{array}
\end{array}
$$

Let's plot a system of strategies and consider the behavior of player ***A***.

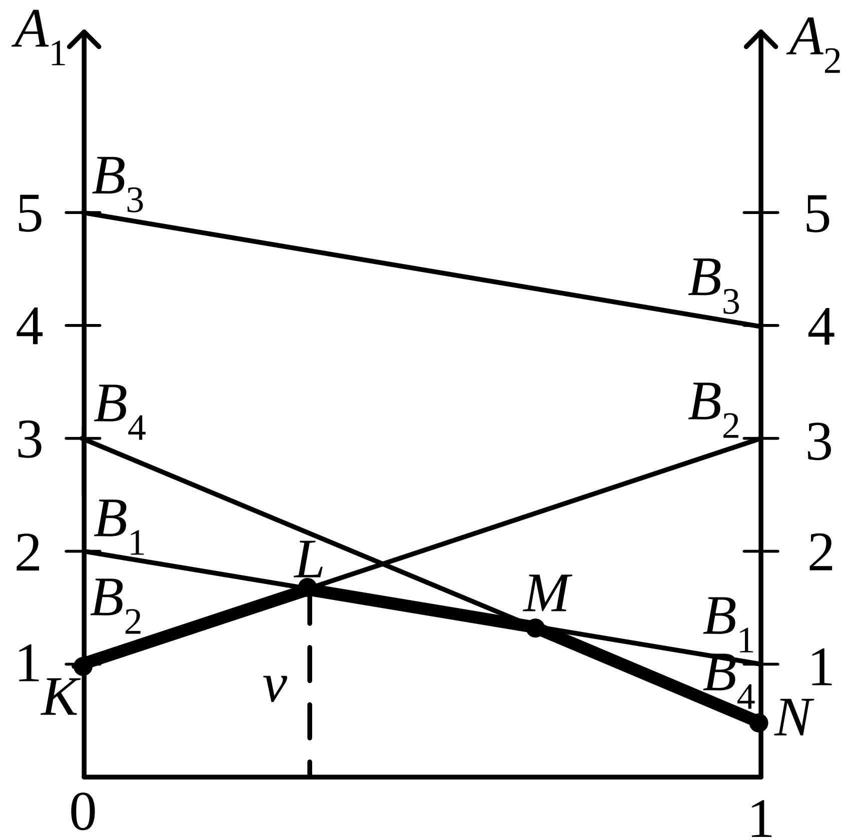
$$C = \begin{pmatrix} 2 & 1 & 5 & 3 \\ 1 & 3 & 4 & 0{,}5 \end{pmatrix}$$



Let's plot strategies of player ***A*** and *choose the principle of maximin, i.e. the lower polyline and the upper point on this polyline (the point with maximal ordinate)*. We denote this *lower polyline as KLMN* and the upper point will be point L.

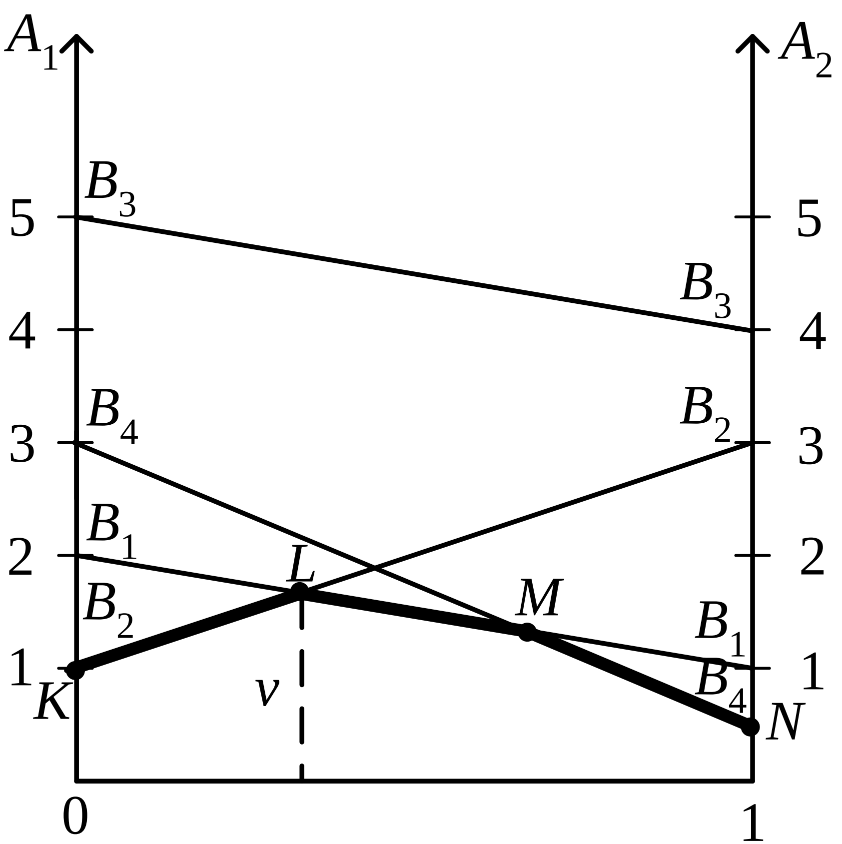
$$C = \begin{pmatrix} 2 & 1 & 5 & 3 \\ 1 & 3 & 4 & 0,5 \end{pmatrix}$$

