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Theme:
Elements of vector algebra and analytic geometry
PART 1. Elements of vector algebra

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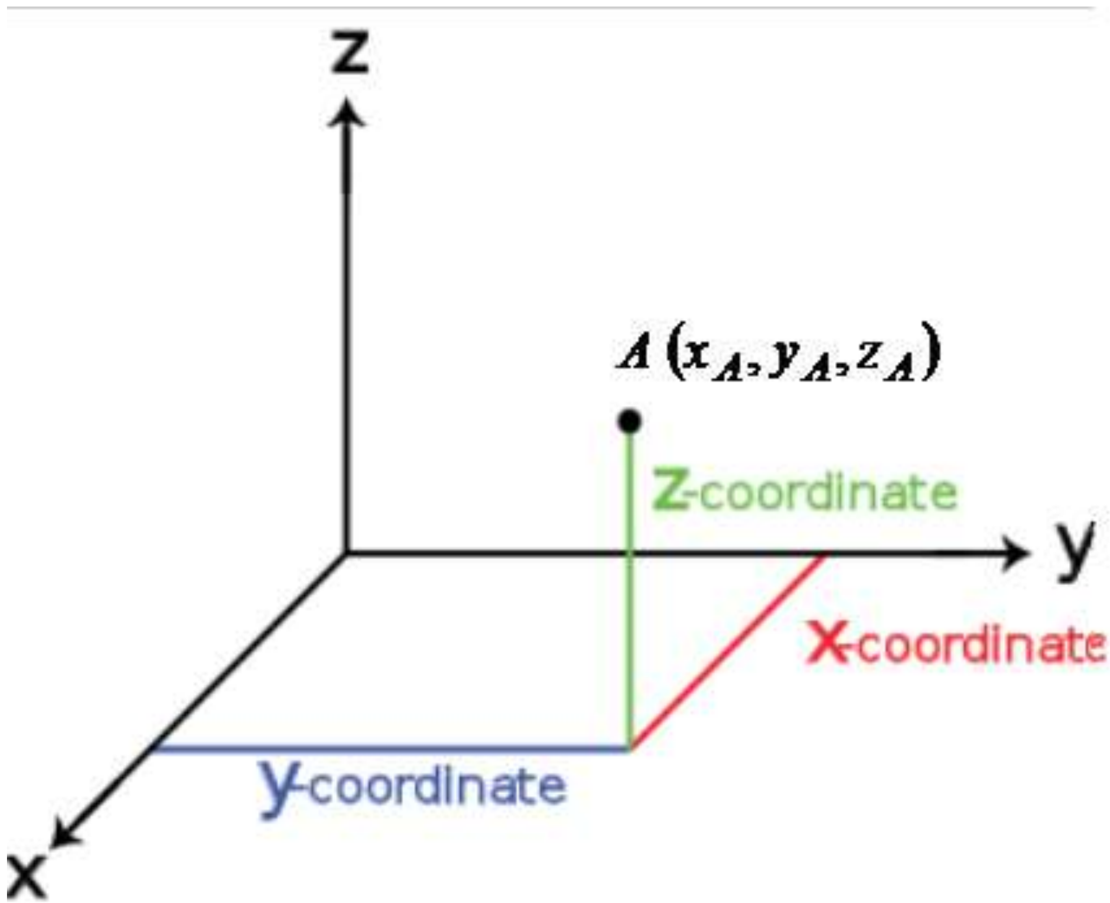
1. Definition of a vector, types of vectors.
2. Basic operations with vectors, properties of these operations
3. Formulas of division of a segment in the given ratio
4. A module of the vector, its properties. The angle between the vectors. A scalar product. A vector product. A mixed product

1. Definition of a vector, types of vectors

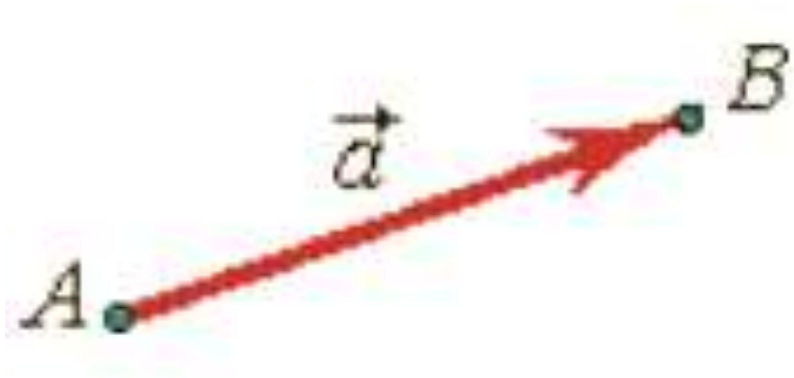
Let a rectangular Cartesian coordinate system be defined in space. Then a position of any spatial point is defined by its coordinates *x, y, z*.



Names of axes

X-axis	or	absciss
Y-axis	or	ordinate
Z-axis	or	applicata
(0; 0; 0)	or	origin

Definition. A directed segment (or an ordered couple of points ***A*** and ***B***) is called *a vector* (geometrical). A vector is denoted by $\overline{AB}$ or $\overline{a}$.



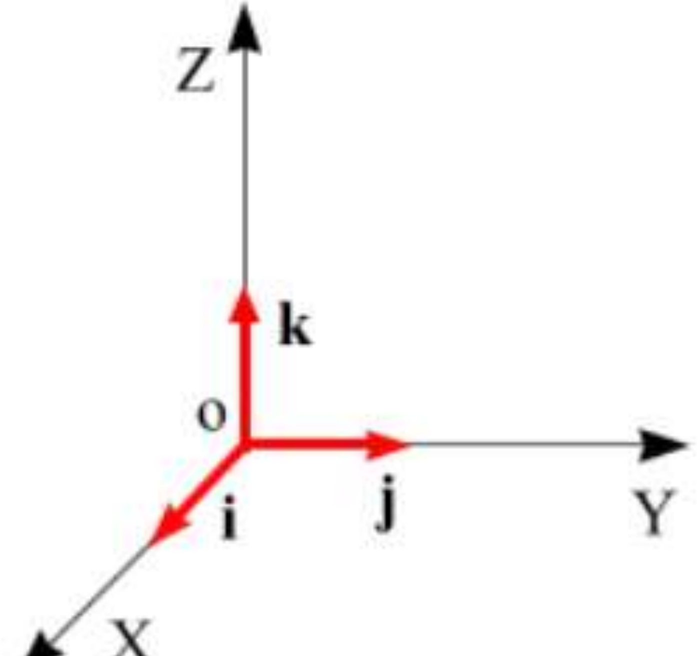
Let the point $A(x_A, y_A, z_A)$ be an origin of the vector and the point $B(x_B, y_B, z_B)$ be its terminus. Coordinates of the vector $\overline{AB}$ are defined as

