PhD Misiura Ie.Iu. (доцент Місюра Є.Ю.)

Theme:
Elements of vector algebra and analytic geometry
PART 2. Elements of analytical geometry

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Straight line on a plane

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Economic examples

Solving Models with Graphs

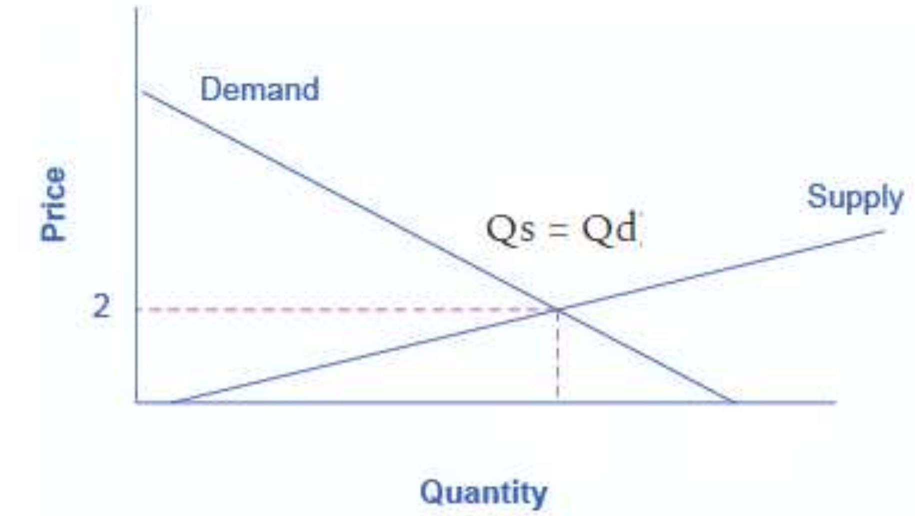

Figure . Supply and Demand Graph. The equations for Qd and Qs are displayed graphically by the sloped lines.

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Economic examples

Demand curve:

Example 1

Let us assume that the demand curve is described by the following line $\boldsymbol{q = mp + b}$. Find its equation given the following information: a promoter discovers that the demand for theater tickets is 1200 when the price is \$60, but decreases to 900 when the price is raised to \$75.

Solution :

The form of the equation $\boldsymbol{q = mp + b}$ indicates that the price $\boldsymbol{p}$, is the independent variable (like $\boldsymbol{x}$), and the quantity $\boldsymbol{q}$, is the dependent variable (like $\boldsymbol{y}$). The problem allows us to deduce two points of the demand line: the points (60\$, 1200) and (75\$, 900). We must identify the slope and the $\boldsymbol{y}$ –intercept of the line.

Slope :

$$m = \frac{\Delta q}{\Delta p} = \frac{q_2 - q_1}{p_2 - p_1} = \frac{900 - 1200}{75 - 60} = \frac{-300}{15} = -20$$

The equation must therefore take on the following form: $\boldsymbol{q = -20p + b}$. It is necessary to find the $\boldsymbol{y}$ -intercept using one of the two points.

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***y*-intercept :**

since $(60\$, 1200)$ is a point on the demand curve, it must satisfy the following equation : $q = -20p + b$. By substitution, we obtain

$$1200 = -20(60) + b$$

$$1200 = -1200 + b$$

$$b = 2400$$

As a result, since $m = -20$ and $b = 2400$, the equation of the demand line is

$$q = -20p + 2400$$

It is interesting to note that once this line is found, we can evaluate what the demand is whatever the price. For example, the demand when the price is at \$40 would be obtained by calculating the variable $\boldsymbol{q}$:

$$q = -20(40) + 2400$$

$$q = -800 + 2400$$

$$q = 1600$$

We could also obtain the price needed for a demand of 1000 tickets.

$$1000 = -20\,p + 2400$$

$$20\,p = 2400 - 1000$$

$$20\,p = 1400$$

$$p = 70\,\$$$

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Economic examples

Example 2

The equilibrium quantity and the equilibrium price of a product are determined by the point where the supply and demand curves intersect. For a given product, the supply is determined by the line

$$q_{supply} = 30p - 45$$

and for the same product, the demand is determined by the line

$$q_{demand} = -15p + 855.$$

Determine the price and the equilibrium quantity and trace the supply and demand curves on the same graph.

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Economic examples

Example 2

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$$q_{demand} = -15p + 855.$$

Determine the price and the equilibrium quantity and trace the supply and demand curves on the same graph.

Solution :

We must determine the coordinates of point (q,p), situated at the intersection of the two lines. This point must therefore satisfy both the supply and the demand equations. The solution to this problem is to solve :

$$q = 30p - 45$$
$$q = -15p + 855$$

Thus,

$$30p - 45 = -15p + 855$$
$$45\,p = 900$$
$$p = 20$$

and

$$q = 30(20) - 45 = 555$$

The equilibrium price and quantity are therefore $20 and 555.

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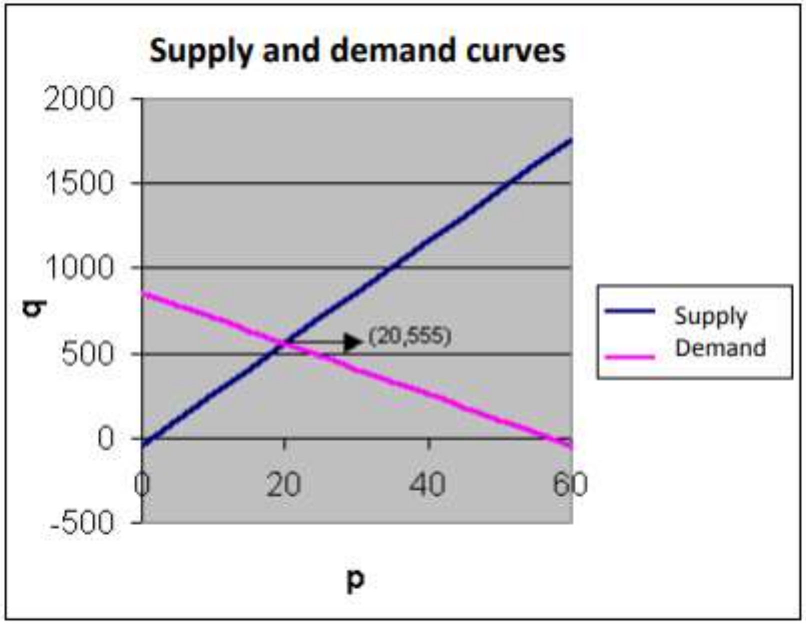
Economic examples



In economics, it is usual to graphically represent the supply and demand curves by placing the price (p) on the ordinate and the quantity (q) on the abscissa.

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Straight line (linear) demand curve

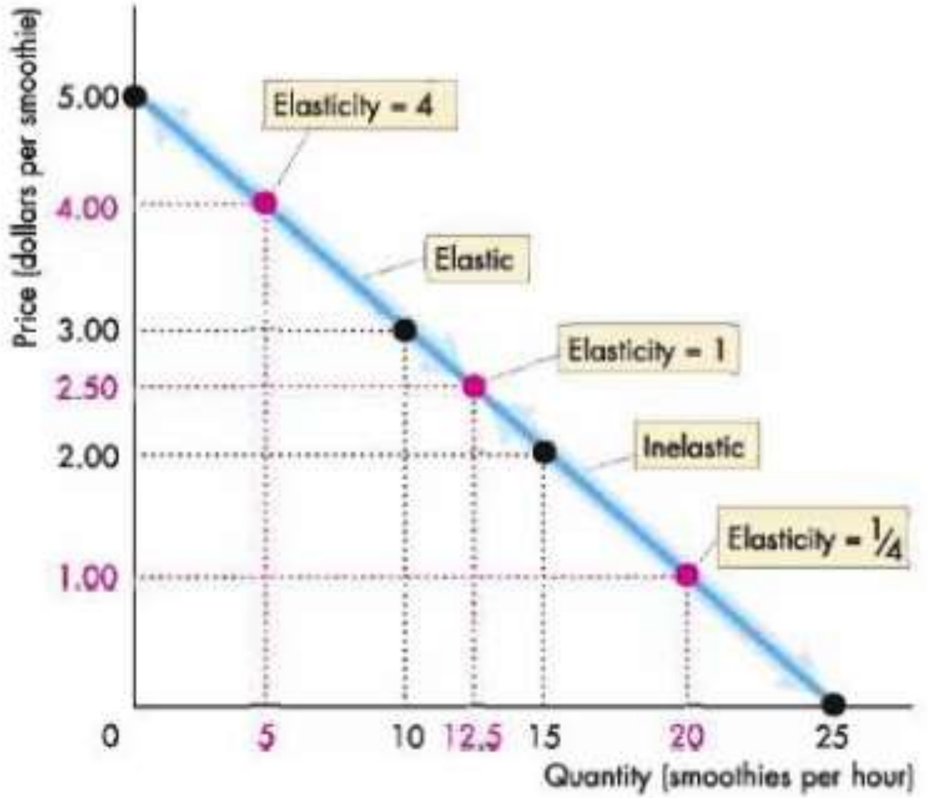
A straight line demand curve will have a different elasticity at each point on it.

The price elasticity of demand can also be measured at any point on the demand curve.

If the demand curve is linear (straight line), it has a unitary elasticity at the midpoint. The total revenue is maximum at this point. Any point above the midpoint has elasticity greater than 1, ($Ed > 1$).

Here, price reduction leads to an increase in the total revenue (expenditure). Below the midpoint elasticity is less than 1. ($Ed < 1$). Price reduction leads to reduction in the total revenue of the firm. Now the question arises, why does a straight line demand curve have different elasticity at each point? The value of PED falls as price falls.

The reason is that low priced products have a more inelastic demand than high priced products, because consumers are not that price sensitive when the product is inexpensive, Similarly the value of PED is higher when the prices increase because consumers are more sensitive to price change when the good is expensive. A mathematical explanation can be given as follows. As we seen in diagram below



On a linear demand curve, elasticity decreases as the price falls and the quantity demanded increases. Demand is unit elastic at the midpoint of the demand curve (elasticity is 1). At prices above the midpoint, demand is elastic; at prices below the midpoint, demand is inelastic.

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What the Slope Means

The concept of slope is very useful in economics, because it measures the relationship between two variables. A **positive slope** means that two variables are positively related—that is, when *x* increases, so does *y*, and when *x* decreases, *y* decreases also. Graphically, a positive slope means that as a line on the line graph moves from left to right, the line rises. We will learn in other sections that "price" and "quantity supplied" have a positive relationship; that is, firms will supply more when the price is higher.

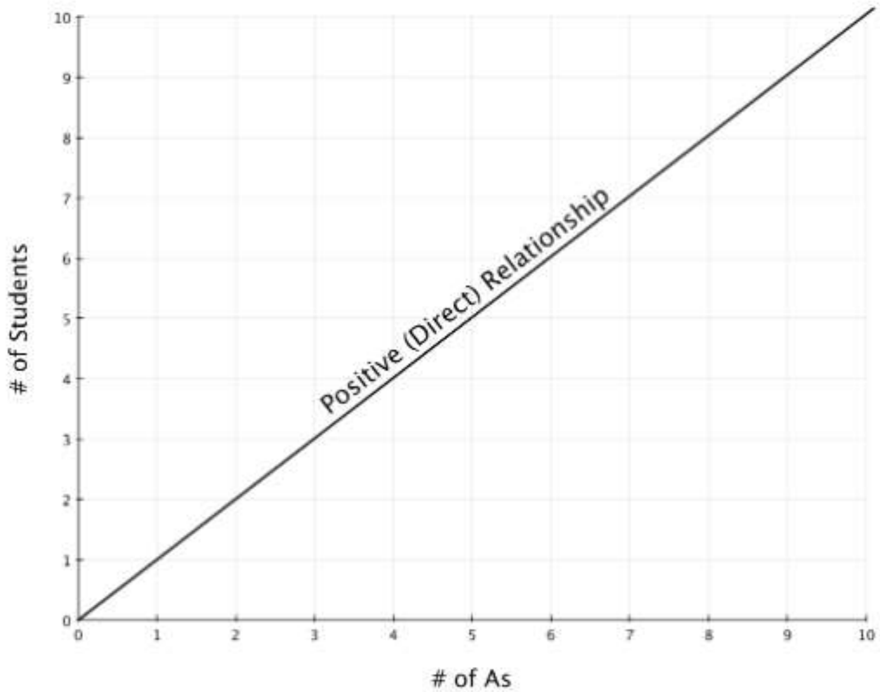

Figure 1. Positive Slope

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A **negative slope** means that two variables are negatively related; that is, when *x* increases, *y* decreases, and when *x* decreases, *y* increases. Graphically, a negative slope means that as the line on the line graph moves from left to right, the line falls. We will learn that "price" and "quantity demanded" have a negative relationship; that is, consumers will purchase less when the price is higher.

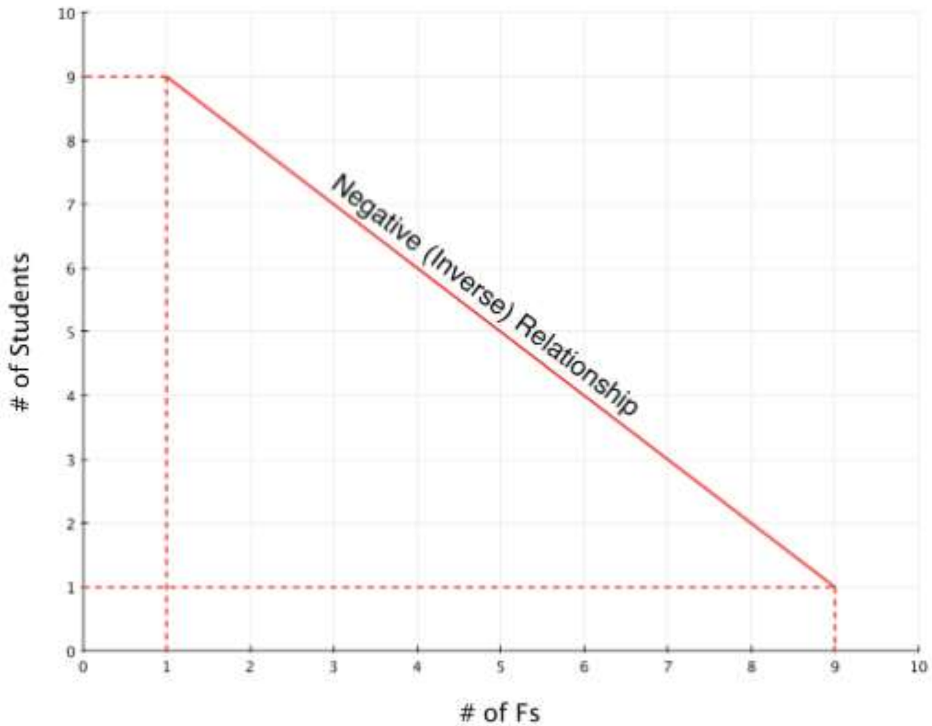

Figure 2. Negative slope

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A **slope of zero** means that there is a constant relationship between *x* and *y*. Graphically, the line is flat; the rise over run is zero.

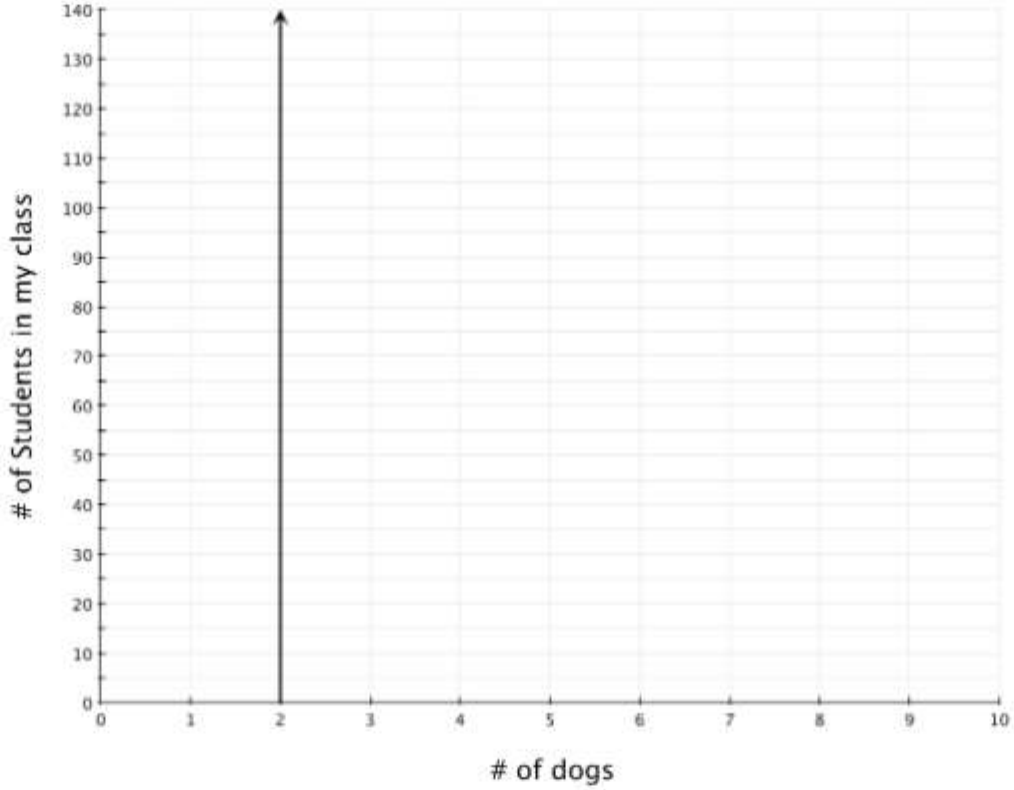


Figure 3. Slope of Zero

Straight line on a plane

Lecture plan

1. **An equation of a straight line on a plane.**
2. **Geometric relationship of two straight lines. Distance from the point to a given straight line.**

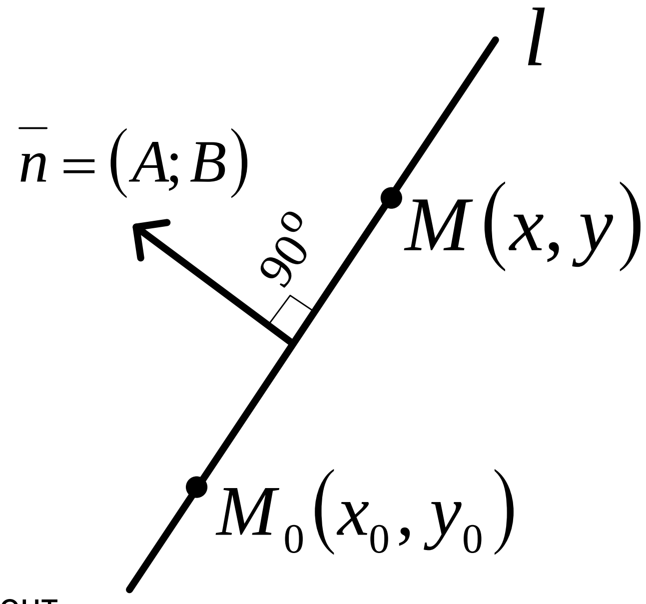
1. An equation of a straight line on a plane

1.1 Equation of a straight line passing through the point $M_0(x_0, y_0)$ with the given normal (perpendicular) vector $\overline{n} = (A; B)$ to this straight line

Let's take the point $M_0(x_0, y_0)$ and pass the line *l* through it. After this we choose any point $M(x, y)$ on this line and plot the vector $\overline{n} = (A; B)$ which is perpendicular to this line *l*.

$$A \cdot (x - x_0) + B \cdot (y - y_0) = 0$$

formula 1



Example

Form a straight line through the point $M(-1,4)$ perpen-dicular to the vector $\vec{n} = (-2,7)$.

1.2. General equation of the straight line

Let's remove the brackets of the previous equation:

$$A\cdot\left(x-x_0\right)+B\cdot\left(y-y_0\right)=0$$

$$A\cdot x-A\cdot x_0+B\cdot y-B\cdot y_0=0$$

$$A\cdot x+B\cdot y\underbrace{-A\cdot x_0-B\cdot y_0}_{C}=0$$

Let's denote the difference as *C* and obtain:

$$Ax+By+C=0$$, if $$A^2+B^2\neq 0$$

formula 2

$$\left|\bar{n}\right|\neq 0$$ or $$\bar{n}=\sqrt{A^2+B^2}\neq 0$$

1.2. General equation of the straight line

$$Ax + By + C = 0$$

If $C = 0$, then the equation of a straight line becomes $Ax + By = 0$ and determines a straight line passing through the origin.

If $B = 0$ and $A \neq 0$, then the equation of a straight line becomes $Ax + C = 0$ and determines a straight line parallel to the axis OY.

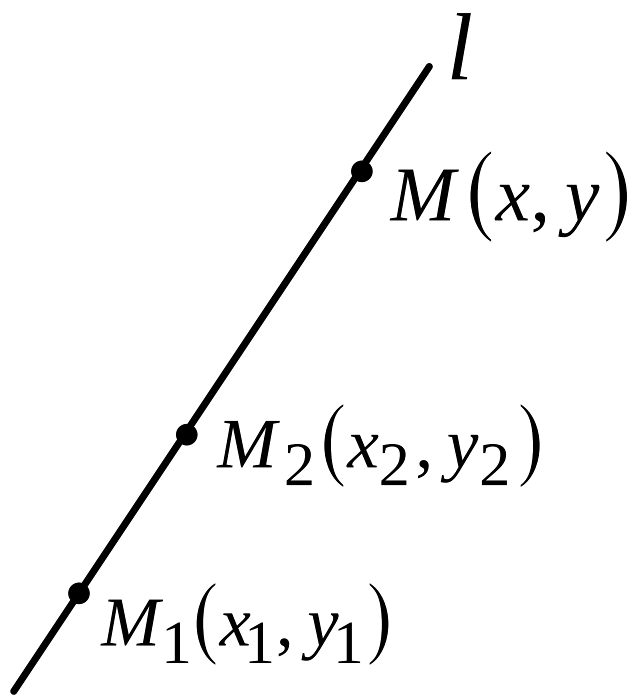
If $A = 0$ and $B \neq 0$, then the equation of a straight line becomes $By + C = 0$ and determines a straight line parallel to the axis OX.