$$\alpha = \max_i(\alpha_i) = \max_i \min_j c_{ij}$$



Let's consider this point L. Point L is intersection of the strategies B_1 and B_2 . Then we have the obtained matrix:

$$C = \begin{pmatrix} 2 & 1 & 5 & 3 \\ 1 & 3 & 4 & 0{,}5 \end{pmatrix}$$

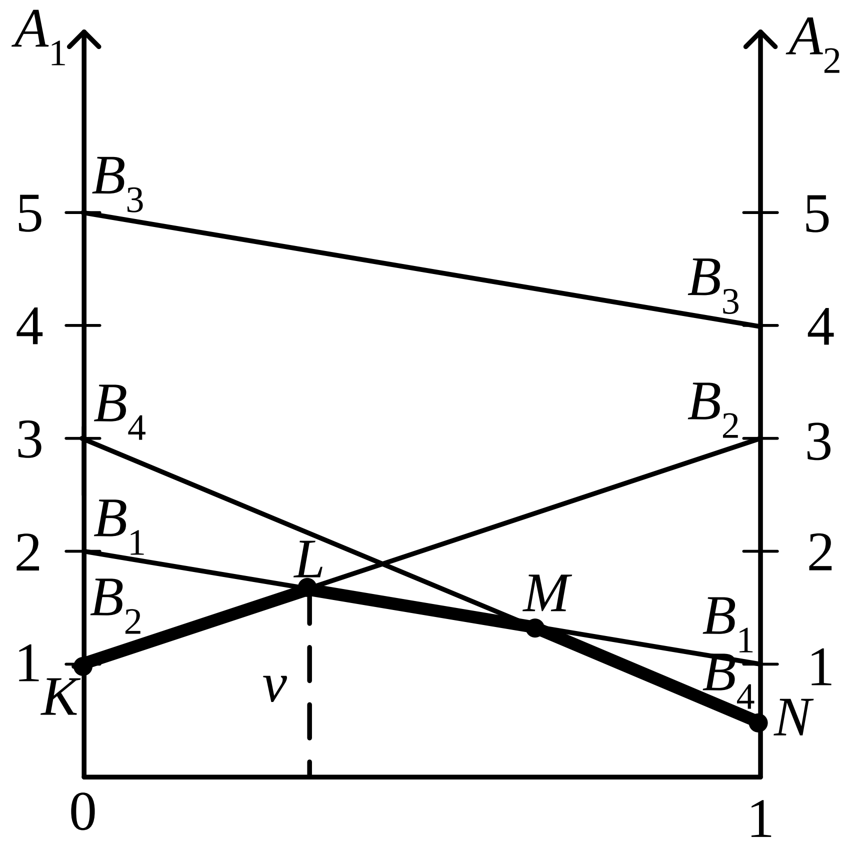


Let's consider this point L. Point L is intersection of the strategies B_1 and B_2. Then we have the obtained matrix:

$$\begin{array}{cc} B_1 & B_2 \end{array}$$

$$C = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{array}{c} A_1 \\ A_2 \end{array}$$

$$C = \begin{pmatrix} 2 & 1 & 5 & 3 \\ 1 & 3 & 4 & 0,5 \end{pmatrix}$$



Let's consider strategies of player ***A***.