$$\overline{AB} = (x_B - x_A, y_B - y_A, z_B - z_A)$$

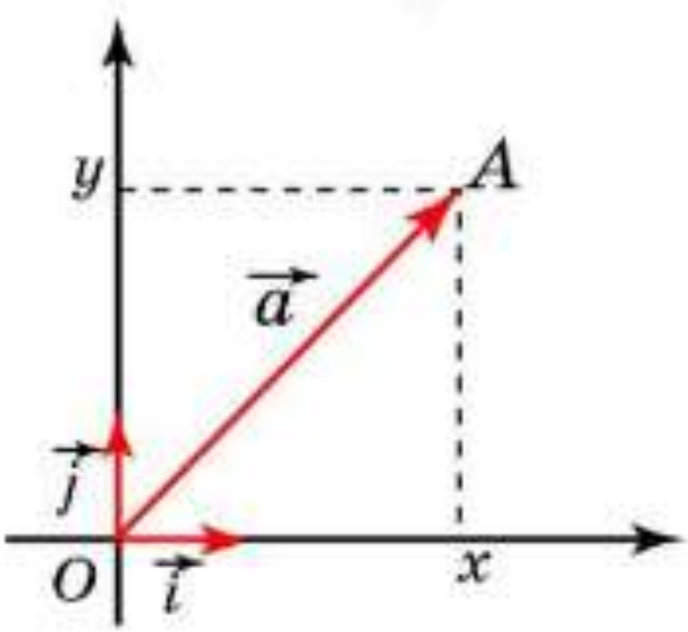
It is known that any vector in space can be presented as

$$\bar{a} = a_x \cdot \bar{i} + a_y \cdot \bar{j} + a_z \cdot \bar{k} \quad \text{or} \quad \bar{a} = (a_x, a_y, a_z)$$

where a_x, a_y, a_z are projections of the vector $\bar{a}$ on the axes **Ox**, **Oy**, **Oz** respectively; $\bar{i}, \bar{j}, \bar{k}$ are unit vectors (orts) whose directions coincide with the direction of the coordinate axes $\bar{i} = (1,0,0)$, $\bar{j} = (0,1,0)$, $\bar{k} = (0,0,1)$.