Example

Use the previous example and get the general equation.

1.3. Straight line equation passing through two points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$

We take two points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ on the straight line and choose any point $M(x, y)$.



$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

formula 3

Example

Form a straight line passing through two points $M_1(3,-1)$ and $M_2(-2,5)$.

1.4. Equation of a straight line with the given intercepts on the axes

This straight line intersects the x-axis at the point $M_1(a;0)$ and the y-axis at the point $M_2(0;b)$, i.e. this line passes through two points M_1 and M_2.

Let's substitute coordinates of these points into the previous formula:

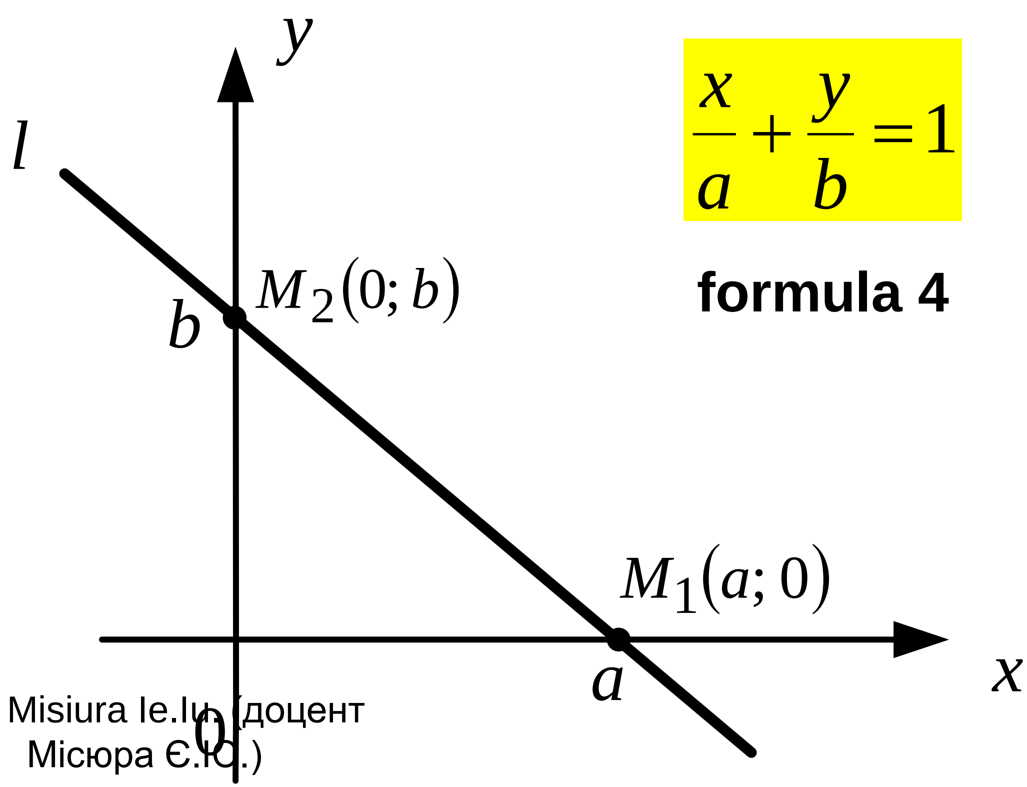
$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

Here $x_1=a$

$y_1=0$

$x_2=0$

$y_2=b$



$$\frac{x}{a}+\frac{y}{b}=1$$

formula 4

Example

Use the previous example and get an equation of a straight line with the given intercepts on the axes

1.5. Equation of the straight line passing through the point $M_0(x_0, y_0)$ and the vector $\bar{a} = (a_x, a_y)$ parallel to this line

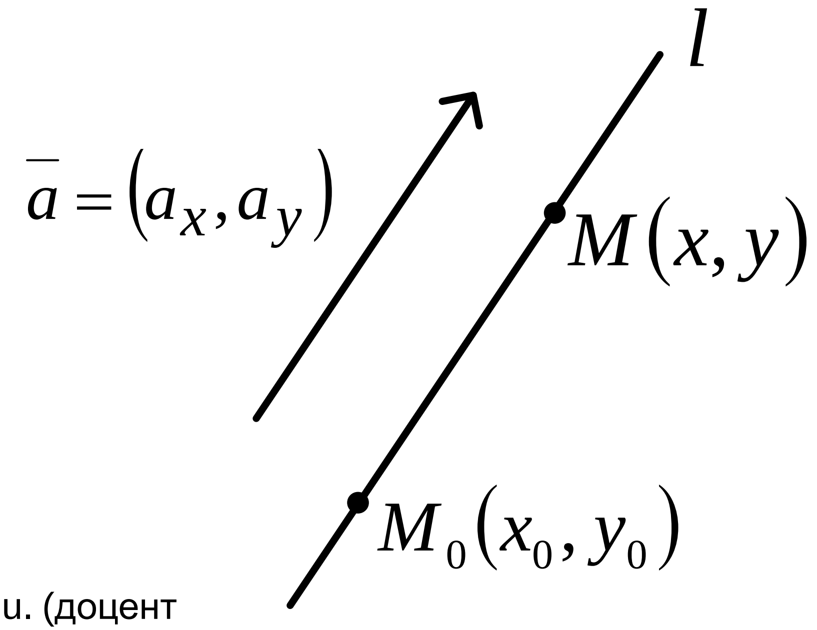
We take any point $M(x, y)$ on this line *l* and plot the vector $\overline{M_0M}$

Let's obtain the vector $\overline{M_0M}$

$$\overline{M_0M} = (x - x_0; y - y_0)$$

$$\frac{y - y_0}{a_y} = \frac{x - x_0}{a_x}$$

formula 5



Example

Form a straight line through the point $M(2,-5)$ parallel to the vector $\vec{a}=(4,-3)$.

1.6. Parametric equation of a straight line

We introduce an arbitrary parameter ***t*** into an equation of a straight line ***l*** passing through the point $M_0(x_0, y_0)$ and the vector $\bar{a} = (a_x, a_y)$ parallel to this line as a coefficient of proportionality:

$$\frac{y - y_0}{a_y} = \frac{x - x_0}{a_x} = t$$

Let’s transform this expression as

$$\frac{y - y_0}{a_y} = t \quad \text{and} \quad \frac{x - x_0}{a_x} = t$$

We express *x* and *y*:

$$y = a_y t + y_0 \quad \text{and} \quad x = a_x t + x_0$$

The obtained expressions are written as

$$\begin{cases} y = a_y t + y_0 \\ x = a_x t + x_0 \end{cases}$$

t is a parameter

formula 6

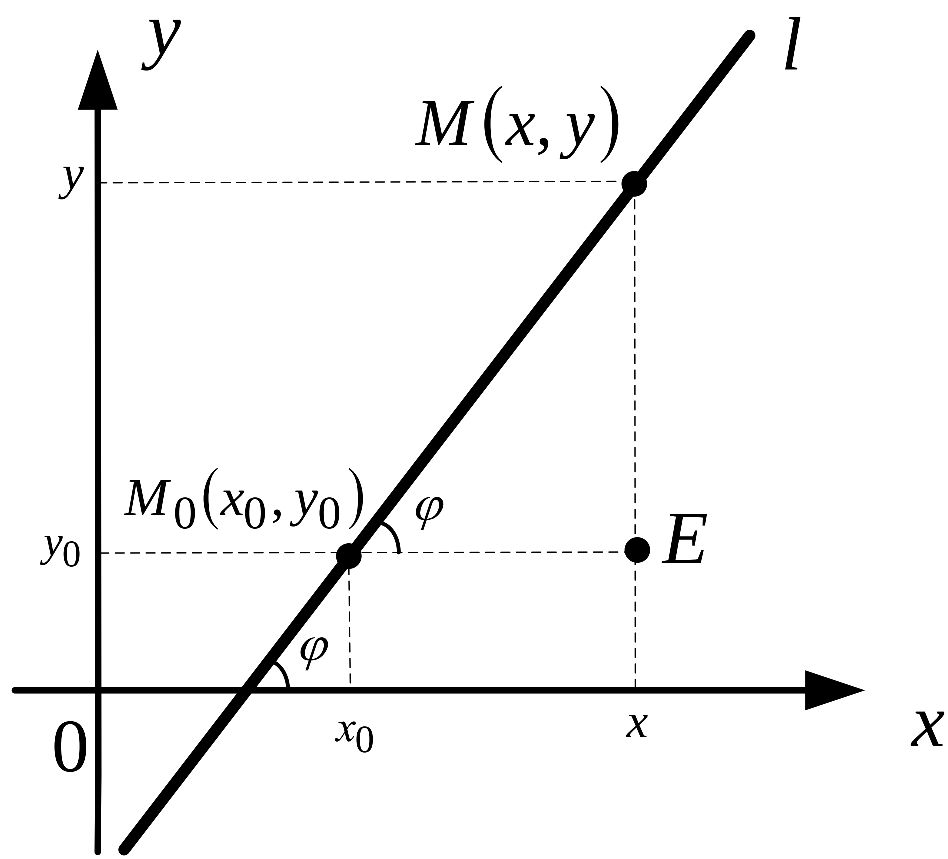
Example

Use the previous example and get a parametric equation of this straight line

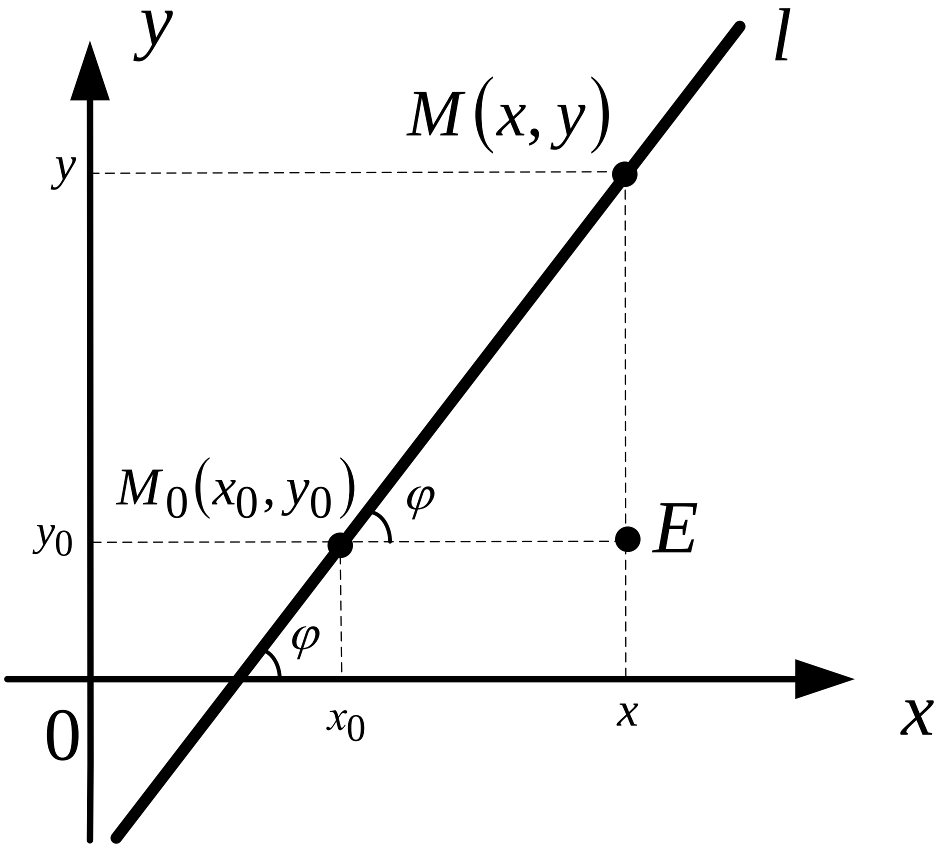
1.7. Equation of a straight line passing through the point $M_0(x_0, y_0)$ with the given angular coefficient k

We plot
this straight line and
the point $M_0(x_0, y_0)$
belonging to it.

Let's look at this picture.
We have
the right triangle M_0EM
with the right angle $\angle E = 90^{o}$
.



Let's find $tg\varphi$:



$$tg\varphi = \frac{ME}{M_0E} = \frac{opposite \quad leg}{adjcent \quad leg}$$

$$tg\varphi = \frac{ME}{M_0E} = \frac{y - y_0}{x - x_0}$$

$$\frac{y - y_0}{x - x_0} = tg\varphi = k$$

$$\frac{y - y_0}{x - x_0} = k$$

$$y - y_0 = k \cdot (x - x_0)$$

$k = tg\varphi \qquad 0 \le \varphi \le \pi$

formula 7

Example

Form the straight line passing through the point $M(5,-1)$

and forming 45° angle with the axis OX.

1.8. Equation of a straight line with the slope k

Let us transform the previous equation and remove the brackets:

$$y - y_0 = k \cdot (x - x_0)$$

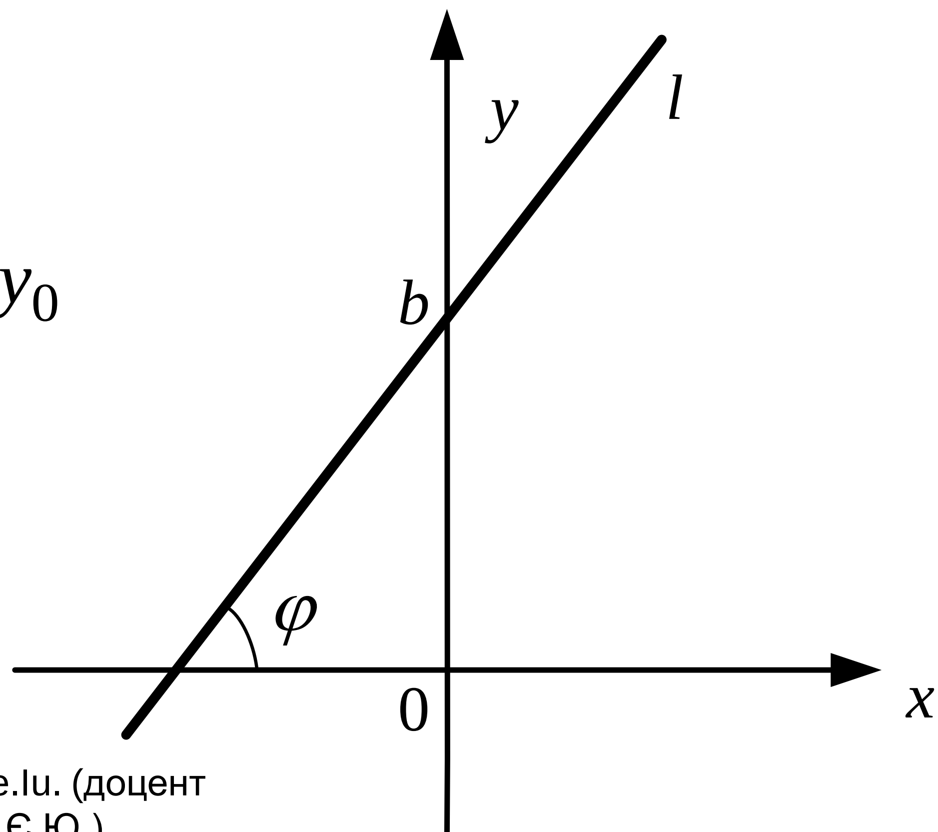
$$y = kx \underbrace{- kx_0 + y_0}_{b}$$

$y = kx + b$, where $b = -kx_0 + y_0$

formula 8

$$k = tg\varphi$$

$$0 \le \varphi \le \pi$$



Particular cases:

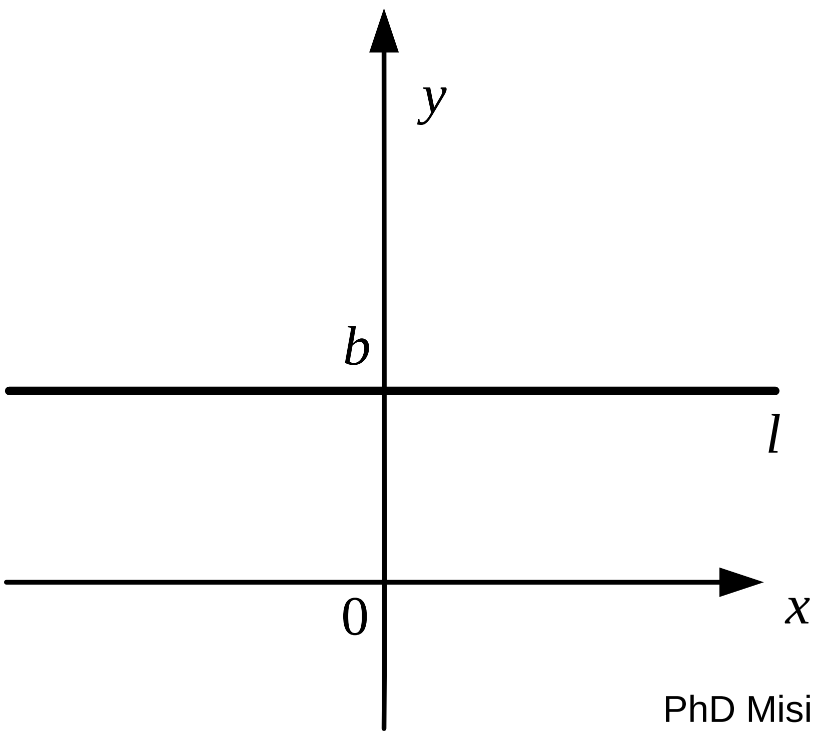
$$y = kx + b$$

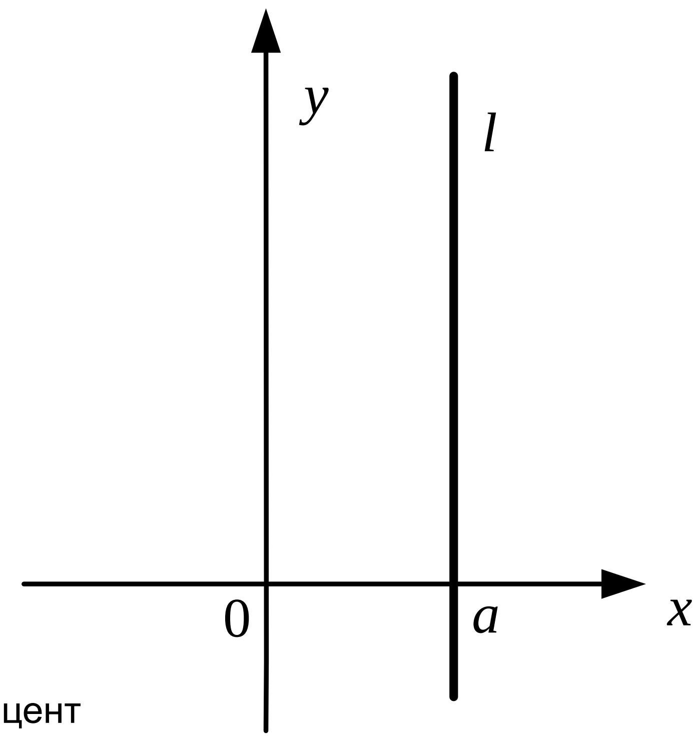
$$x = 0$$

$$y = b$$

$$y = 0$$

$$x = a$$





1.9. Normal equation of a straight line

Let's multiply a general equation of a straight line $Ax + By + C = 0$

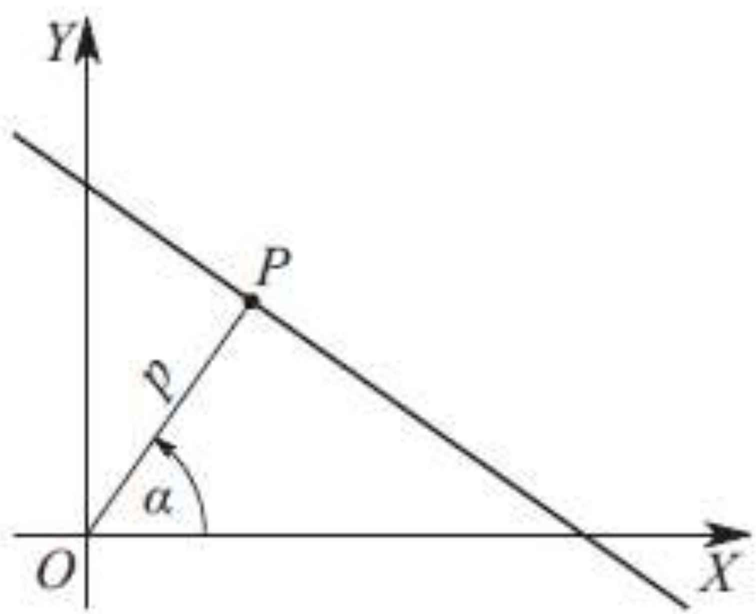
by a normalized multiplier $\mu = \pm \frac{1}{\sqrt{A^2 + B^2}}$ which sign is

opposite to the absolute term *C* sign, and obtain a normal equation of a straight line:

$$x \cdot \cos\alpha + y \cdot \sin\alpha - p = 0$$

formula 9

where p is the length of perpendicular dropped from the beginning of coordinates to the straight line, α is an angle obtained from this perpendicular with the positive direction of *OX* axis.