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ C = & \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{array}{c} A_1 \\ A_2 \end{array} \end{array}$$

Let's consider strategies of player ***A***. Let's denote x_1 as the probability that player ***A*** uses strategy A_1 ; x_2 as the probability that player ***A*** uses strategy A_2 .

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ C = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} & \begin{array}{c} A_1 \\ A_2 \end{array} \end{array}$$

	y_1	y_2
x_1	2	1
x_2	1	3

If player ***B*** uses strategy B_1 , then player ***A*** can choose strategy A_1 with coefficient 2 or strategy A_2 with coefficient 1) :

$$2 \cdot x_1 + 1 \cdot x_2 = v$$

	y_1	y_2
x_1	2	1
x_2	1	3

If player ***B*** uses strategy B_2 , then player ***A*** can choose strategy A_1 with coefficient 1 or strategy A_2 with coefficient 3) :

$$1 \cdot x_1 + 3 \cdot x_2 = v$$

	y_1	y_2
x_1	2	1
x_2	1	3

The sum of probabilities of the complete group of events must be equal to 1:

$$x_1 + x_2 = 1$$

$$2 \cdot x_1 + 1 \cdot x_2 = v$$

$$1 \cdot x_1 + 3 \cdot x_2 = v$$

$$x_1 + x_2 = 1$$

We obtain:

$$X_{opt} = (x_1, x_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$$

$$v = 2 \cdot x_1 + 1 \cdot x_2 = 2 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{5}{3}$$ (units of money)

	y_1	y_2
x_1	2	1
x_2	1	3

If player ***A*** uses strategy A_1 , then player ***B*** can choose strategy B_1 with coefficient 2 or strategy B_2 with coefficient 1) :

$$2 \cdot y_1 + 1 \cdot y_2 = v$$

	y_1	y_2
x_1	2	1
x_2	1	3

If player ***A*** uses strategy A_2 , then player ***B*** can choose strategy B_1 with coefficient 1 or strategy B_2 with coefficient 3) :

$$1 \cdot y_1 + 3 \cdot y_2 = v$$

	y_1	y_2
x_1	2	1
x_2	1	3

The sum of probabilities of the complete group of events must be equal to 1:

$$y_1 + y_2 = 1$$

$$2 \cdot y_1 + 1 \cdot y_2 = v$$

$$1 \cdot y_1 + 3 \cdot y_2 = v$$

$$y_1 + y_2 = 1$$

Let’s obtain: $$Y_{opt} = (y_1, y_2, y_3, y_4) = \left(\frac{2}{3}, \frac{1}{3}, 0, 0\right)$$

In this solution y_3 and y_4 are equal to 0, because strategies B_3 and B_4 are excluded from consideration by the graphical method.

$$2 \cdot y_1 + 1 \cdot y_2 = \nu$$

$$1 \cdot y_1 + 3 \cdot y_2 = \nu$$

$$y_1 + y_2 = 1$$

Let's obtain: $$Y_{opt} = (y_1, y_2, y_3, y_4) = \left(\frac{2}{3}, \frac{1}{3}, 0, 0 \right)$$

Let's calculate the game price:

$$\nu = 2 \cdot y_1 + 1 \cdot y_2 = 2 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{5}{3}$$ (units of money).

Example 3. Solve the given payoff matrix by the analytical and graphical methods.

$$\Pi = \begin{pmatrix} 2 & 5 \\ 7 & 1 \\ 3 & 7 \\ 4 & 6 \end{pmatrix}$$

Solution.

$$
\begin{array}{c c}
 & \begin{array}{cc} B_1 & B_2 \end{array} \\
\Pi = & \begin{pmatrix} 2 & 5 \\ 7 & 1 \\ 3 & 7 \\ 4 & 6 \end{pmatrix}
\end{array}
\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}
$$