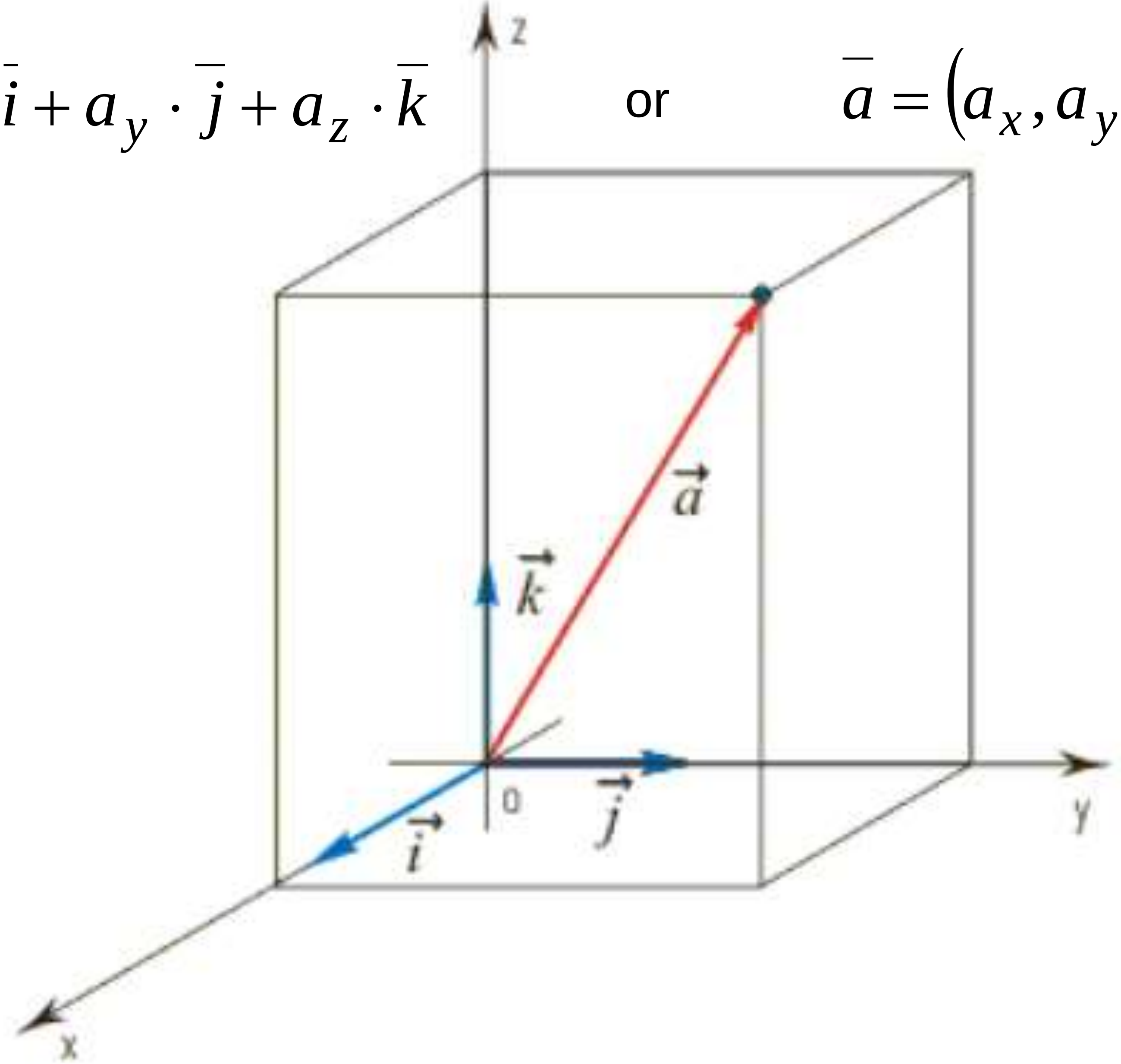
A plane



$$\bar{a} = a_x \cdot \bar{i} + a_y \cdot \bar{j} + a_z \cdot \bar{k}$$

or

$$\bar{a} = (a_x, a_y, a_z)$$

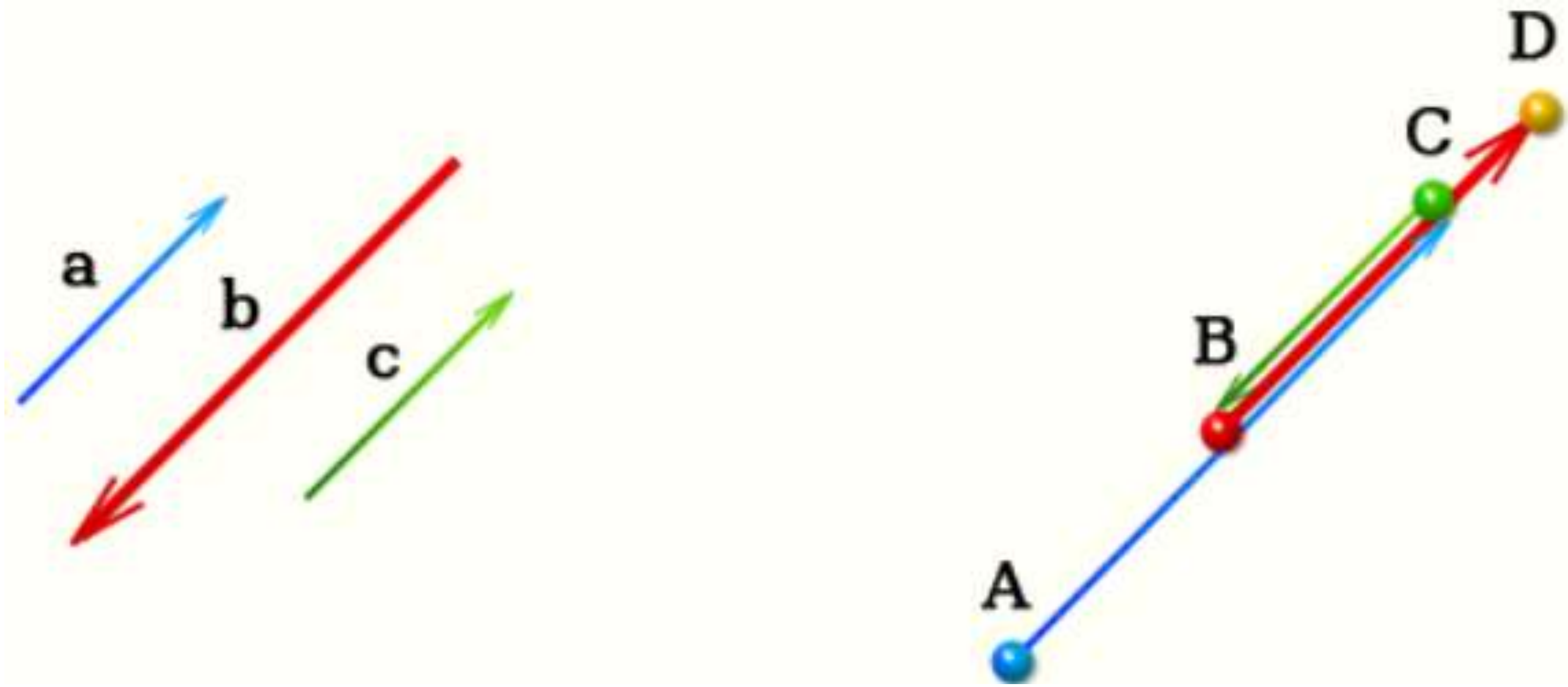


Every of the vectors $\bar{i}, \bar{j}, \bar{k}$ is perpendicular (orthogonal) to the both others. These vectors form a so called orthonormalized basis. The projections a_x, a_y, a_z are coordinates of the vector in the orthonormalized basis.

Definition. So called ***the null vector*** whose origin coincides with its terminus also relates to vectors:

$$\bar{0} = (0,0,0)$$

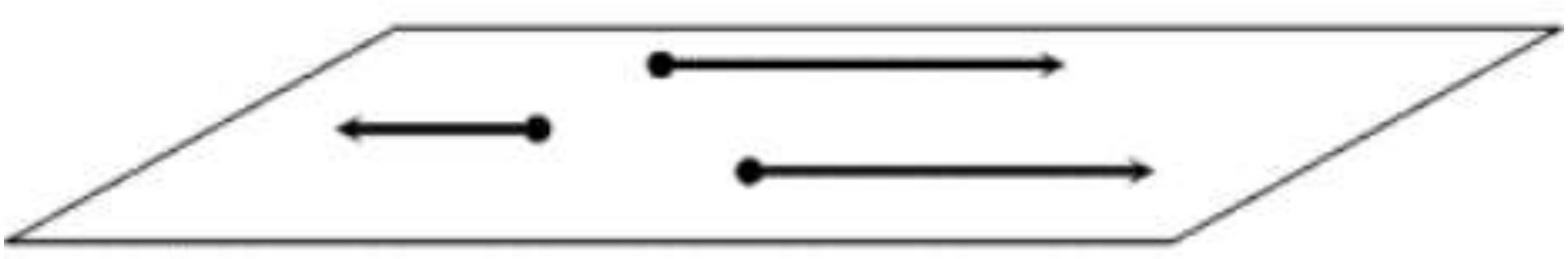
Definition. Vectors located on the same or parallel straight lines are called ***parallel*** or ***collinear***.



Example 1. Vectors $\bar{a}, \bar{b}, \bar{c}$ are collinear,

vectors $\overline{AC}, \overline{BD}, \overline{CB}$ are collinear.

Definition. Vectors are called ***complanar*** if there exists a plane which they are parallel to.



Definition. Two vectors should be considered ***equal*** if they are collinear, equally directed and have equal lengths.

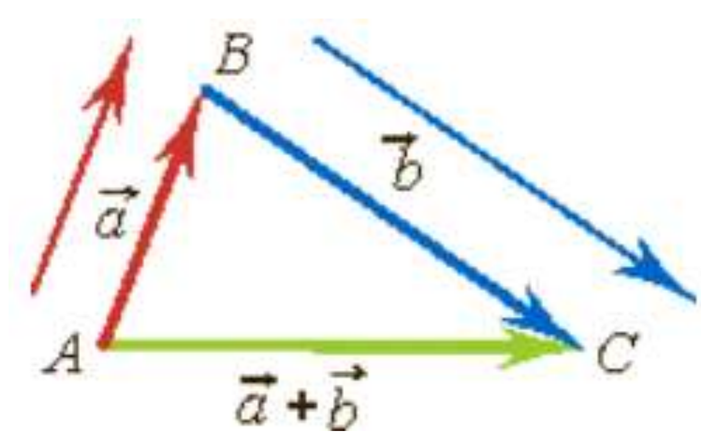
2. Basic operations with vectors and properties of these operations