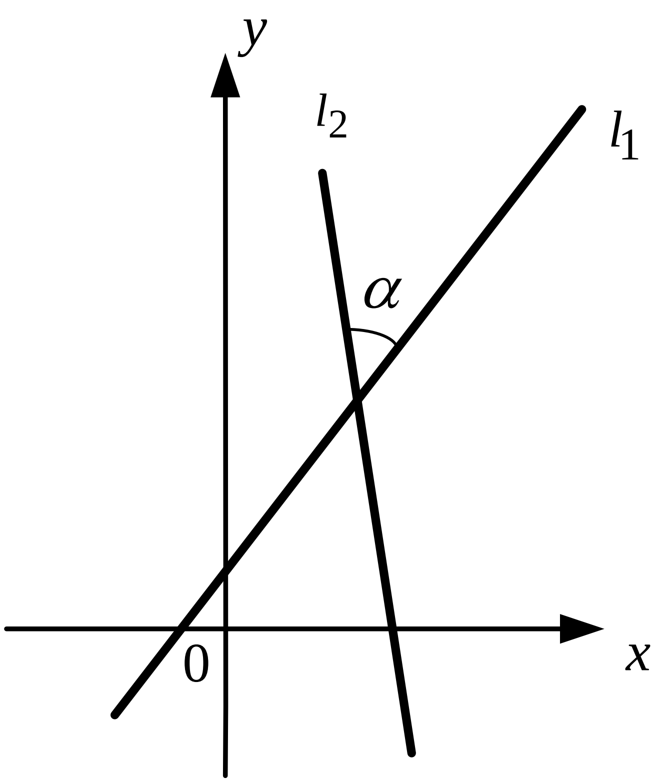
Example

Reduce the equation $12x - 5y + 60 = 0$ to a normal straight line equation.

2. Geometric relationship of two straight lines: an intersection of straight lines; conditions of parallelity and perpendicularity for straight lines; angle between two straight lines. Distance from the point to a given straight line

A tangent of the angle between two straight lines $y = k_1 x + b_1 \quad (l_1)$ and $y = k_2 x + b_2 \quad (l_2)$ is calculated with the help of the formula:

$$tg\alpha = \left| \frac{k_2 - k_1}{1 + k_1 k_2} \right|$$



This formula defines the acute angle between the straight lines.

A tangent of the angle between two straight lines is calculated with the help of the formula:

$$l_1:\ A_1x+B_1y+C_1=0;\quad l_2:\ A_2x+B_2y+C_2=0.$$

$$tg\varphi=\frac{A_1B_2-A_2B_1}{A_1A_2+B_1B_2}$$

This formula defines

the acute angle between

the straight lines.

A cosine of the angle between two straight lines is calculated with the help of the formula:

$$l_1:\ A_1x + B_1y + C_1 = 0; \quad l_2:\ A_2x + B_2y + C_2 = 0.$$

$$\cos\varphi = \frac{A_1A_2 + B_1B_2}{\sqrt{A_1^2 + B_2^2}\sqrt{A_2^2 + B_2^2}}$$

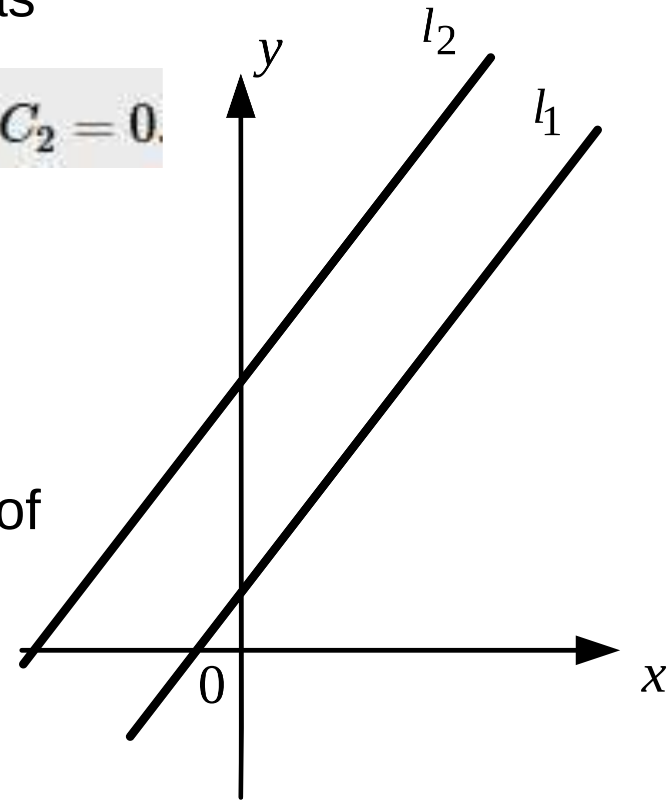
This formula defines the acute angle between the straight lines.

Condition for straight lines to be parallel

If equations of the straight lines are given as

$$l_1:\ A_1x + B_1y + C_1 = 0; \quad l_2:\ A_2x + B_2y + C_2 = 0.$$

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}$$



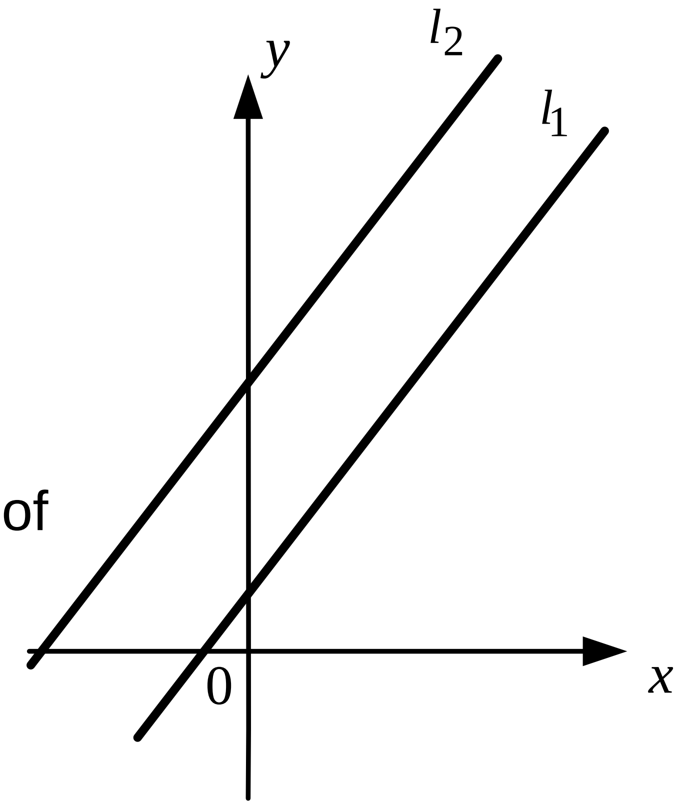
then we obtain the **collinearity condition** of these straight lines.

If $l_1 \parallel l_2$ then the angle between these lines equals $\alpha = 0$.

If equations of the straight lines are given as

$$y = k_1 x + b_1 \qquad (l_1)$$

$$y = k_2 x + b_2 \qquad (l_2)$$



then we obtain the **collinearity condition** of these straight lines.

If $l_1 \parallel l_2$ then the angle between these lines equals $\alpha = 0$.

$$tg\alpha = \left|\frac{k_2 - k_1}{1 + k_1 k_2}\right|$$

$$tg\,0 = 0 = \left|\frac{k_2 - k_1}{1 + k_1 k_2}\right|$$

$$\frac{k_2 - k_1}{1 + k_1 k_2} = 0$$

$$k_2 - k_1 = 0$$

C**ollinearity condition** of two straight lines is:

$$k_2 = k_1 \qquad b_1 \neq b_2.$$

EXAMPLE

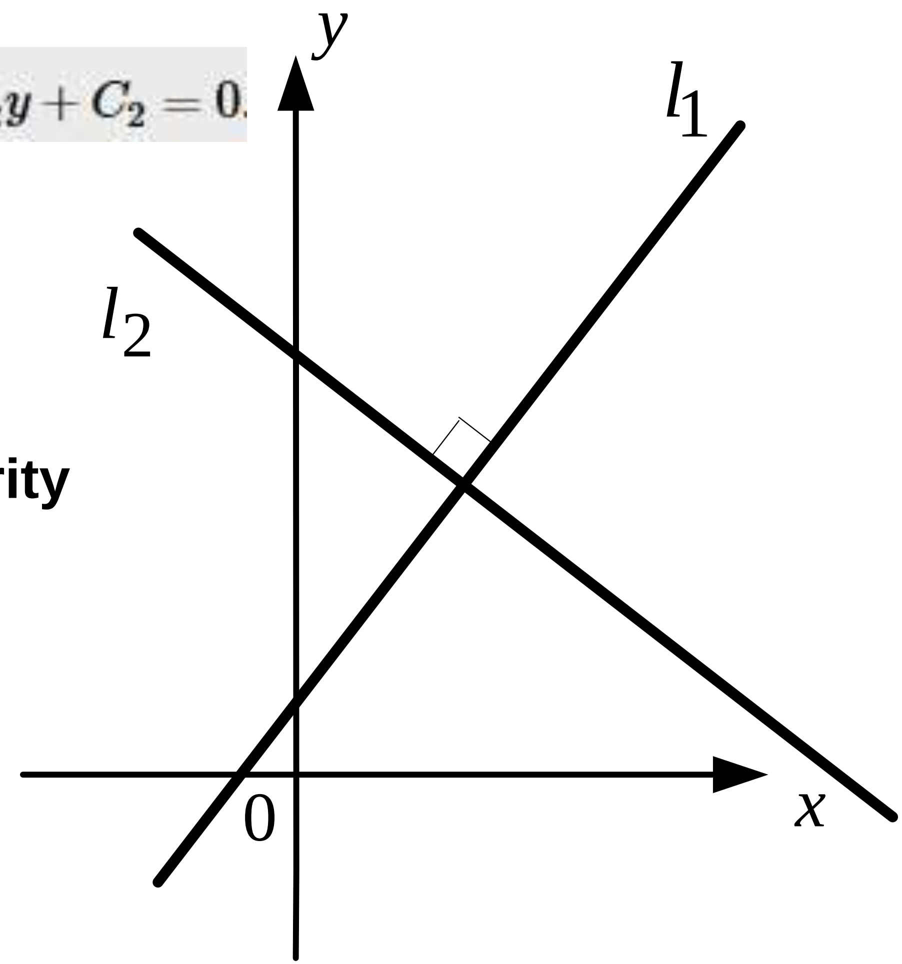
The straight lines $3x + 4y + 5 = 0$ and $3/2\,x + 2y + 6 = 0$ are parallel since the following condition is satisfied:

$$\frac{3}{3/2} = \frac{4}{2} \neq \frac{5}{6}.$$

Condition for straight lines to be perpendicular

If equations of the straight lines are given as

$$l_1:\ A_1x + B_1y + C_1 = 0; \quad l_2:\ A_2x + B_2y + C_2 = 0.$$



$$A_1 \cdot A_2 + B_1 \cdot B_2 = 0$$

then we obtain the **perpendicularity condition** of these straight lines.

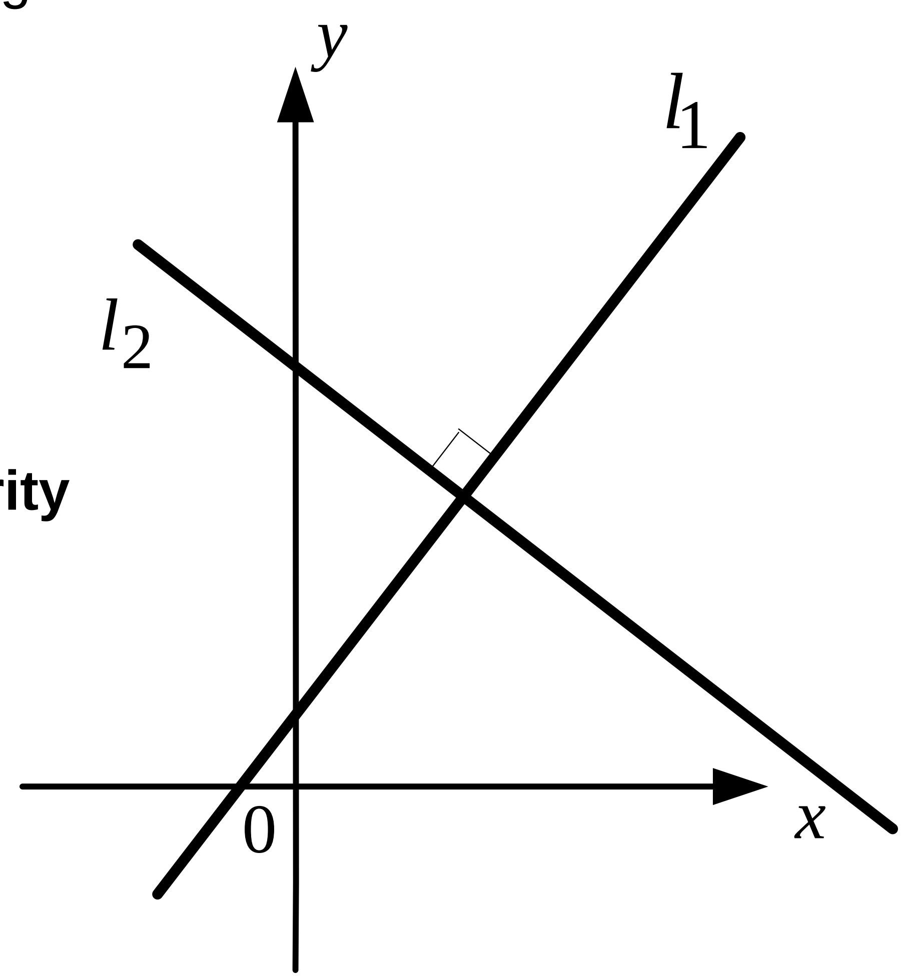
If $l_1 \perp l_2$ then the angle between these lines equals $\alpha = 90^{o}$.

If equations of the straight lines are given as

$$y = k_1 x + b_1 \qquad (l_1)$$

$$y = k_2 x + b_2 \qquad (l_2)$$

then we obtain the **perpendicularity condition** of these straight lines.



If $l_1 \perp l_2$ then the angle between these lines equals $\alpha = 90^{\circ}$.

$$tg\alpha = \left|\frac{k_2 - k_1}{1 + k_1 k_2}\right| \qquad ctg\alpha = \left|\frac{1 + k_1 k_2}{k_2 - k_1}\right|$$

$$ctg90^{o} = 0$$

$$\left|\frac{1 + k_1 k_2}{k_2 - k_1}\right| = 0$$

$$\frac{1 + k_1 k_2}{k_2 - k_1} = 0$$

$$1 + k_1 k_2 = 0$$

Perpendicularity condition of two straight lines is:

$$k_1 k_2 = -1$$

EXAMPLE

The lines $3x+y-3=0$ and $x-3y+8=0$ are perpendicular since they satisfy condition

$$A_1A_2+B_1B_2=3\cdot 1+1\cdot(-3)=0.$$

Condition for straight lines to coincide

For two straight lines given by slope-intercept equations to coincide, it is necessary and sufficient that

$$k_1 = k_2, \quad b_1 = b_2.$$

If the straight lines are given by general equations, then a necessary and sufficient condition for them to coincide has the form

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

Remark. Sometimes the case of coinciding straight lines is considered as a special case of parallel straight lines

Point of intersection of straight lines

Suppose that two straight lines are defined by general equations and consider the system of two first-order algebraic equations:

$$A_1x + B_1y + C_1 = 0,$$
$$A_2x + B_2y + C_2 = 0.$$

Each common solution of equations determines a common point of the two lines:

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = A_1B_2 - A_2B_1 \neq 0,$$

Point of intersection of straight lines

$$A_1x + B_1y + C_1 = 0,$$
$$A_2x + B_2y + C_2 = 0.$$

Each common solution of equations determines a common point of the two lines:

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = A_1B_2 - A_2B_1 \neq 0,$$

then the system is consistent and has a unique solution; hence these straight lines are distinct and nonparallel and meet at the point $A(x_0, y_0)$, , where

$$x_0 = \frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}, \quad y_0 = \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}.$$

Point of intersection of straight lines

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = A_1 B_2 - A_2 B_1 \neq 0,$$

This condition is often written as

$$\frac{A_1}{A_2} \neq \frac{B_1}{B_2} .$$

EXAMPLE

To find the point of intersection of the straight lines $y = 2x - 1$ and $y = -4x + 5$, we solve system

$$2x - y - 1 = 0,$$
$$-4x - y + 5 = 0,$$

and obtain $x = 1$, $y = 1$. Thus the intersection point has the coordinates $(1, 1)$.

Example

Find the angle between two straight lines:

$3x+2y-7=0$ and $3x+2y+15=0$.

Distance between parallel lines

The distance between the parallel lines given by equations

$$A_1x + B_1y + C_1 = 0 \quad \text{and} \quad A_1x + B_1y + C_2 = 0$$

can be found using the formula

$$d = \frac{|C_1 - C_2|}{\sqrt{A_1^2 + B_1^2}}.$$

An angular coefficient of a straight line

is calculated as:

$$k = -\frac{A}{B}$$

or

$$k = \frac{y_2 - y_1}{x_2 - x_1}$$

An area of a triangle

If $A(x_1;y_1)$, $B(x_2;y_2)$ and $C(x_3;y_3)$ are apexes of the triangle ***ABC***, then the area of the triangle is calculated by the formula:

$$S = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{or} \quad S = \pm\frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

The sign "+" or "–" is chosen according to this condition: *S* will be positive**.**

Remark

The condition that three points

$$A(x_1; y_1) \qquad B(x_2; y_2) \qquad C(x_3; y_3)$$

lie on the same plane is

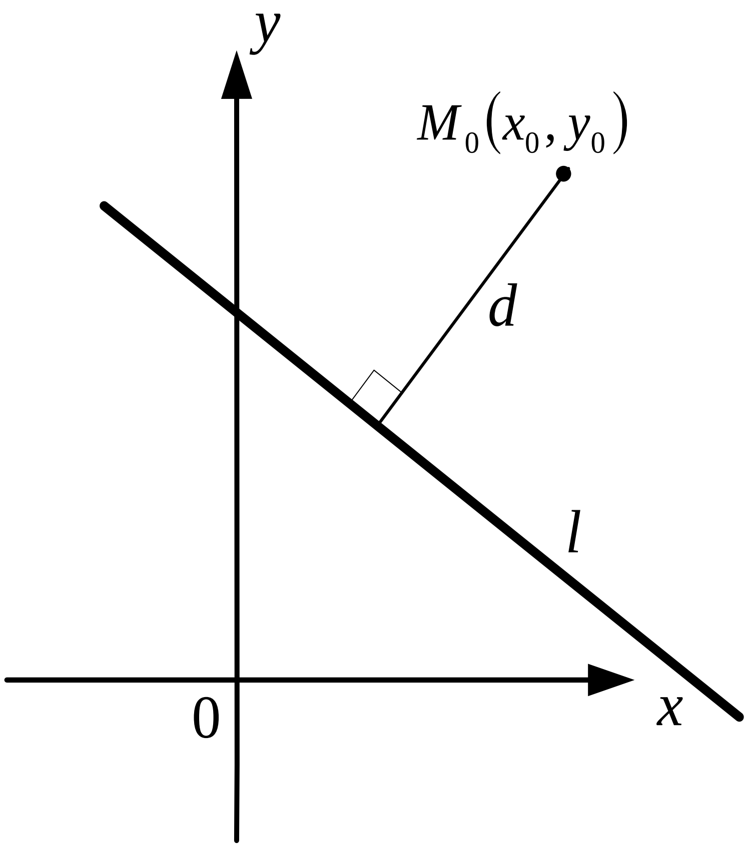
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

(the area of the corresponding triangle equals zero)

Distance *d* from point $M_0(x_0, y_0)$ to the straight line $Ax + By + C = 0$

is calculated according to the formula:

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$



Distance *d* between two points

$A(x_1, y_1)$ **and** $B(x_2, y_2)$

is calculated according to the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example

Find the distance between two parallel straight lines:

$3x + 2y - 7 = 0$ and $3x + 2y + 15 = 0$

Task

The coordinates apexes of a triangle ***ABC*** are given as

$A(-2,-2)$ $B(4,1)$ $C(0,4)$

Using methods of the analytical geometry do the following:

1) find the distance between point *A* and point B ;

1) find the distance between point *A* and point B

$$AB=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}=\sqrt{(4-(-2))^2+(1-(-2))^2}=$$

$$=\sqrt{(4+2)^2+(1+2)^2}=\sqrt{6^2+3^2}=\sqrt{36+9}=\sqrt{45}=3\sqrt{5}$$

Task

The coordinates apexes of a triangle ***ABC*** are given as

$$A(-2,-2) \qquad B(4,1) \qquad C(0,4)$$

Using methods of the analytical geometry do the following:

2) form equation of the sides AB and AC ;

2) form equations of the sides AB and AC ;

$$\frac{y-y_A}{y_B-y_A}=\frac{x-x_A}{x_B-x_A}$$

$$\frac{y-(-2)}{1-(-2)}=\frac{x-(-2)}{4-(-2)}$$

$$y=\frac{1}{2}x-1 \qquad k_{AB}=\frac{1}{2}$$

$$\frac{y-y_A}{y_C-y_A}=\frac{x-x_A}{x_C-x_A}$$

$$\frac{y-(-2)}{4-(-2)}=\frac{x-(-2)}{0-(-2)}$$

$$y=3x+4 \qquad k_{AC}=3$$

Task

The coordinates apexes of a triangle ***ABC*** are given as

$$A(-2,-2) \qquad B(4,1) \qquad C(0,4)$$

Using methods of the analytical geometry do the following:

3) form equation of the altitude dropped from the apex C

3) form equation of the altitude CN dropped from the apex C

$$y - y_C = k_{CN}(x - x_C)$$

$$k_{CN} \cdot k_{AB} = -1$$

$$k_{CN} = -\frac{1}{k_{AB}}$$

$$k_{AB} = \frac{1}{2}$$

$$y - 4 = -\frac{1}{1/2}(x - 0)$$

$$y = -2x + 4$$

Task

The coordinates apexes of a triangle ***ABC*** are given as

$$A(-2,-2) \qquad B(4,1) \qquad C(0,4)$$

Using methods of the analytical geometry do the following:

4) find the inner angle of the triangle at the apex A;