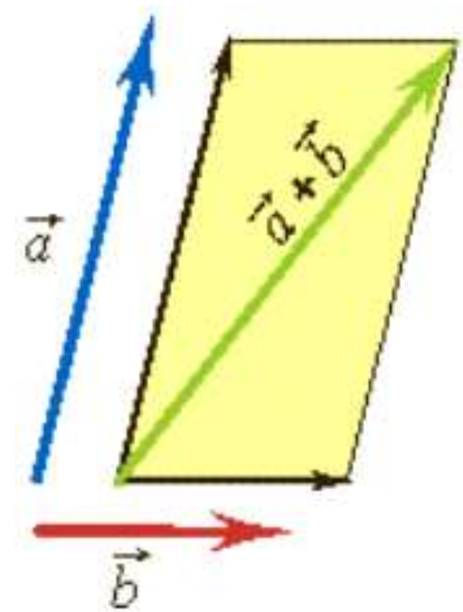
Linear operations with the vectors

Property 1: *a sum or difference of vectors are determined according to the formulas*:

$$\bar{a} \pm \bar{b} = \left(a_x \pm b_x, a_y \pm b_y, a_z \pm b_z\right)$$

Vectors are added and subtracted according to the rule of triangle or parallelogram:





The operations of vector addition (subtraction) satisfy the following laws:

1) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ is a commutative law;

2) $\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$ is an associative law;

3) $\vec{a}+\left(-\vec{a}\right)=\bar{0}$, where $\bar{0}$ is a null vector, $\left(-\vec{a}\right)$ is the opposite vector relative to a vector $\vec{a}$:

$$-\vec{a}=\left(-1\right)\cdot\vec{a}$$

Economic EXAMPLE

Vectors

Ordered n-tuple of objects is called a vector

$$\mathbf{y} = (y_1, y_2, \ldots, y_n).$$

Throughout the text we confine ourselves to vectors the elements y_i of which are real numbers.

In contrast, a variable the value of which is a single number, not a vector, is called *scalar*.

Example 2.1. We can describe some economic unit **EU** by the vector

EU= (output, # of employees, capital stock, profit)

Example 2.2. Let $\mathbf{EU}_1 = (Y_1, L_1, K_1, P_1)$ be a vector representing an economic unit, say, a firm, see Example 2.1 (where, as usually, Y is its output, L is the number of employees, K is the capital stock, and P is the profit). Let us assume that it is merged with another firm represented by a vector $\mathbf{EU}_2 = (Y_2, L_2, K_2, P_2)$ (that is, we should consider two separate units as a single one). The resulting unit will be represented by a sum of two vectors

$$\mathbf{EU}_3 = (Y_1 + Y_2, L_1 + L_2, K_1 + K_2, P_1 + P_2) = \mathbf{EU}_1 + \mathbf{EU}_2.$$

In this situation, we have also $\mathbf{EU}_2 = \mathbf{EU}_3 - \mathbf{EU}_1$. Moreover, if the second firm is similar to the first one, we can assume that $\mathbf{EU}_1 = \mathbf{EU}_2$, hence the unit

$$\mathbf{EU}_3 = (2Y_1, 2L_1, 2K_1, 2P_1) = 2 \cdot \mathbf{EU}_1$$

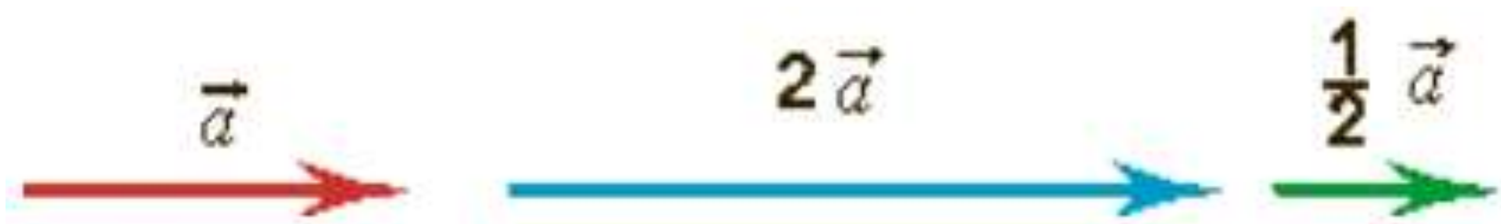
gives also an example of the multiplication by a number 2.

This example, as well as other 'economic' examples in this book has an illustrative nature. Notice, however, that the profit of the merged firm might be higher or lower than the sum of two profits $P_1 + P_2$.

Property 2: *a multiplication of a vector by a number is determined according to the formula*:

$$\alpha \cdot \bar{a} = (\alpha \cdot a_x, \alpha \cdot a_y, \alpha \cdot a_z)$$

If $\alpha > 0$*, then* $\vec{a}$ *and* $\alpha \bar{a}$ *are parallel (collinear) and directed to the same side*



if $\alpha < 0$*, then to the opposite sides.*

Let the vectors $\bar{a}$ and $\bar{b}$ be defined by their coordinates, i.e.

$\bar{a} = \left(a_x, a_y, a_z\right)$ and $\bar{b} = \left(b_x, b_y, b_z\right)$, then the vector

equality $\bar{b} = \alpha \bar{a}$ is equivalent to three numerical ones:

$$b_x = \alpha a_x \qquad b_y = \alpha a_y \qquad b_z = \alpha a_z$$

$$\frac{b_x}{a_x} = \frac{b_y}{a_y} = \frac{b_z}{a_z} = \alpha$$

Rule: thus vectors are *collinear* if their *coordinates are proportional.*

Example 2. Two vectors $\vec{a} = (3,-2,-6)$ and $\vec{b} = (-2,1,0)$ are given. Determine the projections on the coordinate axes of the following vectors:

1) $\vec{a} + \vec{b}$ 2) $\vec{a} - \vec{b}$

3) $2\vec{a}$ 4) $2\vec{a} - 3\vec{b}$

Solution. By the rule of vector addition and vector multiplication by a number we have:

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z) = (3 + (-2), -2 + 1, -6 + 0) = (1,-1,-6)$$

$$\vec{a} - \vec{b} = (a_x - b_x, a_y - b_y, a_z - b_z) = (3 - (-2), -2 - 1, -6 - 0) = (5,-3,-6)$$

$$2 \cdot \vec{a} = (2 \cdot a_x, 2 \cdot a_y, 2 \cdot a_z) = (2 \cdot 3, 2 \cdot (-2), 2 \cdot (-6)) = (6,-4,-12)$$

$$2\vec{a} - 3\vec{b} = \left(2a_x - 3b_x, 2a_y - 3b_y, 2a_z - 3b_z\right) =$$

$$= (2 \cdot 3 - 3 \cdot (-2), 2 \cdot (-2) - 3 \cdot 1, 2 \cdot (-6) - 3 \cdot 0) = (12,-7,-12)$$

3. Formulas of division of a segment in the given ratio

Let us assume that the point $M(x_M, y_M, z_M)$ divides a segment between the points $M_1(x_{M_1}, y_{M_1}, z_{M_1})$ and $M_2(x_{M_2}, y_{M_2}, z_{M_2})$ in the ratio λ, that is

$$\lambda = \frac{|M_1M|}{|MM_2|}$$

Task. Let’s get formulas for coordinates of the point ***M.***

In this case the following formulas should be used to find the coordinates of the point ***M***:

$$x_M = \frac{x_{M_1} + \lambda \cdot x_{M_2}}{1+\lambda}$$

$$y_M = \frac{y_{M_1} + \lambda \cdot y_{M_2}}{1+\lambda}$$

$$z_M = \frac{z_{M_1} + \lambda \cdot z_{M_2}}{1+\lambda}$$

In the particular case, if the point ***M*** bisects the segment M_1M_2 then

$$x_M = \frac{x_{M_1} + x_{M_2}}{2}$$

$$y_M = \frac{y_{M_1} + y_{M_2}}{2}$$

$$z_M = \frac{z_{M_1} + z_{M_2}}{2}$$

4. A module of the vector, its properties. The angle between the vectors.

The distance between the vectors.

The distance between an origin and a terminus of a vector is called its length or module and designated by $\left|\vec{a}\right|$ or $\left|\overrightarrow{AB}\right|$.

The module of a vector $\vec{a}$ is calculated according to the following formula:

$$\left|\vec{a}\right| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}$$

The module of a vector $\overrightarrow{AB}$ is calculated according to the following formula:

$$\left|\overrightarrow{AB}\right| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

Direction cosines are called cosines of the angles between the vector $\vec{a}$ and positive directions of the corresponding coordinate axes and defined as follows:

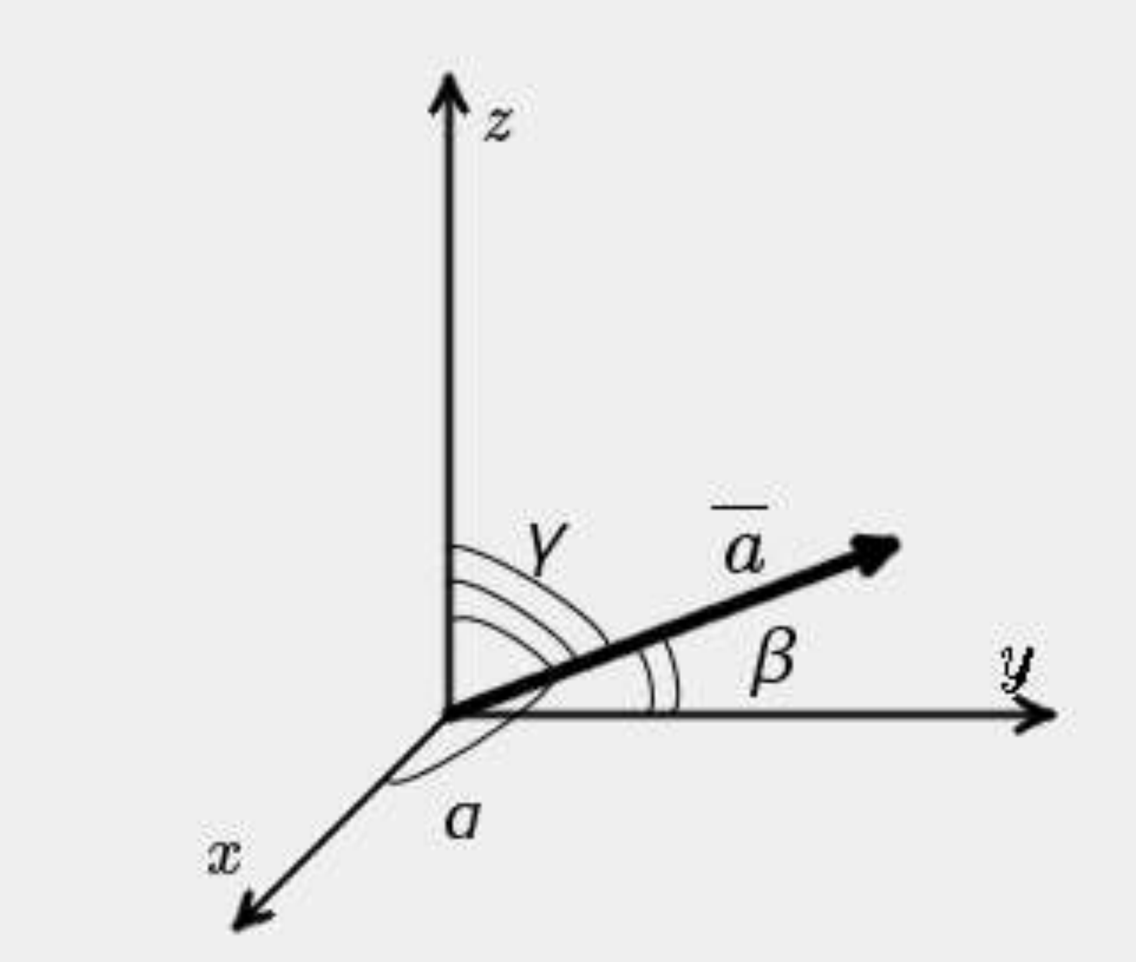
$$\cos\alpha = \frac{a_x}{|\vec{a}|} = \frac{a_x}{\sqrt{a_x^{\,2} + a_y^{\,2} + a_z^{\,2}}} \qquad \cos\beta = \frac{a_y}{|\vec{a}|} = \frac{a_y}{\sqrt{a_x^{\,2} + a_y^{\,2} + a_z^{\,2}}}$$

$$\cos\gamma = \frac{a_z}{|\vec{a}|} = \frac{a_z}{\sqrt{a_x^{\,2} + a_y^{\,2} + a_z^{\,2}}}$$