4) find the inner angle of the triangle at the apex A

$$tg\alpha = \left|\frac{k_{AC} - k_{AB}}{1 + k_{AC} \cdot k_{AB}}\right| = \left|\frac{3 - 1/2}{1 + 3 \cdot 1/2}\right| = \left|\frac{5/2}{5/2}\right| = 1$$

$$\alpha = arctg1 = \frac{\pi}{4}$$

Task

The coordinates apexes of a triangle ***ABC*** are given as

$$A(-2,-2) \qquad B(4,1) \qquad C(0,4)$$

Using methods of the analytical geometry do the following:

5) calculate length of the altitude CN dropped from the apex C;

5) calculate length of the altitude CN dropped from the apex C

$$y = \frac{1}{2}x - 1 \qquad\qquad x - 2y - 1 = 0$$

$$C(0,4)$$

$$CN = \frac{|x_C - 2 \cdot y_C - 2|}{\sqrt{1^2 + (-2)^2}} = \frac{|0 - 2 \cdot 4 - 2|}{\sqrt{1+4}} = \frac{|0 - 8 - 2|}{\sqrt{5}} = \frac{10}{\sqrt{5}}$$

Task

The coordinates apexes of a triangle ***ABC*** are given as

$$A(-2,-2) \qquad B(4,1) \qquad C(0,4)$$

Using methods of the analytical geometry do the following:

6) find the area of the triangle ABC;

6) find the area of the triangle ABC

$$S_{\Delta ABC}=\frac{1}{2}AB\cdot CN=\frac{1}{2}\cdot 3\sqrt{5}\cdot\frac{10}{\sqrt{5}}=15$$

Task

The coordinates apexes of a triangle ***ABC*** are given as

$$A(-2,-2) \qquad B(4,1) \qquad C(0,4)$$

Using methods of the analytical geometry do the following:

7) form equation of the median CM dropped from the apex C;

7) form equation of the median CM dropped from the apex C

$$x_M = \frac{x_A + x_B}{2} = \frac{-2+4}{2} = \frac{2}{2} = 1$$

$$y_M = \frac{y_A + y_B}{2} = \frac{-2+1}{2} = -\frac{1}{2}$$

$$\frac{y - y_M}{y_C - y_M} = \frac{x - x_M}{x_C - x_M}$$

$$\frac{y - \left(-\frac{1}{2}\right)}{4 - \left(-\frac{1}{2}\right)} = \frac{x-1}{0-1}$$

$$y = -\frac{9}{2}x + 4$$

Task

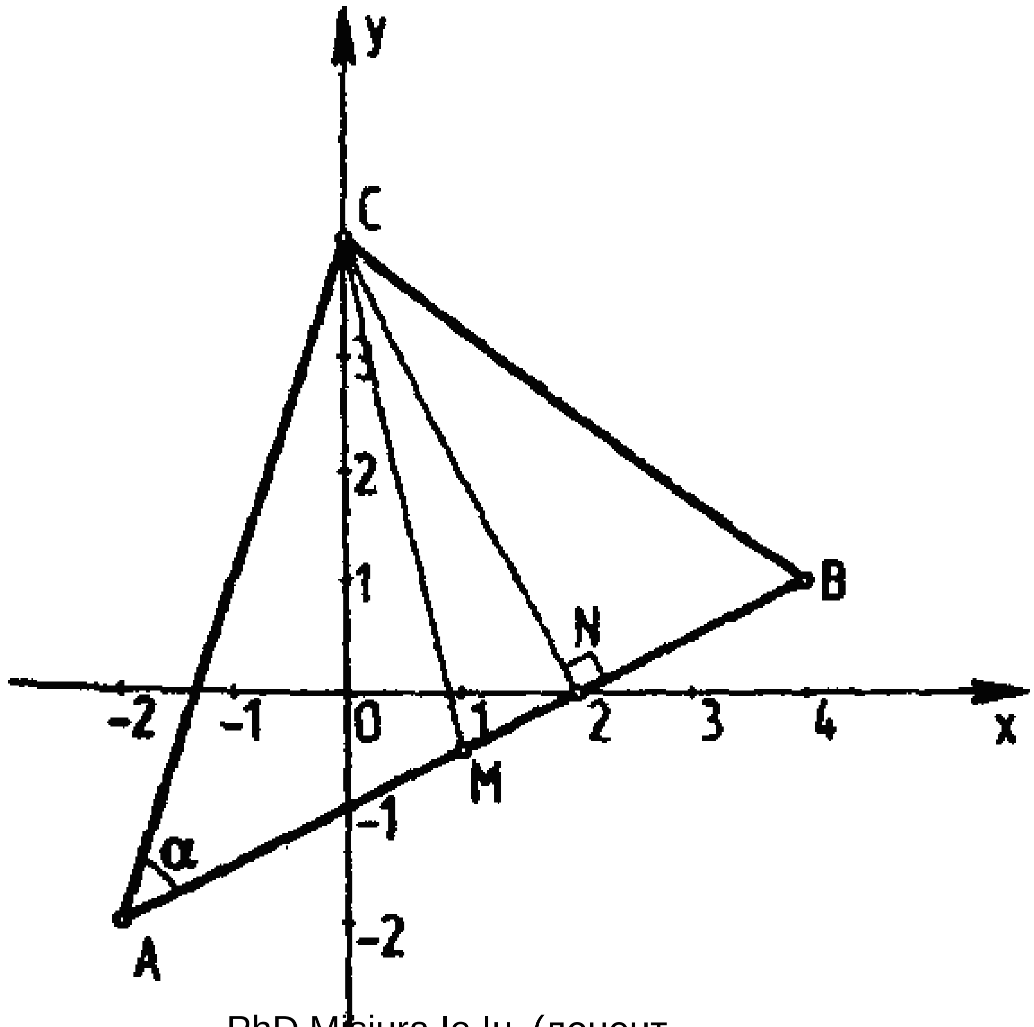
The coordinates apexes of a triangle ***ABC*** are given as

$$A(-2,-2) \qquad B(4,1) \qquad C(0,4)$$

Using methods of the analytical geometry do the following:

8) draw the triangle ABC

8) the picture



TASK 1

Find the distance from the point *A*(4; 3) to the straight line: *x* = 3+ 2*t*, *y* = −4 + 5*t*, *t*є***R***.

$$t = \frac{x-3}{2} = \frac{y+4}{5} \Rightarrow 5(x-3) = 2(y+4) \Leftrightarrow 5x - 2y - 23 = 0$$

$$\rho(A, \ell) = \frac{|5 \cdot 4 - 2 \cdot 3 - 23|}{\sqrt{5^2 + (-2)^2}} = \frac{9}{\sqrt{29}}.$$

TASK 2

Write down the equation of the straight line passing through the point $M(-6;\ 12/5)$ and forming the triangle with the area equal to 30.

$$(1)\ \frac{x}{30}+\frac{y}{2}=1 \Leftrightarrow x+15y-30=0;\quad (2)\ \frac{x}{-5}+\frac{y}{-12}=1 \Leftrightarrow 12x+5y+60=0;$$

$$(3)\ \frac{x}{-15}+\frac{y}{4}=1 \Leftrightarrow 4x-15y+60=0;\quad (4)\ \frac{x}{-10}+\frac{y}{6}=1 \Leftrightarrow 3x-5y+30=0$$

TASK 3

Three points $A(-1;\ 4)$, $B(5;1)$ $C(5;\ 6)$ are given. Find:

a) coordinates of the point D as the projection of the point C on the side AB;

b) coordinates of the point E as the symmetrical point C relative to the side AB.

Curves of the second order

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Lecture plan

1. Curve of the second order. A general equation
2. Circle. The canonical equation of circle
3. Ellipse. The canonical equation of ellipse
4. Hyperbola. The canonical equation of hyperbola
5. Parabola. The canonical equation of parabola

1. Curve of the second order. A general equation

Definition. ***Curve of the second order*** on a plane is called a set of points which coordinates are of the same system as the Cartesian coordinates satisfy the following equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E, F are any numbers, and

$$A^2 + B^2 + C^2 \neq 0$$

According to the sign of the value $AC - B^2$ the curves of the second order are divided by three types:

1) elliptic if $AC - B^2 > 0$

2) hyperbolic if $AC - B^2 < 0$

3) parabolic if $AC - B^2 = 0$

Example 1

What type is it?

$$x^2 + y^2 - 4x + 8y - 16 = 0$$

Example 2

What type is it?

$$9x^2 + 4y^2 - 18x - 8y - 23 = 0$$

Example 3

What type is it?

$$x^2 - 4y^2 + 6x + 16y - 11 = 0$$

Example 4

What type is it?

$$2y^2 + x - 8y + 3 = 0$$

Basic formulas:

$$(x-x_0)^2+(y-y_0)^2=R^2$$

$$(y-y_0)^2=2p(x-x_0)$$

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$$

$$(x-x_0)^2=2p(y-y_0)$$

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$$

2. Circle.

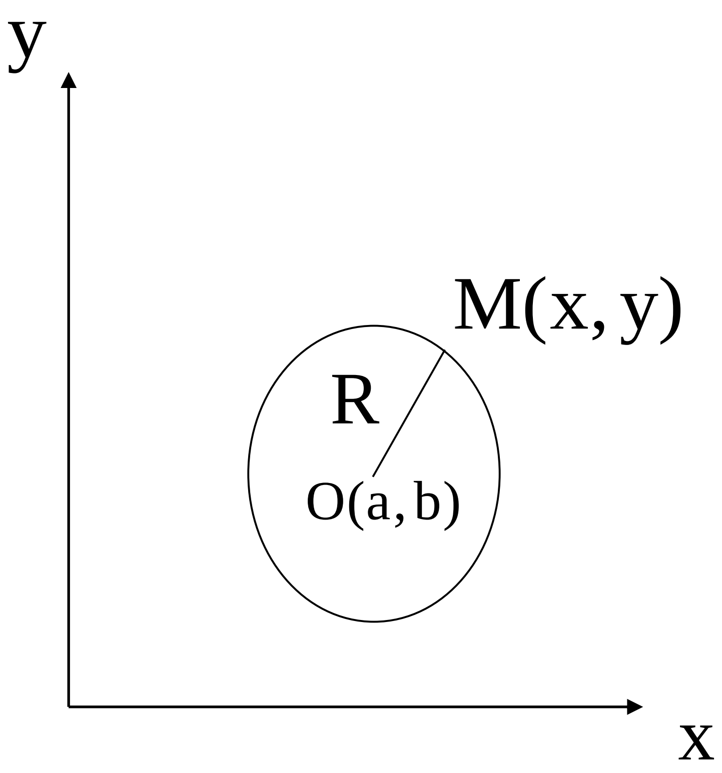
The canonical equation of circle

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

If $A = C$ and $B = 0$, we have an equation of *circle*.

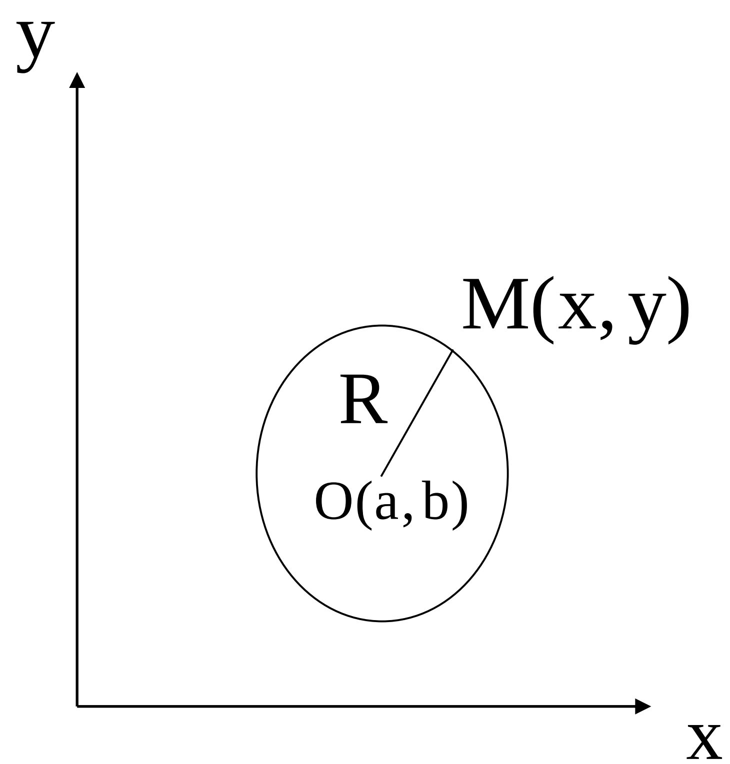
Definition. *Circle* is a geometrical place of points equidistant from a certain point called centre.

Definition. *Circle* is a geometrical place of points equidistant from a certain point called centre.



$O(a,b)$ is the center; $M(x,y)$ is any point of the circle

R is radius;



$O(a,b)$ is the center; $M(x, y)$ is any point of the circle

R is radius;

$$(x-a)^2+(y-b)^2=R^2$$

It is the desired ***equation of a circle***. If the center is the origin of coordinates, then we have the following equation of a circle:

$$x^2+y^2=R^2$$

Example 5

Find coordinates of centre ***O*** and radius ***R*** of circle if the second order equation is known as

$$2x^2+2y^2-8x+5y-4=0$$

Example 5

Find coordinates of centre ***O*** and radius ***R*** of circle if the second order equation is known as

$$2x^2+2y^2-8x+5y-4=0$$

$$(x-2)^2+\left(y+\frac{5}{4}\right)^2=\frac{121}{16}$$

$$O\left(2,-\frac{5}{4}\right)$$

$$R=\frac{11}{4}$$

3. Ellipse.
The canonical equation of ellipse

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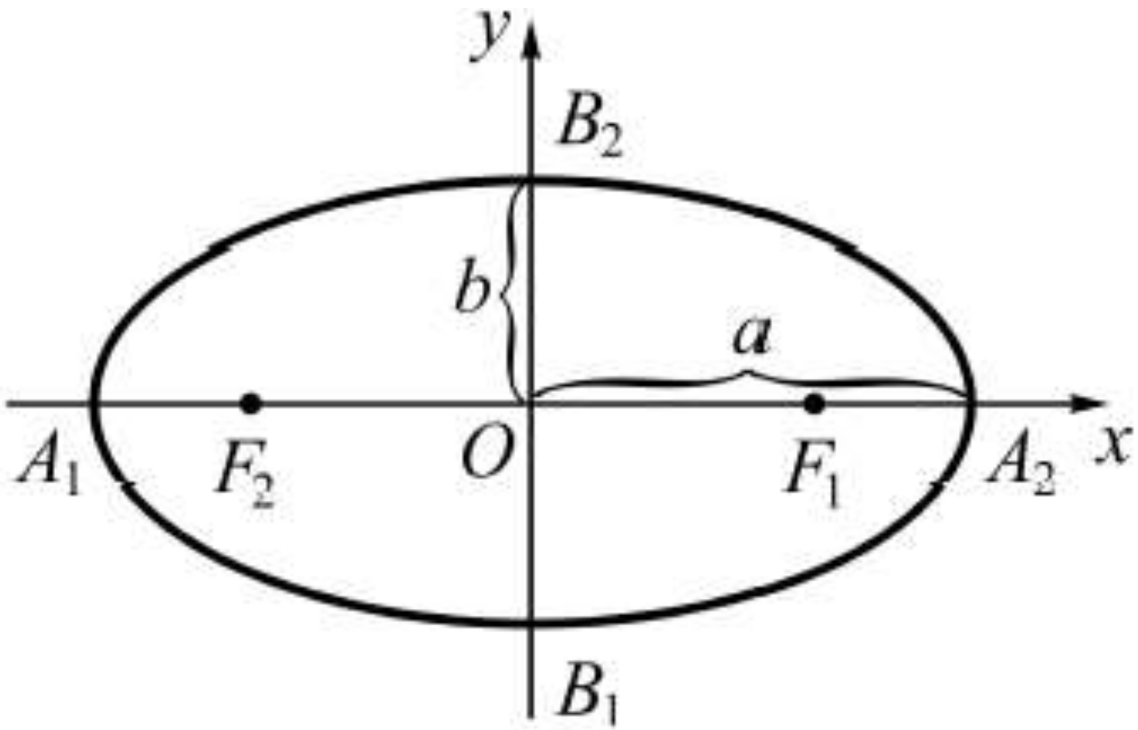
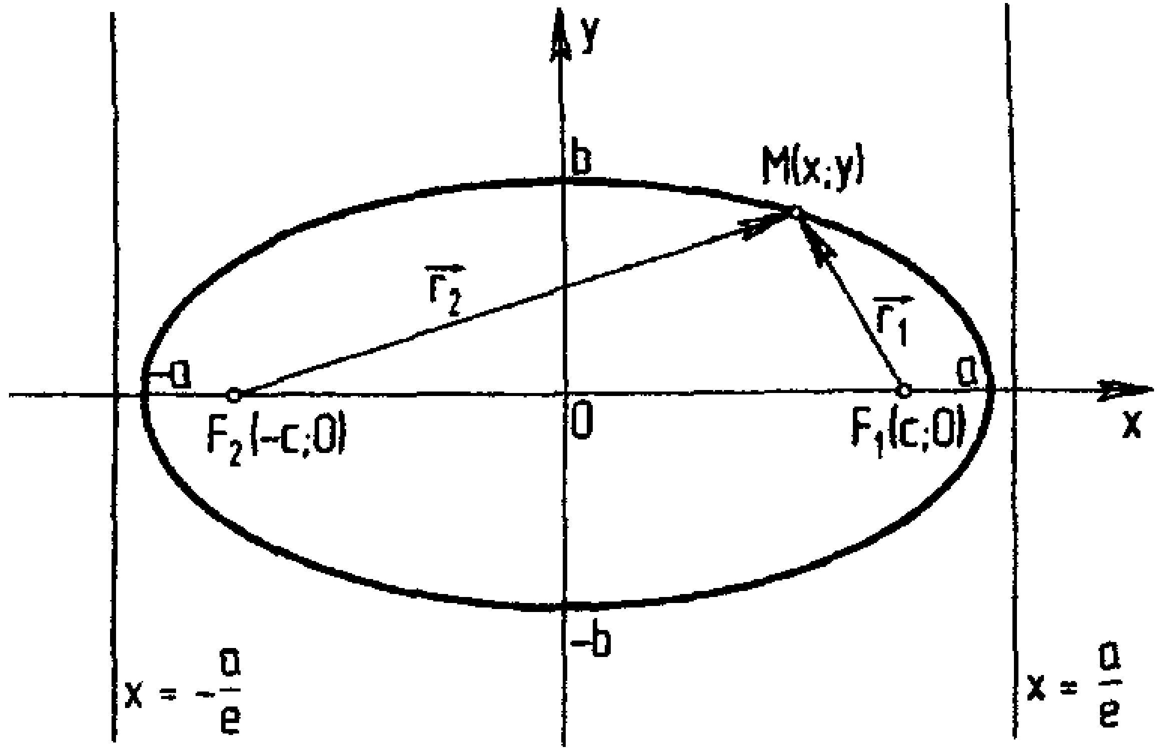
Definition. ***Ellipse*** is a geometrical place of the points which sum of distances to two certain points called focuses is constant (usually considered equal to $2a$) greater than the distance between focuses.



$$A_1(-a,0),\ A_2(a,0),\ B_1(0,-b),\ B_2(0,b)$$

are called ***apexes of an ellipse***



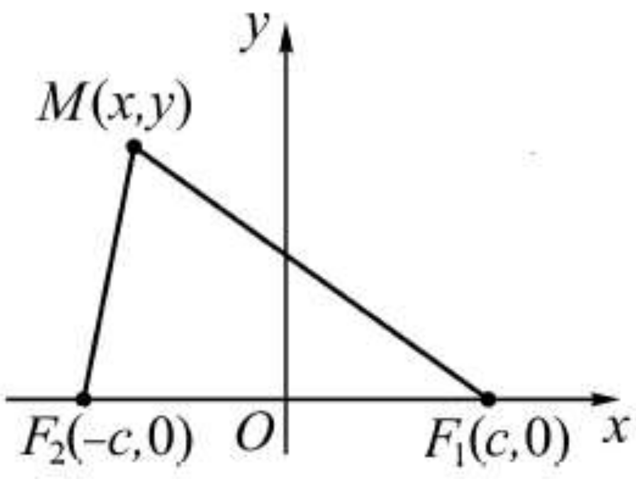
$$|MF_1| + |MF_2| = 2a \qquad (2a > 2c)$$

$2c$ is the distance between the focuses

Proofing

We have

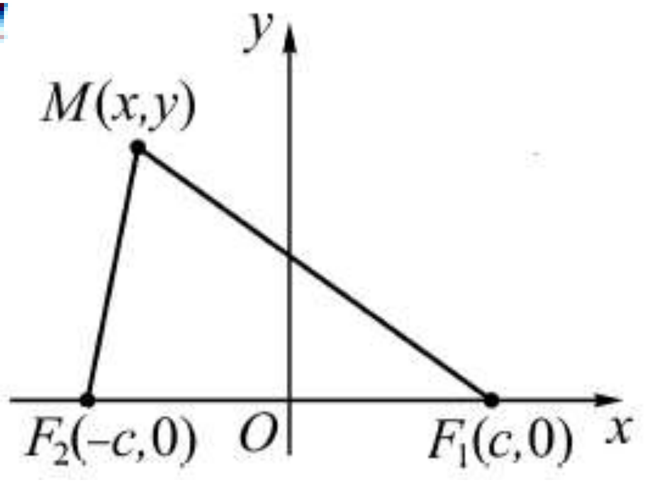
$$F_1M + F_2M = 2a$$



Proofing

We have $F_1M + F_2M = 2a$

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

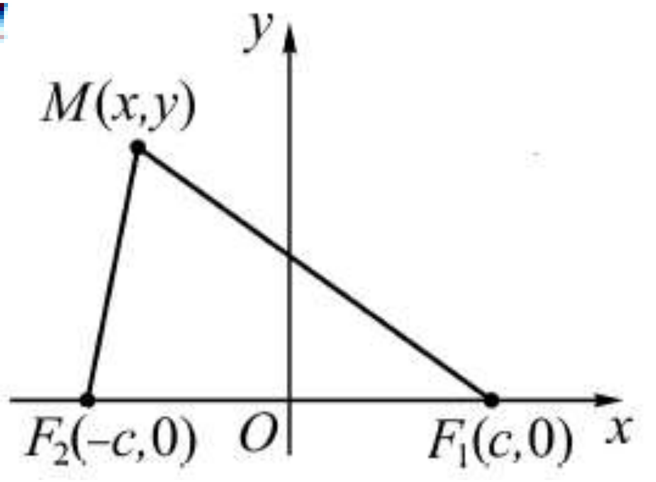


Proofing

We have $F_1M + F_2M = 2a$

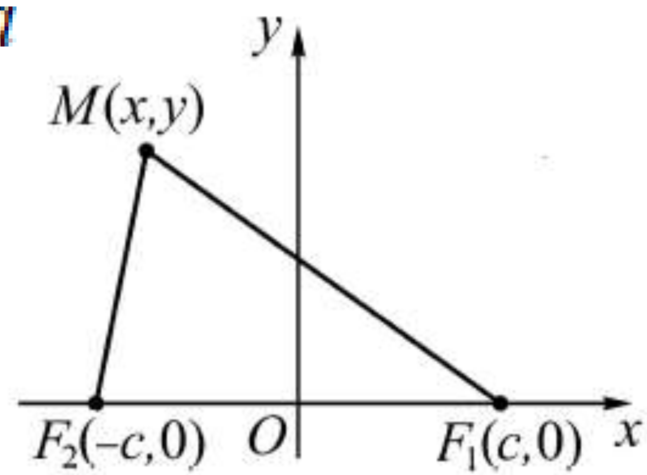
$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\sqrt{(x-c)^2+y^2}=2a-\sqrt{(x+c)^2+y^2}$$