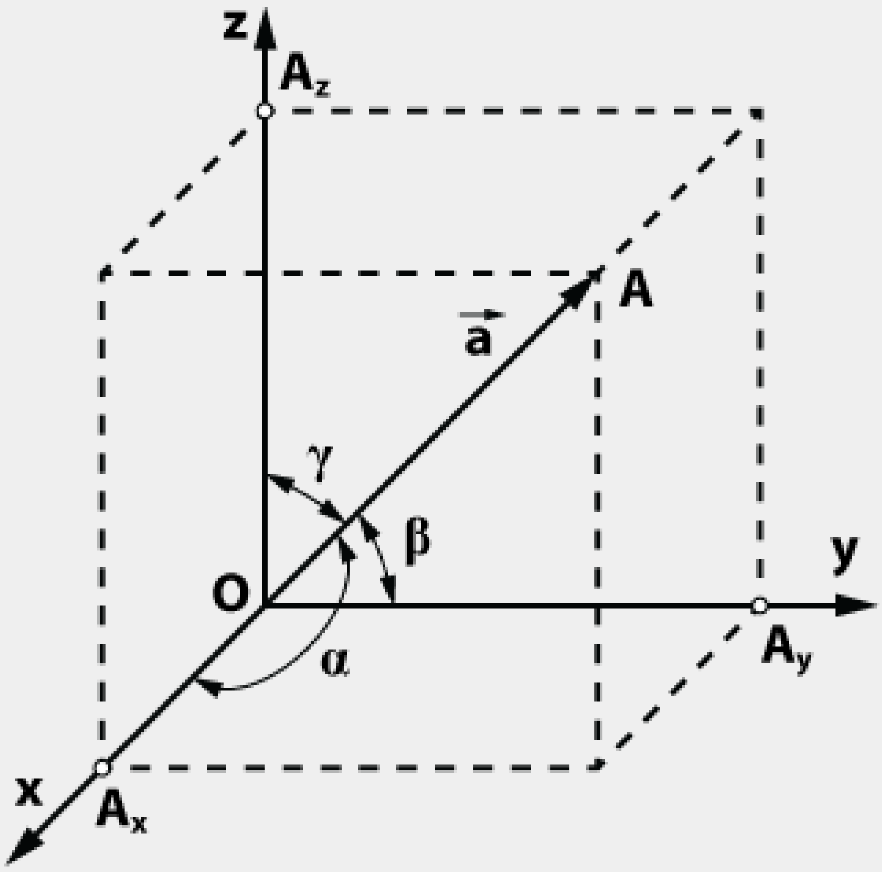
They are related to the equality

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Angles between the vector and positive directions



Angles between the vector and positive directions



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Example 3. Find the direction cosines of the vector $\overrightarrow{AB}$ if the points $A(1,2,0)$ and $B(3,1,-2)$ are given.

Solution. The coordinates of the vector $\overrightarrow{AB}$ are calculated in this way:

$$\overrightarrow{AB} = (x_B - x_A; y_B - y_A; z_B - z_A) = (3-1, 1-2, -2-0) = (2, -1, -2)$$

Its length is

$$\left|\overrightarrow{AB}\right| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} = \sqrt{2^2 + (-1)^2 + (-2)^2} =$$

$$= \sqrt{9} = 3$$

The direction cosines are

$$\cos\alpha = \frac{a_x}{\left|\vec{a}\right|} = \frac{2}{3} \qquad \cos\beta = \frac{a_y}{\left|\vec{a}\right|} = -\frac{1}{3} \qquad \cos\gamma = \frac{a_z}{\left|\vec{a}\right|} = -\frac{2}{3}$$

Nonlinear operations with vectors

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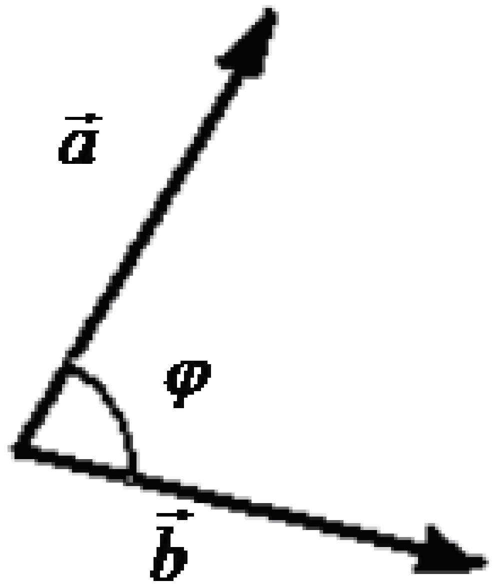
Scalar product

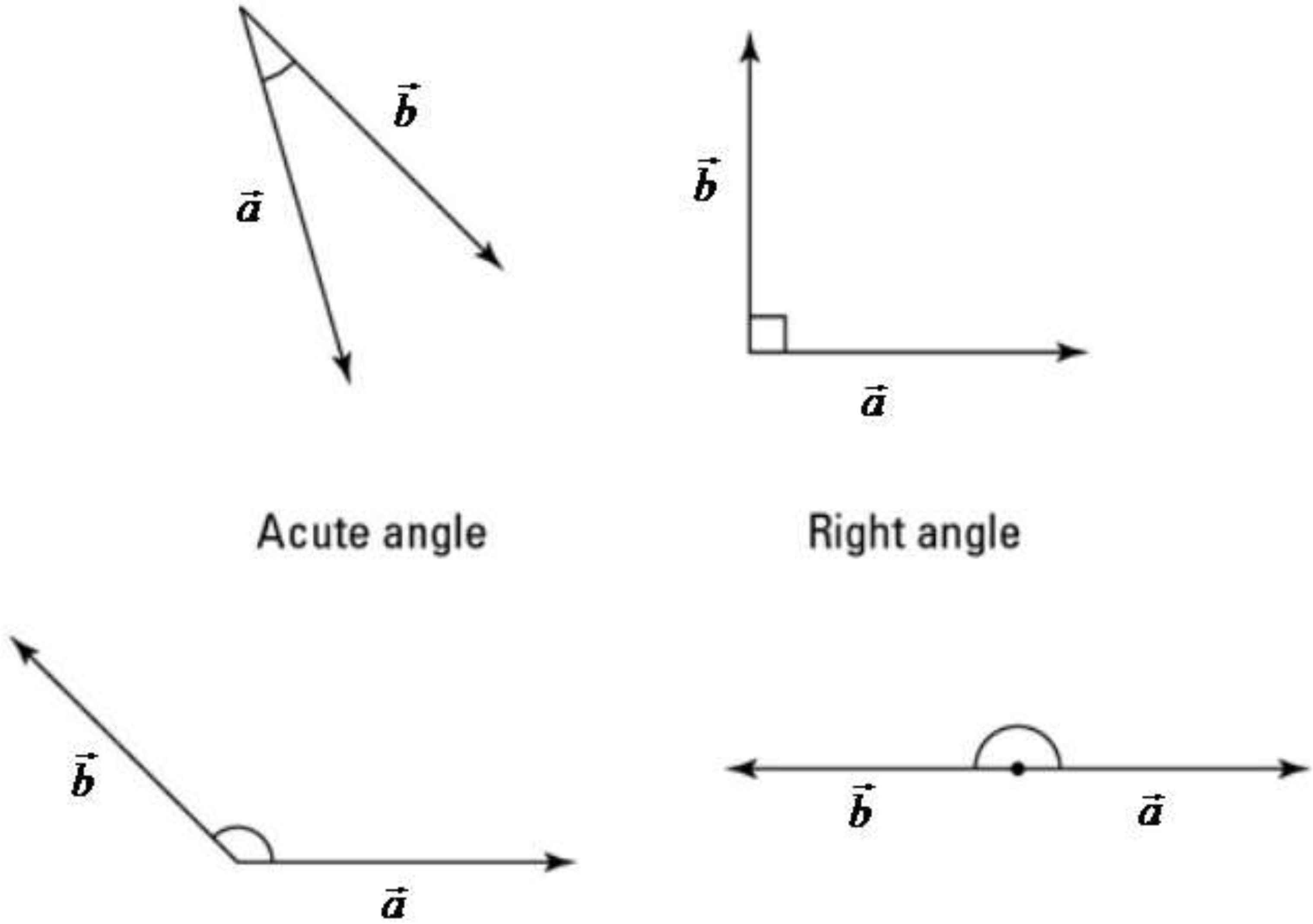
Scalar product. The value $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi$ is called *a scalar product* of the vectors $\vec{a}$ and $\vec{b}$.