Proofing

We have $F_1M + F_2M = 2a$



$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\sqrt{(x-c)^2+y^2}=2a-\sqrt{(x+c)^2+y^2}$$

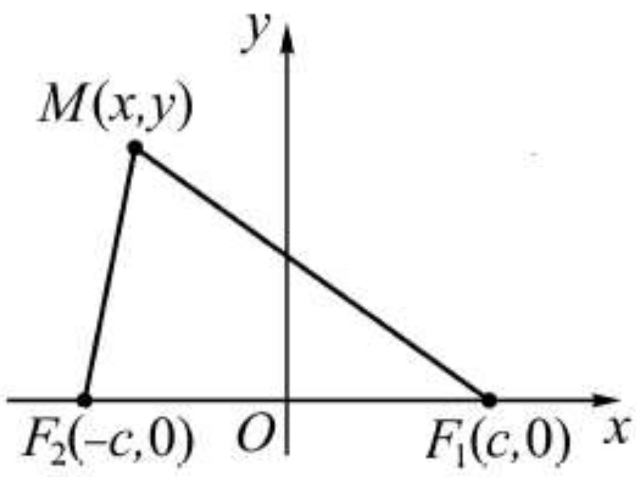
$$(x-c)^2+y^2=4a^2-4a\sqrt{(x+c)^2+y^2}+(x+c)^2+y^2 \Rightarrow$$

$$\Rightarrow 4a\sqrt{(x+c)^2+y^2}=4a^2+4cx \Rightarrow$$

Proofing

$$\Rightarrow\ a^2\left(x^2+2cx+c^2+y^2\right)=a^4+2a^2cx+c^2x^2\ \Rightarrow$$

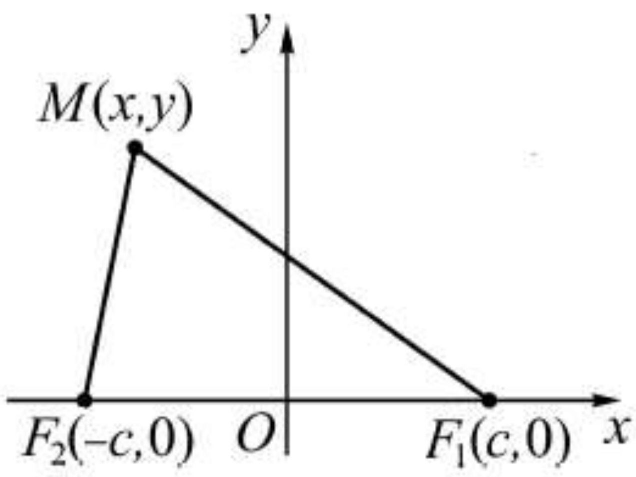
$$\Rightarrow\ a^2x^2+a^2c^2+a^2y^2=a^4+c^2x^2\ \Rightarrow$$



Proofing

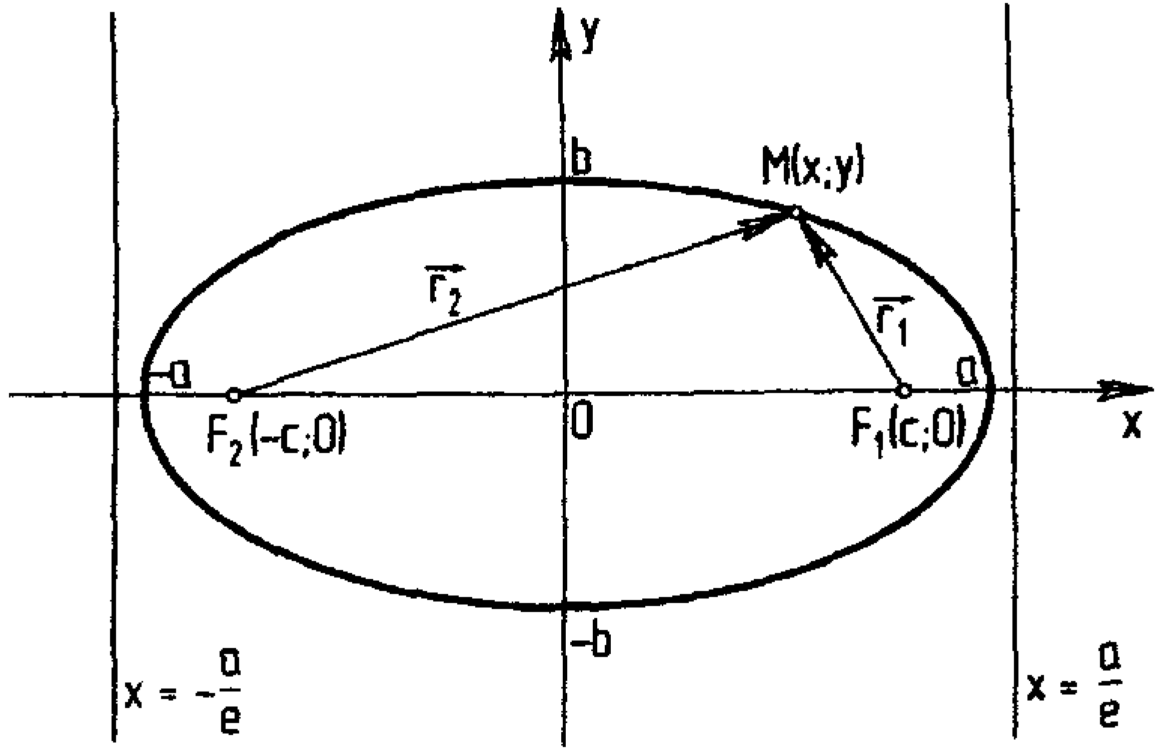
$$\Rightarrow\ a^2\left(x^2+2cx+c^2+y^2\right)=a^4+2a^2cx+c^2x^2\ \Rightarrow$$

$$\Rightarrow\ a^2x^2+a^2c^2+a^2y^2=a^4+c^2x^2\ \Rightarrow$$



$$\Rightarrow\ \left(a^2-c^2\right)x^2+a^2y^2=a^2\left(a^2-c^2\right)\Rightarrow\left|\ a^2-c^2=b^2\ \right|\Rightarrow$$

$$\Rightarrow\ b^2x^2+a^2y^2=a^2b^2\ \Rightarrow\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

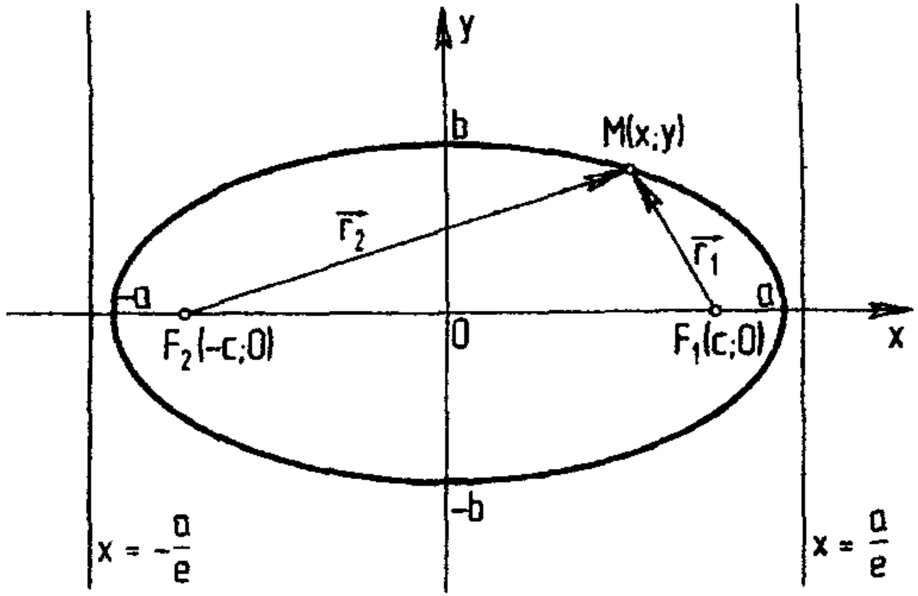
Its equation is called ***a canonical equation of an ellipse***

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Its equation is called ***a canonical equation of an ellipse***

$2a$ is the length of major axis of an ellipse

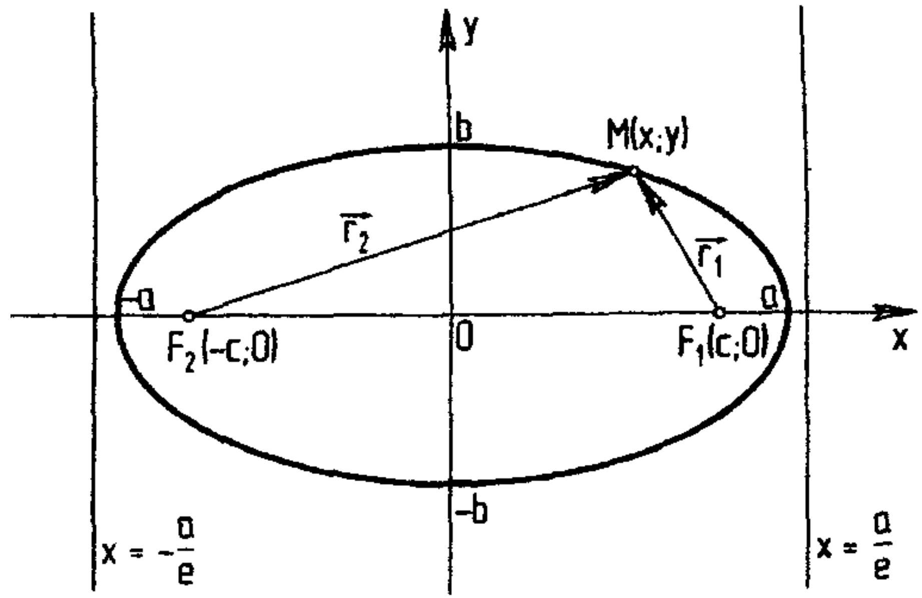
$2b$ is the length of minor axis of an ellipse



$$a > 0 \qquad b > 0$$

are called ***the major and minor semi-axes of an ellipse***. They are connected by the following equality:

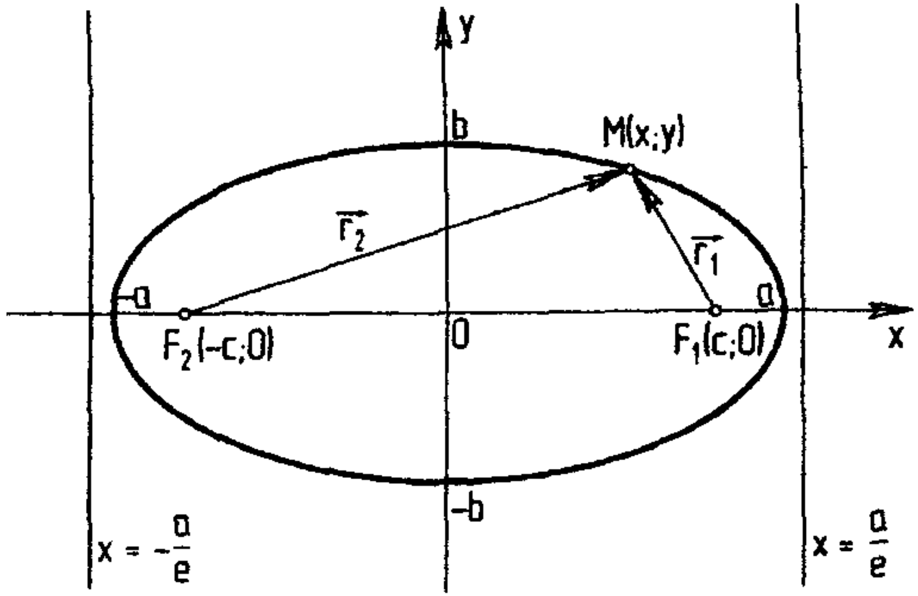
$$c^2 = a^2 - b^2 \qquad c = \sqrt{a^2 - b^2}$$



Ratio of half of focal distance to the length of half of a major axis is called ***eccentricity*** of an ellipse

$$e = \frac{c}{a}, \quad 0 < e < 1$$

$$0 < e < 1$$



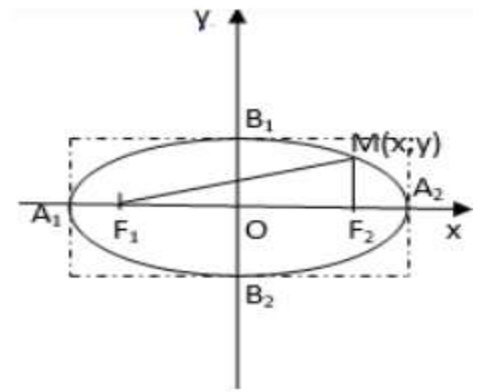
Ratio of half of focal distance to the length of half of a major axis is called ***eccentricity*** of an ellipse

$$e = \frac{c}{a} \Rightarrow \left| c = \sqrt{a^2 - b^2} \right| \Rightarrow \left(e = \frac{1}{a}\sqrt{a^2 - b^2},\; e^2 = 1 - \left(\frac{b}{a}\right)^2 \right)$$

A GENERAL NOTE: STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER $(0, 0)$

The standard form of the equation of an ellipse with center $(0, 0)$ and **major axis** parallel to the *x*-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



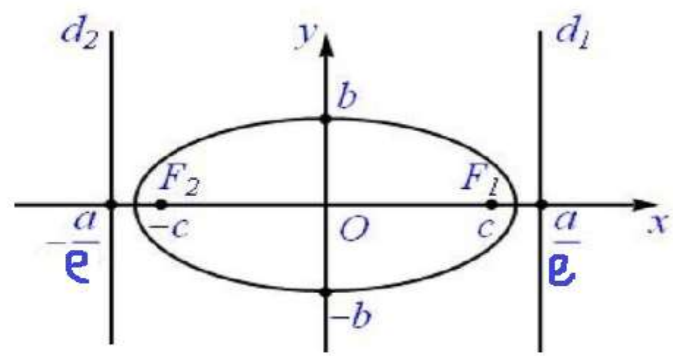
where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the coordinates of the foci are $F_1; F_2 \in Ox$ where $a^2 - b^2 = c^2$

  $F_1(-c; 0)$, $F_2(c; 0)$

- an eccentricity is $e = c/a$
- directrices are

$$d_1 : x = \frac{a}{e}, \ d_2 : x = -\frac{a}{e}$$



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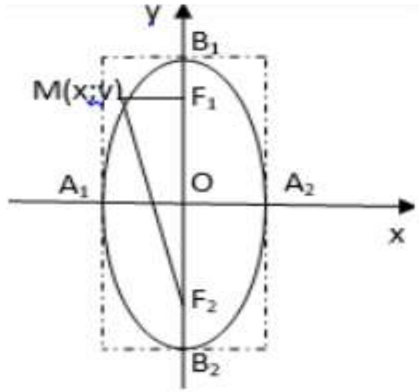
A GENERAL NOTE: STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER $(0,0)$

The standard form of the equation of an ellipse with center $(0,0)$ and **major axis** parallel to the y-axis is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



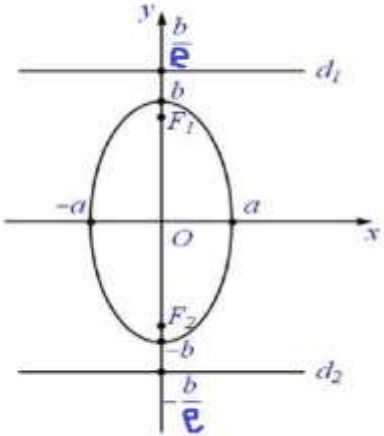
where

- $b>a$
- the length of the major axis is $2b$
- the coordinates of the co-vertices are $(\pm a,0)$
- the length of the minor axis is $2a$
- the coordinates of the vertices are $(0,\pm b)$
- the coordinates of the foci are $F_1; F_2 \in Oy$ where $b^2-a^2=c^2$

$F_1(0;\ c),\ F_2(0;\ -c)$

- an eccentricity is **e=c/b**
- directrices are

$$d_1: y=\frac{b}{e},\quad d_2: y=-\frac{b}{e}$$



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Example : Find the vertices of ellipse having the equation

$$\frac{x^2}{36} + \frac{y^2}{25} = 1.$$

Solution:

The given equation of the ellipse is $\frac{x^2}{36} + \frac{y^2}{25} = 1.$

Comparing this with the standard equation of the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we have $a^2 = 36$, $b^2 = 25$.

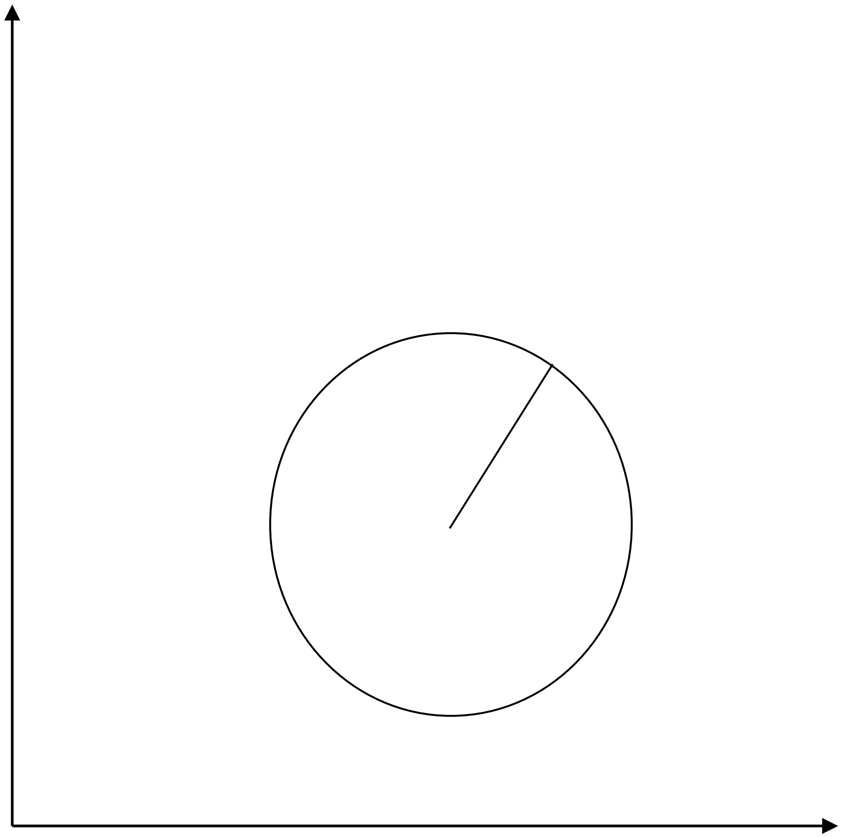
Hence we have a = 6, and b = 5.

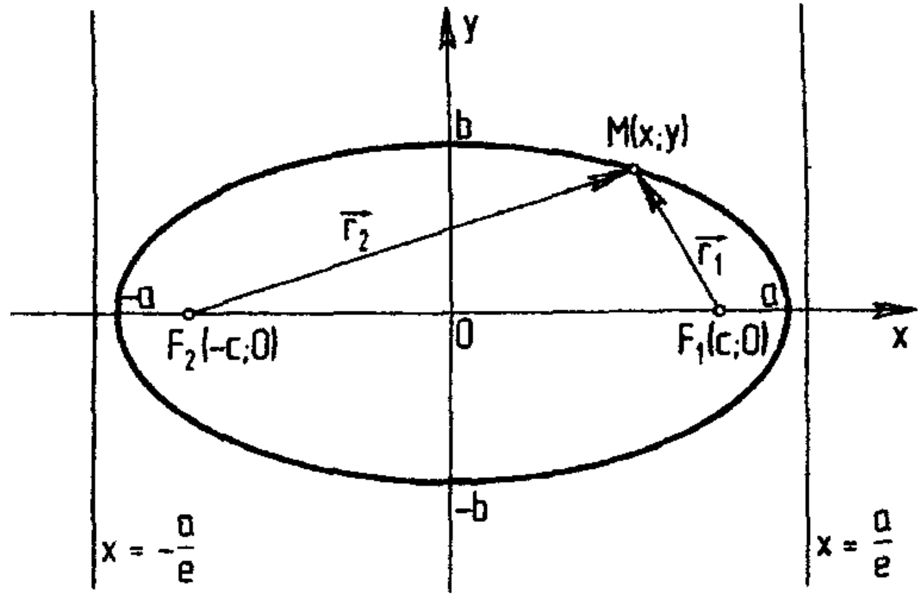
From this we can derive that the vertex of the ellipse is $(\pm a, 0) = (\pm 6, 0)$.

Therefore, the vertices of the ellipse are (+6, 0), (-6, 0).

$a = b$

$e = 0$





The lines $x = \pm \dfrac{a}{e}$ are called directrixes of an ellipse.

TASK 2

Reduce the given equation

$$16x^2 + 25y^2 - 32x + 50y - 359 = 0$$

to a canonical form and draw the curve.

TASK 2

Reduce the given equation

$$16x^2 + 25y^2 - 32x + 50y - 359 = 0$$

to a canonical form and draw the curve.

$$A \cdot C - B^2 = 16 \cdot 25 - 0^2 = 400 > 0$$

TASK 2

Reduce the given equation

$$16x^2+25y^2-32x+50y-359=0$$

to a canonical form and draw the curve.

$$16\left(x^2-2x+1\right)-16+25\left(y^2+2y+1\right)-25-359=0$$

TASK 2

Reduce the given equation

$$16x^2 + 25y^2 - 32x + 50y - 359 = 0$$

to a canonical form and draw the curve.

$$\frac{(x-1)^2}{25} + \frac{(y+1)^2}{16} = 1$$

4. Hyperbola.
The canonical equation of hyperbola

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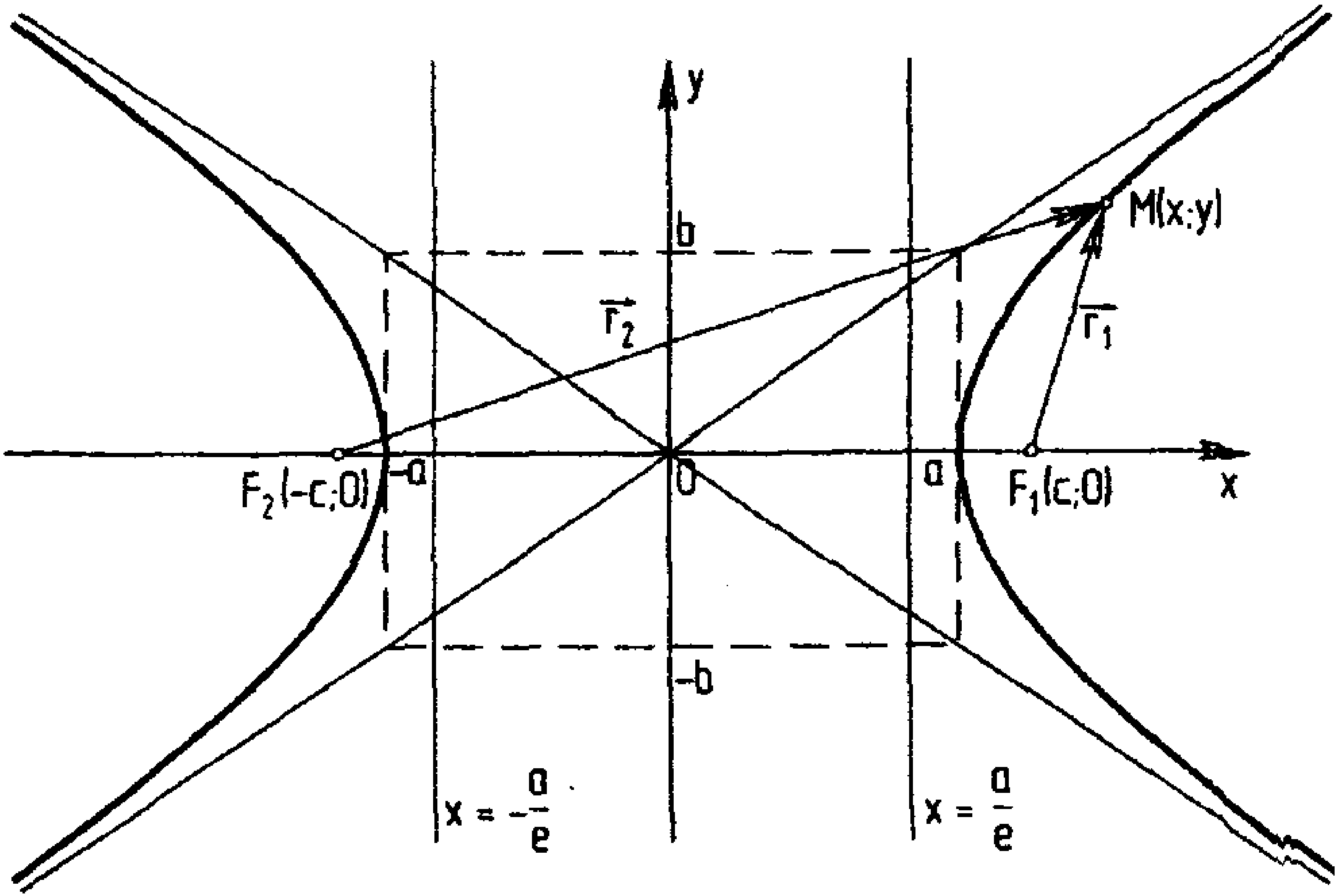
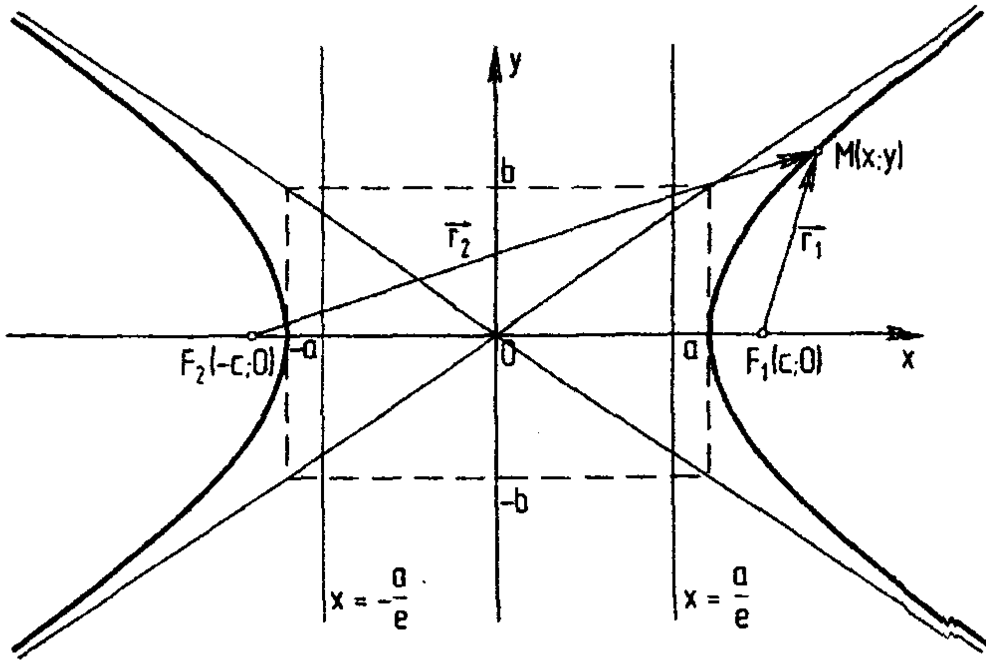
Definition. ***Hyperbola*** is a geometrical place of points which absolute difference of distances to two certain points called focuses is constant (usually considered equal $2a$) less than the distance between focuses.



y
M(x;y)
b
$\vec{r_2}$
$\vec{r_1}$
$F_2(-c;0)$
-a
0
a
$F_1(c;0)$
x
-b
$x = -\frac{a}{e}$
$x = \frac{a}{e}$



$$|MF_1| - |MF_2| = \pm 2a$$

$2c$ is the distance between the focuses

Proofing

We have

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a\,,$$

$$\sqrt{(x+c)^2+y^2}=\sqrt{(x-c)^2+y^2}\pm 2a\,.$$

Proofing

We have $\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a$,

$$\sqrt{(x+c)^2+y^2}=\sqrt{(x-c)^2+y^2}\pm 2a.$$

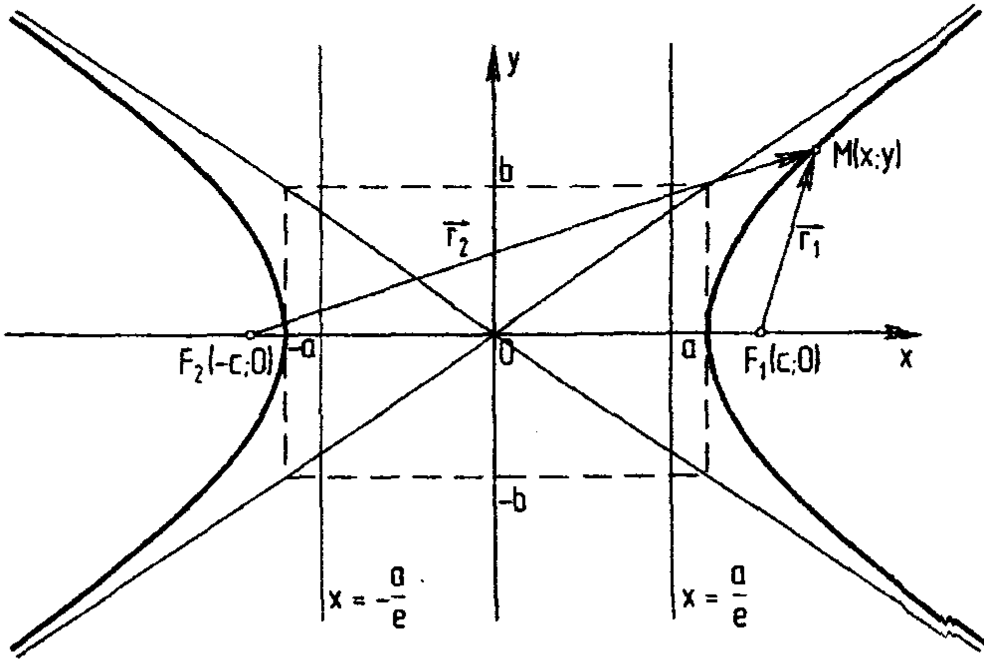
$$(x+c)^2+y^2=(x-c)^2+y^2\pm 4a\sqrt{(x-c)^2+y^2}+4a^2 \Rightarrow$$

$$\Rightarrow 4cx-4a^2=\pm 4a\sqrt{(x-c)^2+y^2} \Rightarrow$$

$$\Rightarrow c^2x^2-2a^2cx+a^4=a^2\left(x^2-2cx+c^2+y^2\right)\Rightarrow$$

$$\Rightarrow \left(c^2-a^2\right)x^2-a^2y^2=a^2\left(c^2-a^2\right)\Rightarrow \left| c^2-a^2=b^2 \right| \Rightarrow$$

$$\Rightarrow b^2x^2-a^2y^2=a^2b^2 \Rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Its equation is called ***a canonical equation of hyperbola***

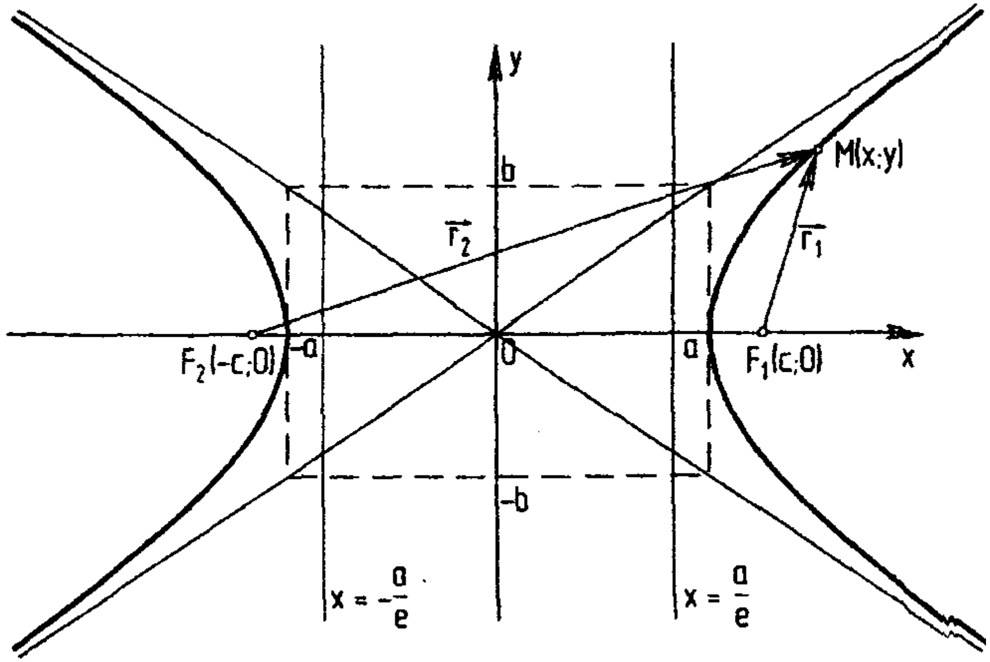
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Its equation is called ***a canonical equation of an ellipse***

$2a$ is the length of real axis of hyperbola

$2b$ is the length of imaginary axis of hyperbola

$$c^2 = a^2 + b^2$$

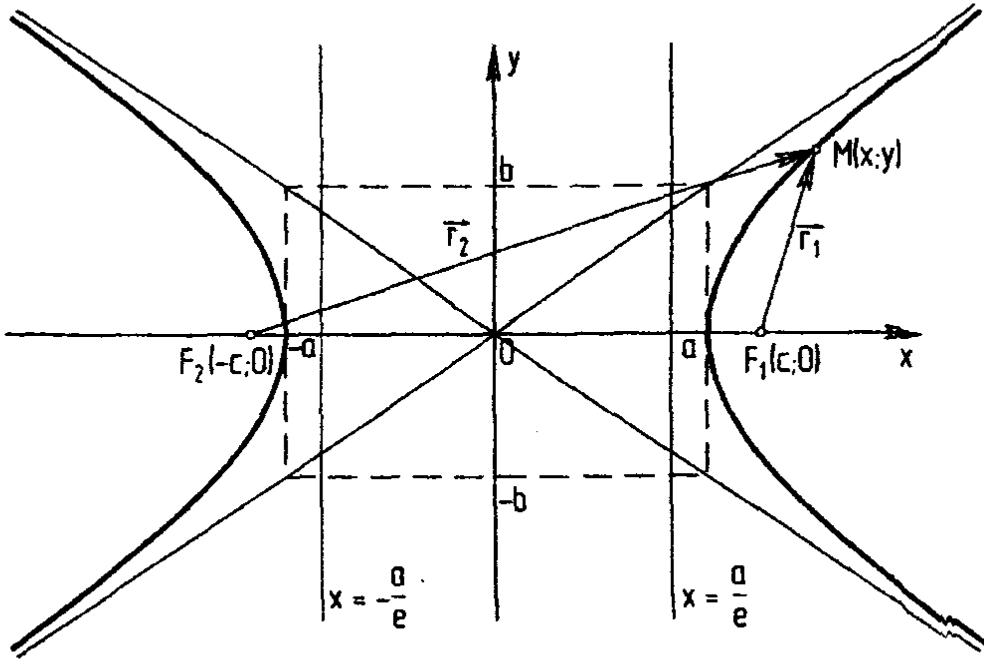


$$A_1(-a,0),\ A_2(a,0)$$

are called ***apexes of hyperbola***

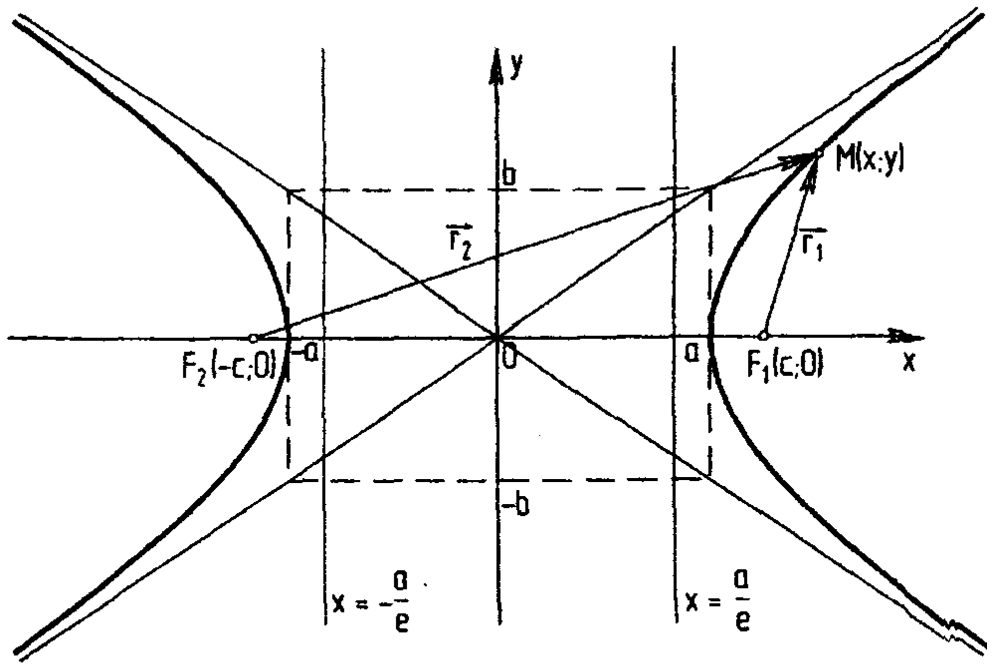
$$e = \frac{c}{a}$$

$$e > 1$$

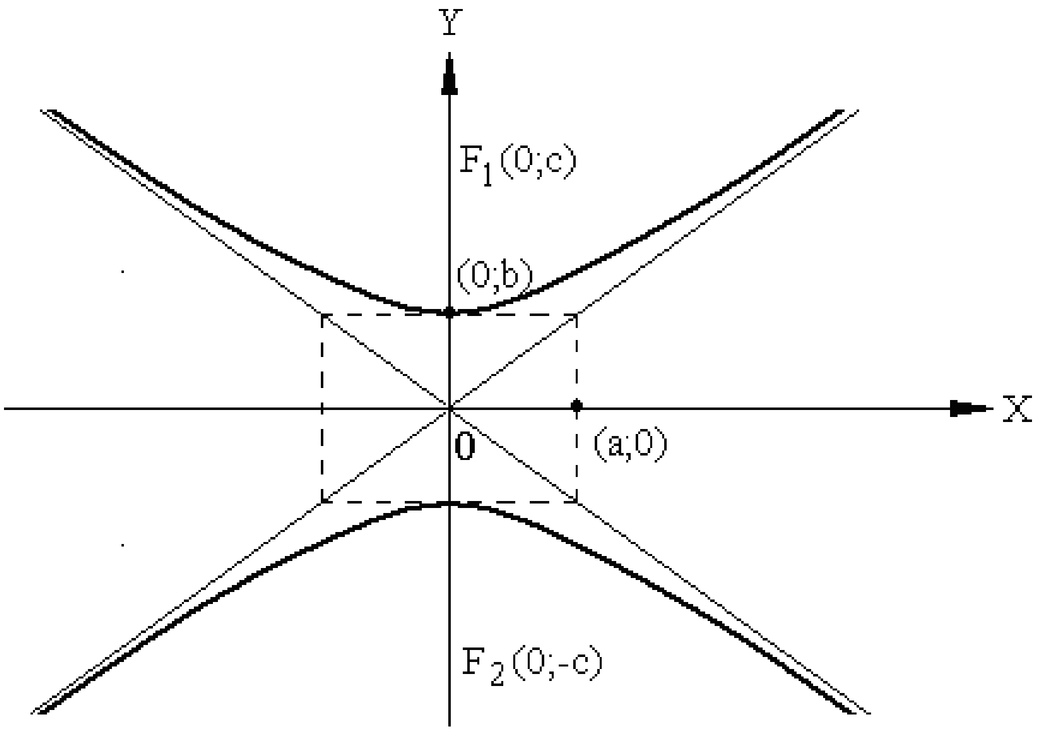


The ratio of half of focal distance to the length of half real axis is called ***eccentricity*** of hyperbola.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