If the vectors $\vec{a}$ and $\vec{b}$ are given by their coordinates then their scalar product is as follows:

$$\vec{a}\cdot\vec{b}=a_x\cdot b_x+a_y\cdot b_y+a_z\cdot b_z$$

Types of angles between vectors



Obtuse angle Straight angle

It is obvious that if vectors are perpendicular then their *scalar product is equal to zero*, i.e.

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi=|\vec{a}|\cdot|\vec{b}|\cdot\cos 90^{o}=0$$

$$\vec{a}\cdot\vec{b}=0 \qquad a_x\cdot b_x+a_y\cdot b_y+a_z\cdot b_z=0$$

The **basic properties** of a scalar product:

1) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

2) a length of the vector $\vec{a}$ is defined by means of a scalar product as

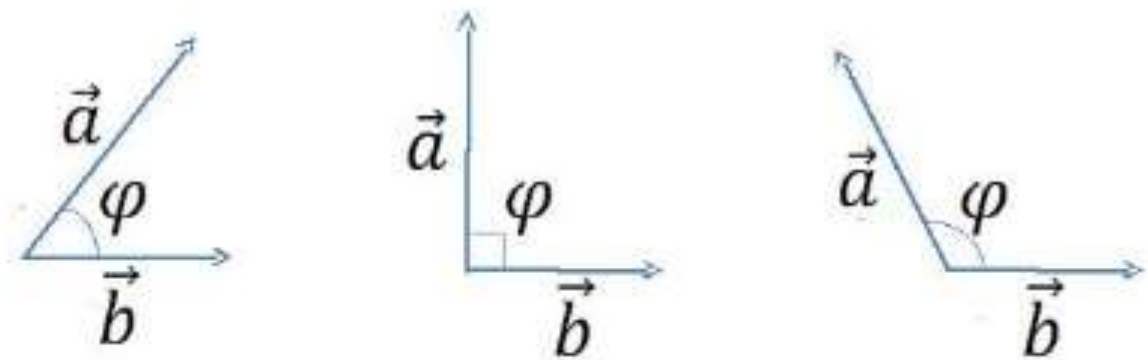
$$|\vec{a}|=\sqrt{\vec{a}\cdot\vec{a}} \qquad \vec{a}\cdot\vec{a}=|\vec{a}|^2$$

3) the cosine of an angle between the vectors $\vec{a}$ and $\vec{b}$ is calculated as

$$\cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\frac{a_x\cdot b_x+a_y\cdot b_y+a_z\cdot b_z}{\sqrt{a_x{}^2+a_y{}^2+a_z{}^2}\sqrt{b_x{}^2+b_y{}^2+b_z{}^2}}$$

A scalar product

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi$$



$\vec{a} \cdot \vec{b} > 0$ $\quad$ $\vec{a} \cdot \vec{b} = 0$ $\quad$ $\vec{a} \cdot \vec{b} < 0$

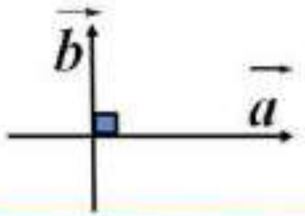
Properties of a scalar product

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a};$$
$$\alpha \vec{a} \cdot \vec{b} = \alpha(\vec{a} \cdot \vec{b});$$
$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c};$$
$$|\vec{a}| = \sqrt{\vec{a}^{\,2}};$$
$$(\vec{a} + \vec{b})^2 = \vec{a}^{\,2} + 2 \cdot (\vec{a} \cdot \vec{b}) + \vec{b}^{\,2};$$

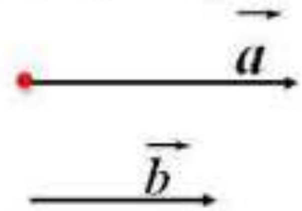
$$(\vec{a} \cdot \vec{b})^2 \leqslant \vec{a}^{\,2} \cdot \vec{b}^{\,2}$$

Properties

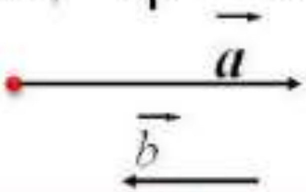
$\vec{a} \perp \vec{b}, \quad \varphi = 90^0;$



$\vec{a} \uparrow\uparrow \vec{b}, \quad \varphi = 0^0;$



$\vec{a} \uparrow\downarrow \vec{b}, \quad \varphi = 180^0;$



Economic EXAMPLE

Dot Product of Two Vectors

Definition 2.1. For any two vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$, the *dot product*[4] of $\mathbf{x}$ and $\mathbf{y}$ is denoted by $(\mathbf{x}, \mathbf{y})$, and is defined as

$$(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i . \qquad (2.2)$$

[4]Other terms for dot product are *scalar product* and *inner product*.