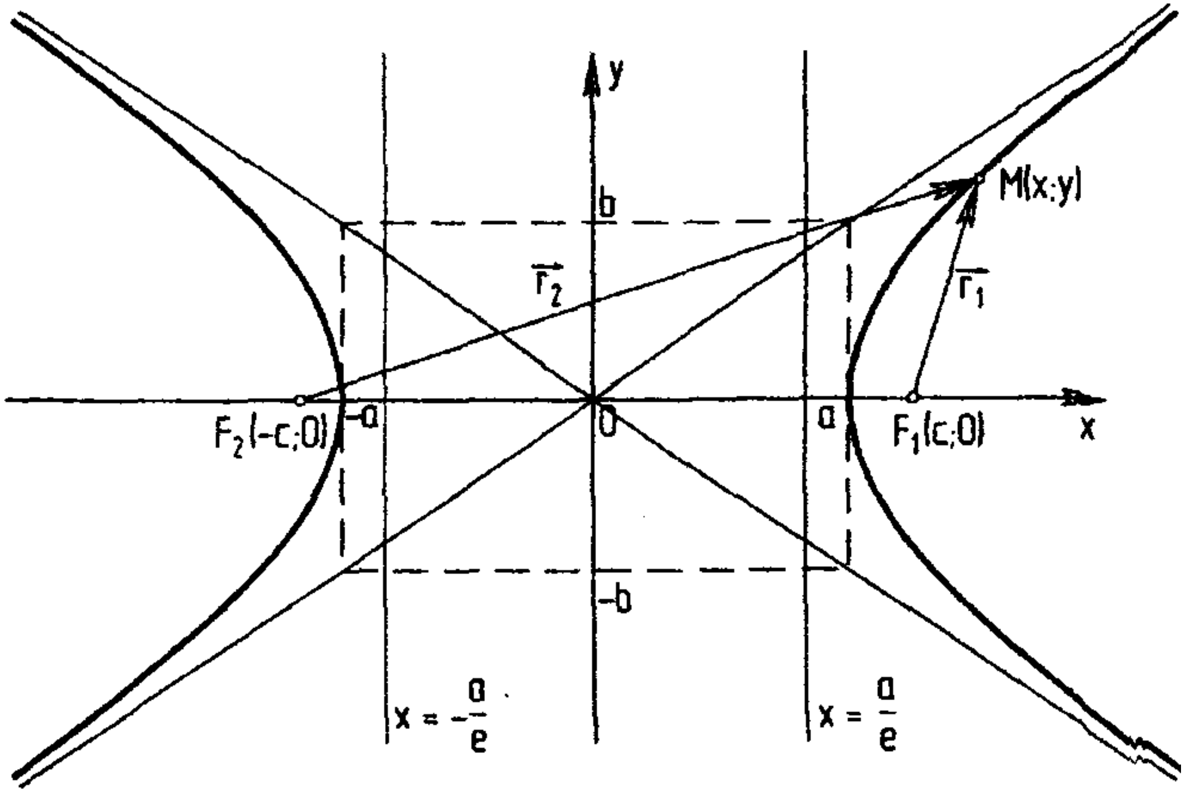


$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

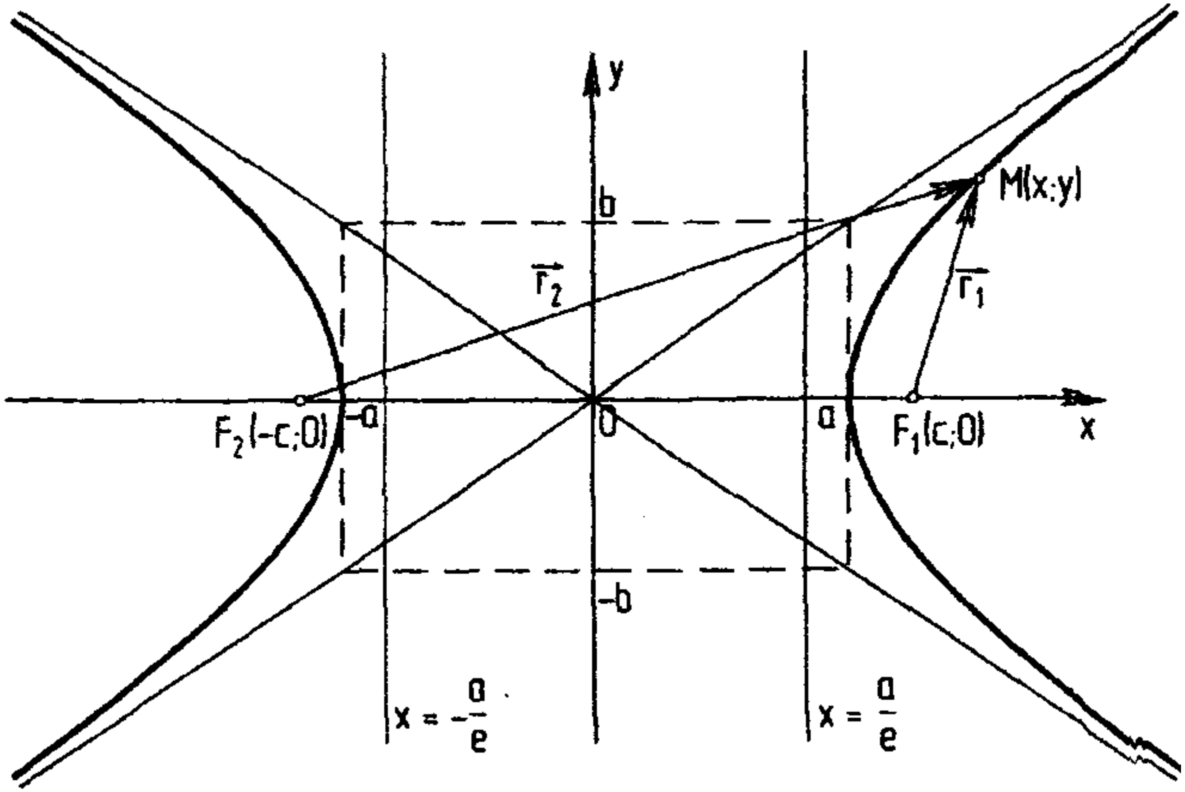


If $a = b$,
then *a* hyperbola is called **rectangular**.

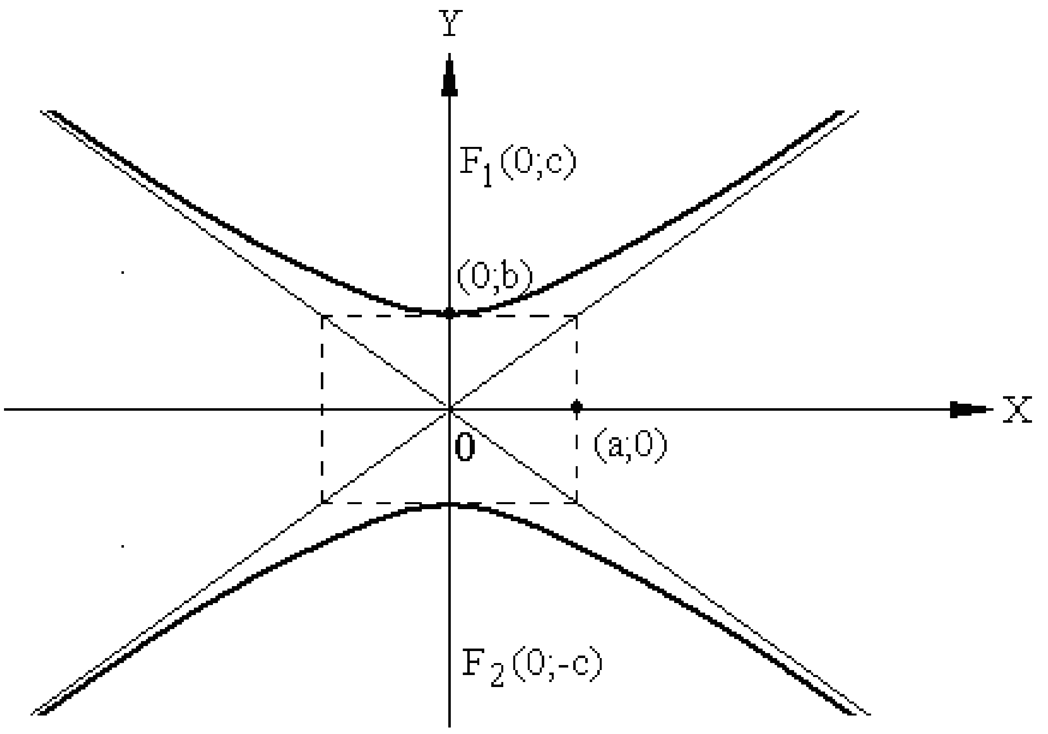
$$x^2 - y^2 = a^2 \qquad\qquad x^2 - y^2 = -a^2$$



The lines $y = \pm \dfrac{b}{a} x$ are called asymptotes of hyperbola



For this hyperbola equations of directrixes are $x = \pm\frac{a}{e}$



For this hyperbola equations of directrixes are $y = \pm \frac{b}{e}$

Task 3

Reduce the given equation

$$9x^2 - 4y^2 + 18x + 8y - 31 = 0$$

to a canonical form and draw the curve.

5. Parabola.
The canonical equation of parabola

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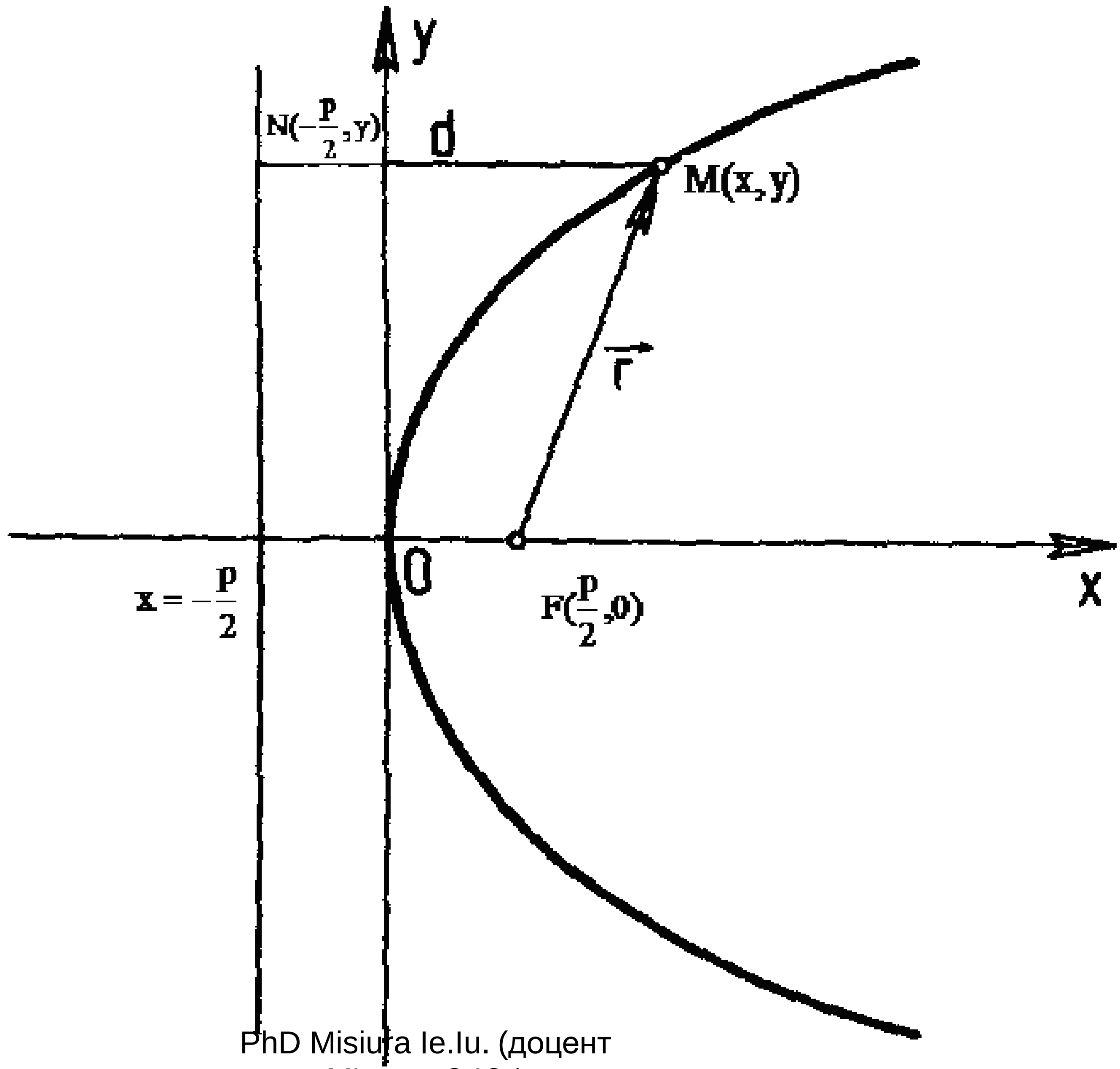
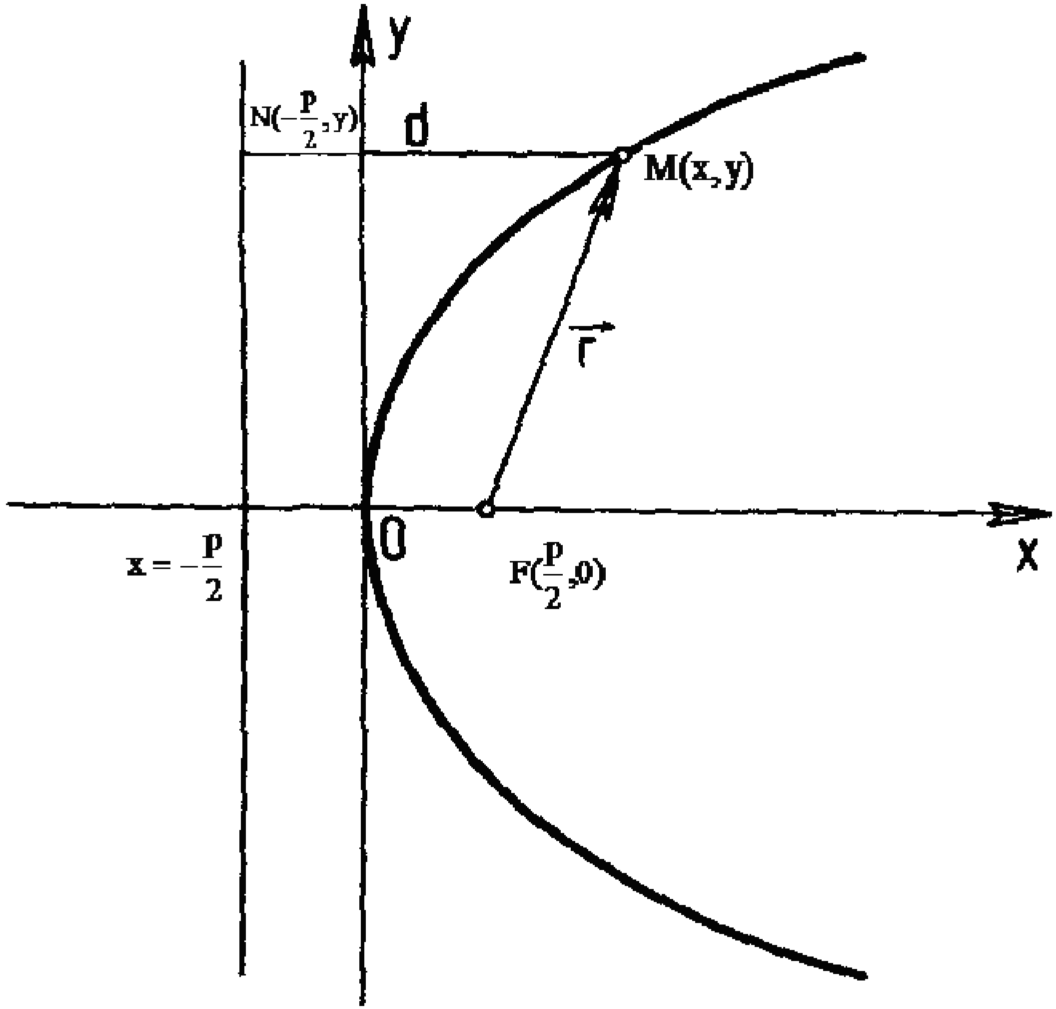
Definition. *Parabola* is a geometrical place of points equidistant from one certain point called focus and a certain line called directrix.


y
N(−p/2, y)
d
M(x, y)
r
0
x = −p/2
F(p/2, 0)
X

$p > 0$

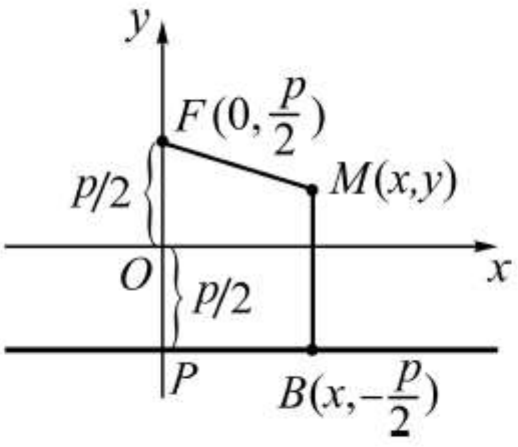


$$|MF| = |MN|$$

Proofing

We have

$$\sqrt{x^2+\left(y-\frac{p}{2}\right)^2}=y+\frac{p}{2}$$



Proofing

We have

$$\sqrt{x^2+\left(y-\frac{p}{2}\right)^2}=y+\frac{p}{2}$$

$$x^2+y^2-py+\frac{p^2}{4}=y^2+py+\frac{p^2}{4},$$

$$x^2=2py.$$

Proofing

We have

$$\sqrt{x^2 + \left(y - \frac{p}{2}\right)^2} = y + \frac{p}{2}$$

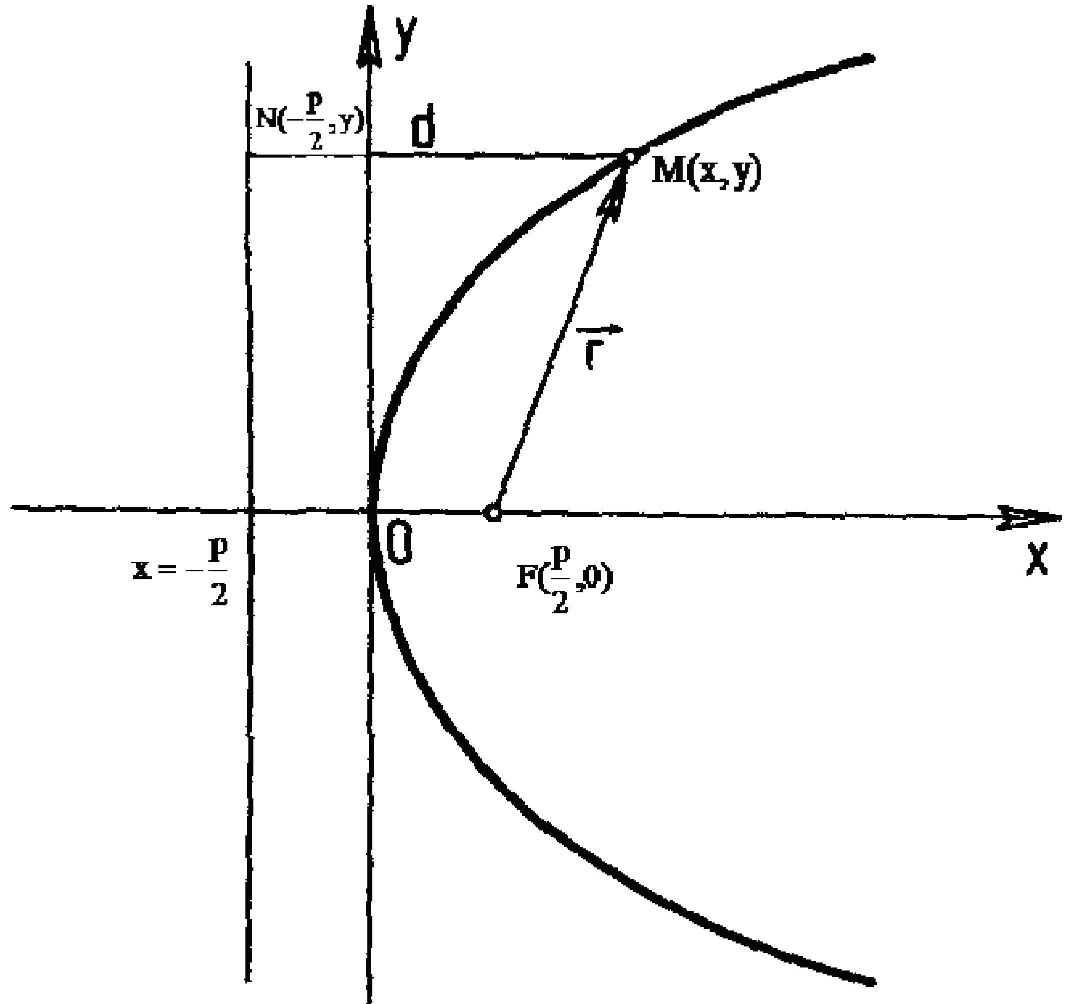
$$x^2 + y^2 - py + \frac{p^2}{4} = y^2 + py + \frac{p^2}{4},$$

$$x^2 = 2py.$$

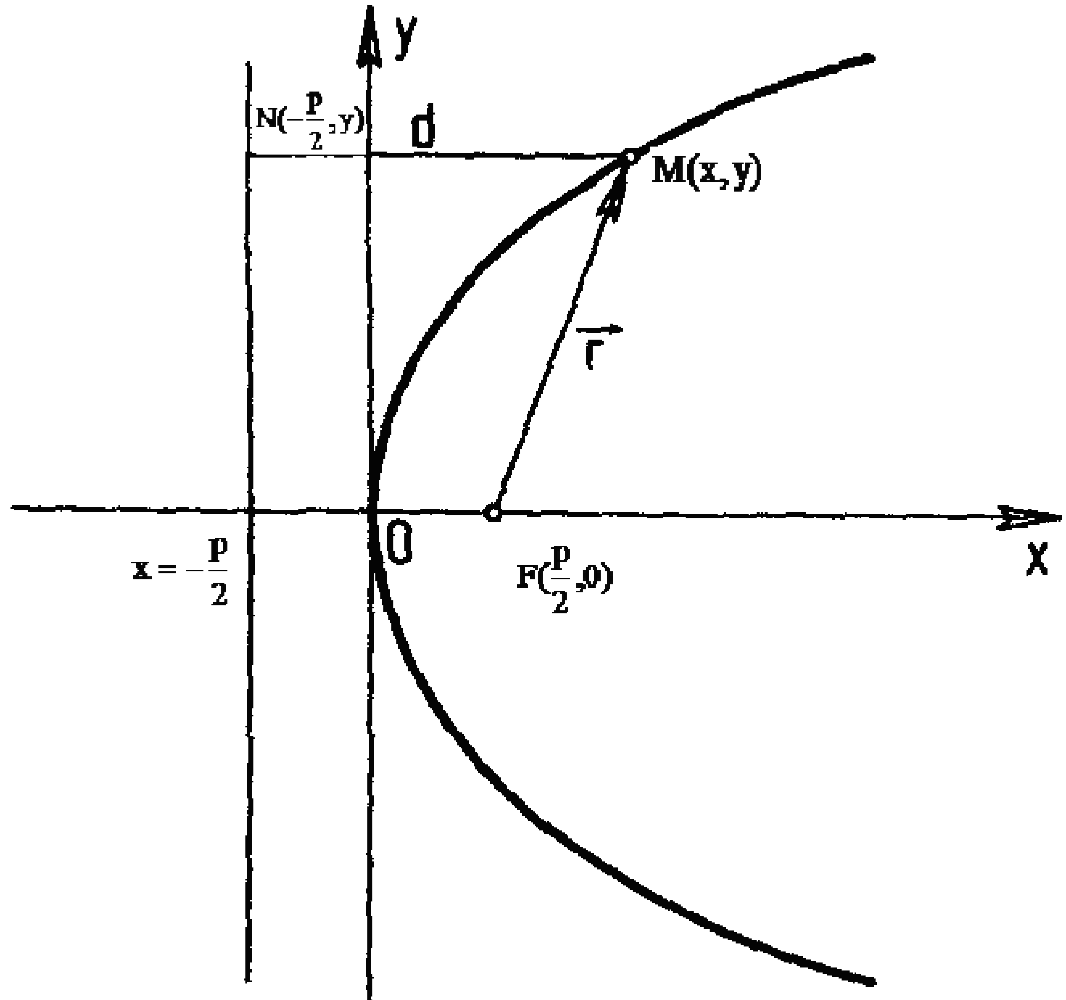
$$y^2 = 2px$$

$$x^2 = 2py$$

This parabola has the focus $F\left(0;\frac{p}{2}\right)$ and the directrix $y=-\frac{p}{2}$.

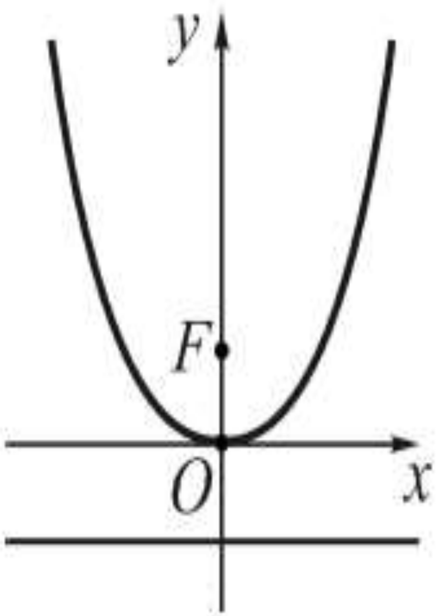


This parabola has the focus $F\left(0;\frac{p}{2}\right)$ and the directrix $y=-\frac{p}{2}$.

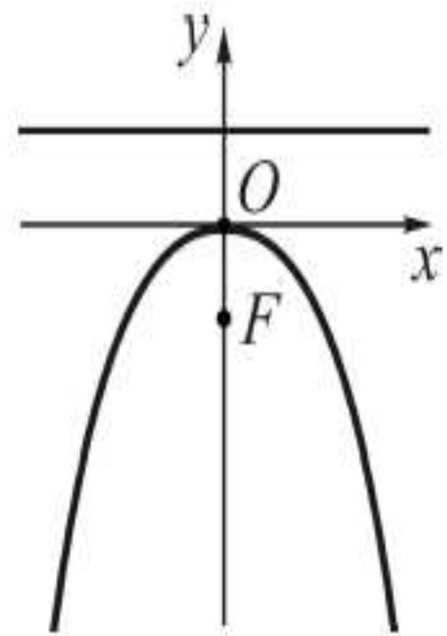


BASIC VARIANTS

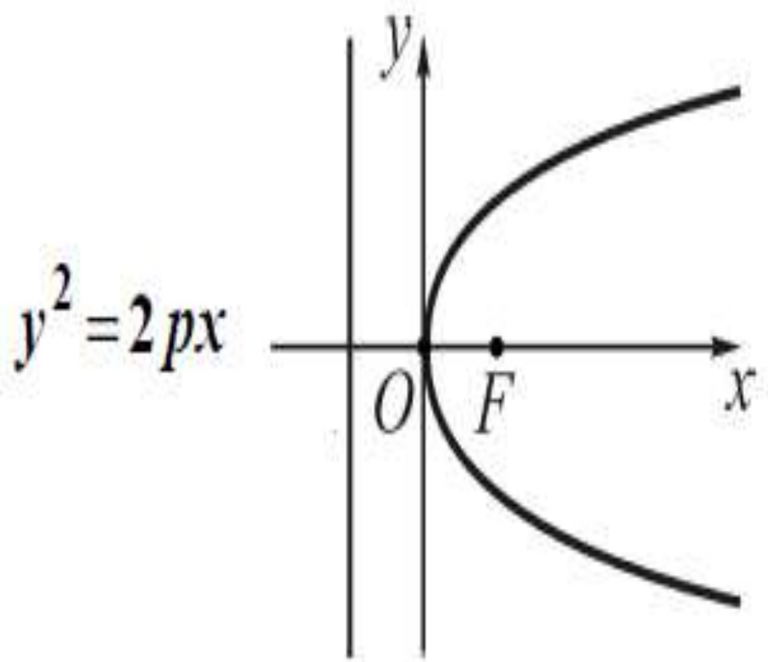
$x^2 = 2py$

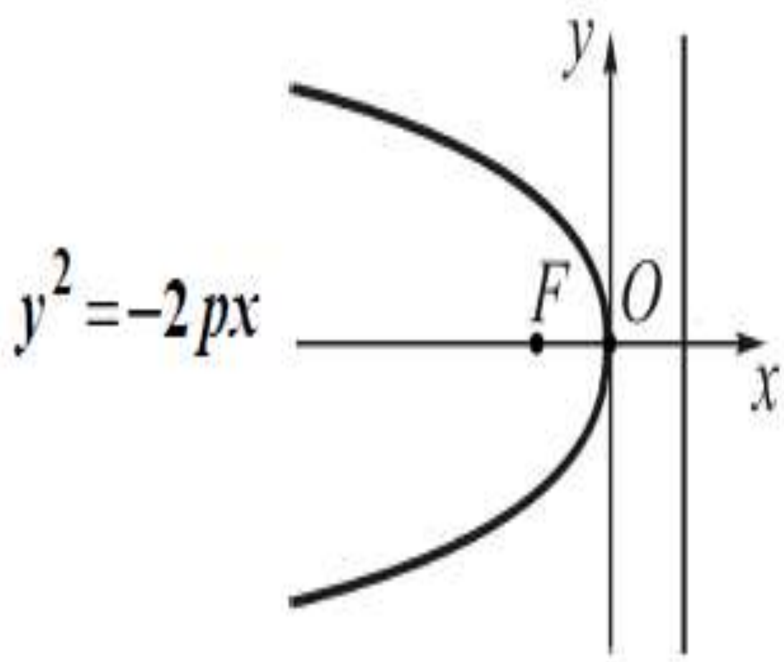


$x^2 = -2py$



BASIC VARIANTS





Task 4

Reduce the given equation

$$y^2 + 8y - 2x + 44 = 0$$

to a canonical form and draw the curve.

Conditions on eccentricity for various sections

$e = 1$	Parabola
$e < 1$	Ellipse
$e > 1$	Hyperbola
$e = 0$	Circle
$e = \infty$	Pair of Straight Lines

Reduction of equations to the canonical forms of curves of the second order

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Let's allocate the perfect square:

$$ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a}$$

$$x^2 + 6x - 7$$

A general equation of the second order curve

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E, F are any numbers,

and $A^2 + B^2 + C^2 \neq 0$

A general equation of the circle

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Here $A = C\ ,\ \ B = 0$ and

$$Ax^2 + Ay^2 + Dx + Ey + F = 0$$

A canonical equation of a circle

is $$(x - x_0)^2 + (y - y_0)^2 = R^2$$

where R is the radius;

O (x_0; y_0) is the center

Let's reduce the general equation to the canonical form

We have

$$Ax^2 + Ay^2 + Dx + Ey + F = 0$$

Let's reduce the general equation to the canonical form

We have

$$Ax^2 + Ay^2 + Dx + Ey + F = 0$$

Let's take out the common factor ***A*** and divide this equation by it:

$$x^2 + y^2 + \frac{D}{A}x + \frac{E}{A}y + \frac{F}{A} = 0$$

Let’s reduce the general equation to the canonical form

Let’s take out the common factor ***A*** and divide this equation by it:

$$x^2 + y^2 + \frac{D}{A}x + \frac{E}{A}y + \frac{F}{A} = 0$$

Let’s denote fractions as:

$$D/A = 2a$$
$$E/A = 2b$$
$$F/A = c$$

Let's reduce the general equation to the canonical form

Let's take out the common factor ***A*** and divide this equation by it:

$$x^2 + y^2 + \frac{D}{A}x + \frac{E}{A}y + \frac{F}{A} = 0$$

Let's denote fractions as:

$$D/A = 2a$$
$$E/A = 2b$$
$$F/A = c$$

and get $\left(x^2 + 2ax\right) + \left(y^2 + 2by\right) + c = 0$

Let's reduce the general equation to the canonical form

Let's denote fractions as: $$D/A = 2a$$
$$E/A = 2b$$
$$F/A = c$$

and get $$\left(x^2 + 2ax\right) + \left(y^2 + 2by\right) + c = 0$$

Let's allocate perfect squares:

$$(x+a)^2 - a^2 + (y+b)^2 - b^2 + c = 0$$

Let's reduce the general equation to the canonical form

Let's allocate perfect squares:

$$(x+a)^2 - a^2 + (y+b)^2 - b^2 + c = 0$$

and get $$(x-x_0)^2 + (y-y_0)^2 = R^2$$

where $x_0 = -a\,,\ y_0 = -b\,,\ R = \sqrt{a^2 + b^2 - c}$

Let's reduce the general equation to the canonical form

and get $$(x-x_0)^2+(y-y_0)^2=R^2$$

where

$$x_0=-a\,,\;y_0=-b\,,\;R=\sqrt{a^2+b^2-c}$$

Remark. Let's use the formula for R and get the following inequality:

$$a^2+b^2-c>0$$

Let's reduce the general equation to the canonical form

and get $$(x-x_0)^2+(y-y_0)^2=R^2$$

where

$$x_0=-a\,,\; y_0=-b\,,\; R=\sqrt{a^2+b^2-c}$$

Remark. Let's use the formula for R and get the following inequality:

$$a^2+b^2-c>0$$

$$D/A=2a$$
$$E/A=2b$$
$$F/A=c$$

Let's reduce the general equation to the canonical form

and get $$(x-x_0)^2+(y-y_0)^2=R^2$$

where

$$x_0=-a\,,\ y_0=-b\,,\ R=\sqrt{a^2+b^2-c}$$

Remark. Let's use the formula for R and get the following inequality:

$$a^2+b^2-c>0$$

$$\boxed{D^2+E^2-4AF>0}$$

$$D/A=2a$$

$$E/A=2b$$

$$F/A=c$$

EXAMPLE 1

Let's reduce the general equation to the canonical form

$$3x^2 + 3y^2 + 6x - 7y + 3 = 0 \Rightarrow$$

EXAMPLE 1

Let's reduce the general equation to the canonical form

$$3x^2 + 3y^2 + 6x - 7y + 3 = 0 \Rightarrow$$

$$\Rightarrow x^2 + y^2 + 2x - \frac{7}{3}y + 1 = 0 \Rightarrow$$

EXAMPLE 1

Let's reduce the general equation to the canonical form

$$3x^2 + 3y^2 + 6x - 7y + 3 = 0 \Rightarrow$$

$$\Rightarrow x^2 + y^2 + 2x - \frac{7}{3}y + 1 = 0 \Rightarrow$$

$$\Rightarrow (x+1)^2 + \left(y - \frac{7}{6}\right)^2 - \frac{49}{36} = 0 \Rightarrow$$

EXAMPLE 1

Let's reduce the general equation to the canonical form

$$3x^2+3y^2+6x-7y+3=0 \Rightarrow$$

$$\Rightarrow x^2+y^2+2x-\frac{7}{3}y+1=0 \Rightarrow$$

$$\Rightarrow (x+1)^2+\left(y-\frac{7}{6}\right)^2-\frac{49}{36}=0 \Rightarrow$$

$$\Rightarrow (x+1)^2+\left(y-\frac{7}{6}\right)^2=\frac{49}{36}$$

EXAMPLE 1

Let's reduce the general equation to the canonical form

$$\Rightarrow (x+1)^2 + \left(y - \frac{7}{6}\right)^2 = \frac{49}{36}$$

It's the canonical equation of the circle with the

radius $R = 7/6$ and the center at the point

Transformations

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Nine canonical second-order curves

There exists a rectangular Cartesian coordinate system in which equations can be reduced to one of the following *nine canonical forms*:

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, an ellipse;
2. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, a hyperbola;
3. $y = 2px$, a parabola;
4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$, an imaginary ellipse;
5. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, a pair of intersecting straight lines;
6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$, a pair of imaginary intersecting straight lines;
7. $x^2 - a^2 = 0$, a pair of parallel straight lines;
8. $x^2 + a^2 = 0$, a pair of imaginary parallel straight lines;
9. $x^2 = 0$, a pair of coinciding straight lines.

According to the sign of the value $AC - B^2$ the curves of the second order are divided by three types:

1) elliptic if $AC - B^2 > 0$

(if $A = C$ *it is a circle*)

2) hyperbolic if $AC - B^2 < 0$

3) parabolic if $AC - B^2 = 0$

Let's consider examples if $B = 0$

According to the sign of the value $\boldsymbol{A \cdot C}$ the curves of the second order are divided by three types:

1) elliptic if $\boldsymbol{A \cdot C > 0}$ (if $\boldsymbol{A = C}$ *it is a circle*)

2) hyperbolic if $\boldsymbol{A \cdot C < 0}$

3) parabolic if $\boldsymbol{A \cdot C = 0}$

EXAMPLE 2

Reduce this equation

$$4x^2 - 25y^2 - 24x - 50y - 89 = 0$$

to the canonical form and use transformations of coordinates.

EXAMPLE 2

$$4x^2 - 25y^2 - 24x - 50y - 89 = 0$$

Let’s define the type of this curve:

$$A = 4,\ C = -25 \quad \Rightarrow \quad A \cdot C < 0$$

EXAMPLE 2

$$4x^2 - 25y^2 - 24x - 50y - 89 = 0$$

Let’s define the type of this curve:

$$A = 4,\ C = -25 \ \Rightarrow \ A \cdot C < 0$$

It means that it is the hyperbolic type.

EXAMPLE 2

$$4x^2 - 25y^2 - 24x - 50y - 89 = 0$$

Let's group summands and allocate perfect squares:

$$4\left(x^2 - 6x\right) - 25\left(y^2 + 2y\right) - 89 = 0 \Rightarrow$$

EXAMPLE 2

$$4x^2 - 25y^2 - 24x - 50y - 89 = 0$$

Let's group summands and allocate perfect squares:

$$4\left(x^2 - 6x\right) - 25\left(y^2 + 2y\right) - 89 = 0 \Rightarrow$$

$$\Rightarrow 4\left[\left(x^2 - 2\cdot 3x + 9\right) - 9\right] - 25\left[\left(y^2 + 2y + 1\right) - 1\right] - 89 = 0 \Rightarrow$$

$$\Rightarrow 4\left(x-3\right)^2 - 25\left(y+1\right)^2 = 100 \Rightarrow \frac{\left(x-3\right)^2}{25} - \frac{\left(y+1\right)^2}{4} = 1$$

EXAMPLE 2

$$4x^2 - 25y^2 - 24x - 50y - 89 = 0$$

Let’s group summands and allocate perfect squares:

$$\Rightarrow 4(x-3)^2 - 25(y+1)^2 = 100 \Rightarrow \frac{(x-3)^2}{25} - \frac{(y+1)^2}{4} = 1$$

Let’s use this formula $\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$

and get $x_0 = 3;\, y_0 = -1$

EXAMPLE 2

$$\frac{(x-3)^2}{25}-\frac{(y+1)^2}{4}=1$$

Let's use this formula $$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$$

and get $x_0=3; y_0=-1$

EXAMPLE 2

$$\frac{(x-3)^2}{25}-\frac{(y+1)^2}{4}=1 \qquad \frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$$

Let’s get $x_0=3;\, y_0=-1$

and the new coordinates: $x-3=X$

$$y+1=Y$$

EXAMPLE 2

$$\frac{(x-3)^2}{25}-\frac{(y+1)^2}{4}=1$$

We have the hyperbola $\frac{X^2}{25}-\frac{Y^2}{4}=1$

with the center $(3,-1)$ and semiaxes $a=5,\ b=2$.

EXAMPLE 3

Reduce this equation

$$2x^2 + 3y^2 + 16x - 18y + 59 = 0$$

to the canonical form and use transformations of coordinates.

EXAMPLE 3

$$2x^2 + 3y^2 + 16x - 18y + 59 = 0$$

Let’s define the type of this curve:

$$A = 2,\ C = 3 \quad \Rightarrow \quad A \cdot C > 0$$

EXAMPLE 3

$$2x^2+3y^2+16x-18y+59=0$$

Let’s define the type of this curve:

$$A=2,\ C=3 \quad \Rightarrow \quad A\cdot C>0$$

It means that it is the elliptic type.

EXAMPLE 3

$$2x^2+3y^2+16x-18y+59=0$$

Let's group summands and allocate perfect squares:

$$2(x^2+8x)+3(y^2-6y)+59=0 \Rightarrow$$

$$\Rightarrow 2\left[(x+4)^2-16\right]+3\left[(y-3)^2-9\right]+59=0 \Rightarrow$$

$$\Rightarrow 2(x+4)^2+3(y-3)^2=0 \Rightarrow \frac{(x+4)^2}{3}+\frac{(y-3)^2}{2}=0 \Rightarrow$$

EXAMPLE 3

$$2x^2 + 3y^2 + 16x - 18y + 59 = 0$$

Let’s group summands and allocate perfect squares:

$$2\left(x^2 + 8x\right) + 3\left(y^2 - 6y\right) + 59 = 0 \;\Rightarrow$$

$$\Rightarrow\; 2\left[(x+4)^2 - 16\right] + 3\left[(y-3)^2 - 9\right] + 59 = 0 \;\Rightarrow$$

$$\Rightarrow 2(x+4)^2 + 3(y-3)^2 = 0 \;\Rightarrow\; \frac{(x+4)^2}{3} + \frac{(y-3)^2}{2} = 0 \;\Rightarrow$$

EXAMPLE 3

$$2x^2+3y^2+16x-18y+59=0$$

Let's get

$$2\left(x^2+8x\right)+3\left(y^2-6y\right)+59=0 \quad \Rightarrow$$

$$\Rightarrow \quad 2\left[(x+4)^2-16\right]+3\left[(y-3)^2-9\right]+59=0 \quad \Rightarrow$$

$$\Rightarrow 2\left(x+4\right)^2+3\left(y-3\right)^2=0 \;\Rightarrow\; \frac{(x+4)^2}{3}+\frac{(y-3)^2}{2}=0 \quad \Rightarrow$$

$$\Rightarrow \left|\begin{array}{l} x+4=X \Rightarrow x_0=-4 \\ y-3=Y \Rightarrow y_0=3 \end{array}\right| \Rightarrow \frac{X^2}{3}+\frac{Y^2}{2}=0.$$

EXAMPLE 3

$$2x^2+3y^2+16x-18y+59=0$$

Let's get

$$\Rightarrow 2(x+4)^2+3(y-3)^2=0 \Rightarrow \frac{(x+4)^2}{3}+\frac{(y-3)^2}{2}=0 \Rightarrow$$

$$\Rightarrow \left| \begin{array}{l} x+4=X \Rightarrow x_0=-4 \\ y-3=Y \Rightarrow y_0=3 \end{array} \right| \Rightarrow \frac{X^2}{3}+\frac{Y^2}{2}=0.$$

It will be the ellipse which is singulared at the point $(-4,\ 3)$

EXAMPLE 4

Reduce this equation

$$2x^2 - y - 4x + 8 = 0$$

to the canonical form and use transformations of coordinates.

EXAMPLE 4

$$2x^2 - y - 4x + 8 = 0$$

Let's define the type of this curve:

$$A = 2,\ C = 0 \quad \Rightarrow \quad A \cdot C = 0$$

It means that it is the parabolic type.

EXAMPLE 4

$$2x^2 - y - 4x + 8 = 0$$

Let's group summands and allocate perfect squares:

$$y = 2x^2 - 4x + 8 \Rightarrow y = 2\left(x^2 - 2x + 1\right) - 2 + 8 \Rightarrow$$

$$\Rightarrow y = 2(x-1)^2 + 6 \Rightarrow y - 6 = 2(x-1)^2 \Rightarrow (x-1)^2 = \frac{1}{2}(y-6) \Rightarrow$$

$$\Rightarrow \left| \begin{array}{l} x - 1 = X \Rightarrow x_0 = 1 \\ y - 6 = Y \Rightarrow y_0 = 6 \end{array} \right| \Rightarrow X^2 = \frac{1}{2}Y \Rightarrow Y = 2X^2.$$

EXAMPLE 4

$$2x^2 - y - 4x + 8 = 0$$

Let's group summands and allocate perfect squares:

$$\Rightarrow (x-1)^2 = \frac{1}{2}(y-6) \Rightarrow$$

$$\Rightarrow \left| \begin{matrix} x-1 = X \Rightarrow x_0 = 1 \\ y-6 = Y \Rightarrow y_0 = 6 \end{matrix} \right| \Rightarrow X^2 = \frac{1}{2}Y \Rightarrow Y = 2X^2$$

It's the parabola with the vertex $O_1(1,6)$ (it is symmetrical relative to O_1Y) and the parameter equals $1/4$ $(p = 1/4)$.

TASK 1

Get the canonical equation:

$$x^2 + y^2 - 4x + 8y - 16 = 0$$

TASK 2

Get the canonical equation:

$$9x^2 + 4y^2 - 18x - 8y - 23 = 0$$

TASK 3

Get the canonical equation:

$$x^2 - 4y^2 + 6x + 16y - 11 = 0$$

TASK 4

Get the canonical equation:

$$2y^2 + x - 8y + 3 = 0$$

Basic formulas:

$$(x-x_0)^2+(y-y_0)^2=R^2$$

$$(y-y_0)^2=2p(x-x_0)$$

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$$

$$(x-x_0)^2=2p(y-y_0)$$

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$$