Example 2.5 (Household expenditures). Suppose the family consumes n goods. Let $\mathbf{p}$ be the vector of prices of these commodities (we assume competitive economy and take them as given), and $\mathbf{q}$ be the vector of the amounts of commodities consumed by this household. Then the total expenditure of the household can be obtained by dot product of these two vectors

$$\mathrm{E} = (\mathbf{p}, \mathbf{q}).$$

Example 4. Find the angle between the vectors $\vec{a} = (2,-1,-2)$ and $\vec{b} = (0,3,4)$

Solution. Let's use the formula:

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{a_x\cdot b_x + a_y\cdot b_y + a_z\cdot b_z}{\sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}\sqrt{b_x{}^2 + b_y{}^2 + b_z{}^2}}$$

We have

$$\vec{a}\cdot\vec{b} = 2\cdot 0 + (-1)\cdot 3 + (-2)\cdot 4 = -11$$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$|\vec{b}| = \sqrt{0^2 + 3^2 + 4^2} = \sqrt{25} = 5$$

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{-11}{3\cdot 5} = -\frac{11}{15}$$

$$\varphi = \pi - \arccos\frac{11}{15}$$

Example 5. The vectors $\vec{a}$ and $\vec{b}$ form an angle $\varphi = 60^o$, $|\vec{a}| = 3$, $|\vec{b}| = 2$. Find a scalar product of the vectors $\vec{a}$ and $\vec{b}$

Solution.

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi = 3 \cdot 2 \cdot \cos 60^o = 6 \cdot \frac{1}{2} = 3$$

Vector (cross) product

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Vector product. *The vector product* of the vectors $\vec{a}$ and $\vec{b}$ is the vector $\vec{c}$ which satisfies the following conditions:

1) it is perpendicular to both the vector-multiplicands $\vec{a}$ and $\vec{b}$
2) it is directed in such a way that, looking from its terminus, the shortest turn on the angle φ from $\vec{a}$ to $\vec{b}$ occurs anticlockwise, i.e. the triple of vectors $\vec{a}$, $\vec{b}$, $\vec{a}\times\vec{b}$ is right-hand triple,
3) the length of this vector is equal to

$$\left|\vec{c}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin\varphi$$

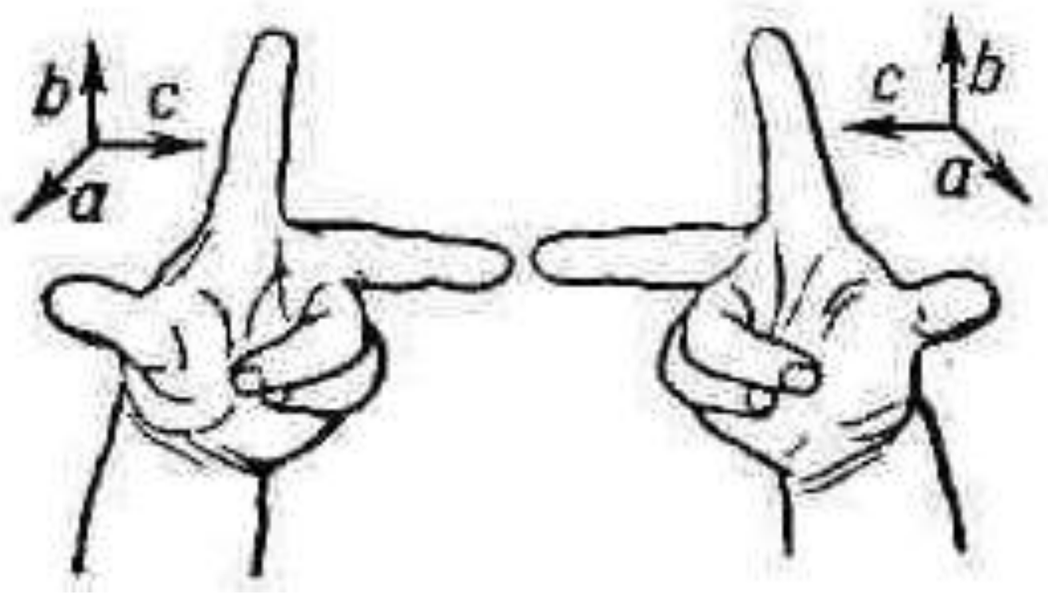
за годинниковою стрілкою

clockwise

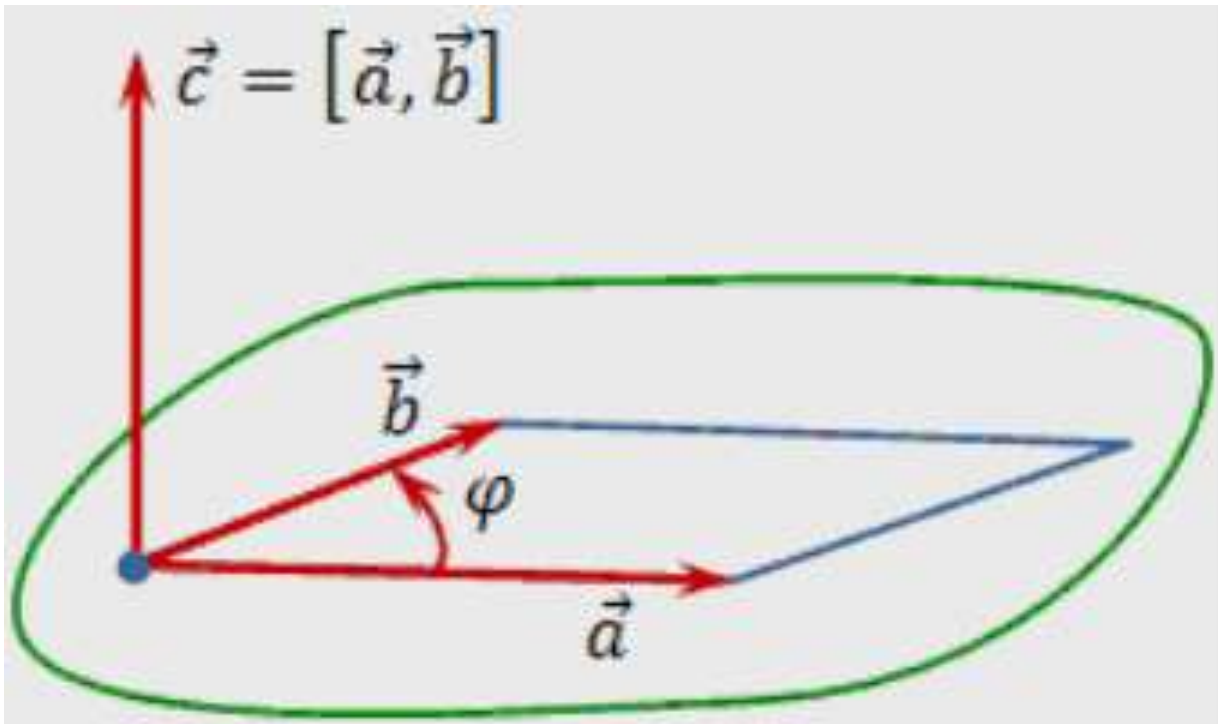
проти за годинникової стрілки

anticlockwise

Right-hand triple of vectors and left-hand triple of vectors

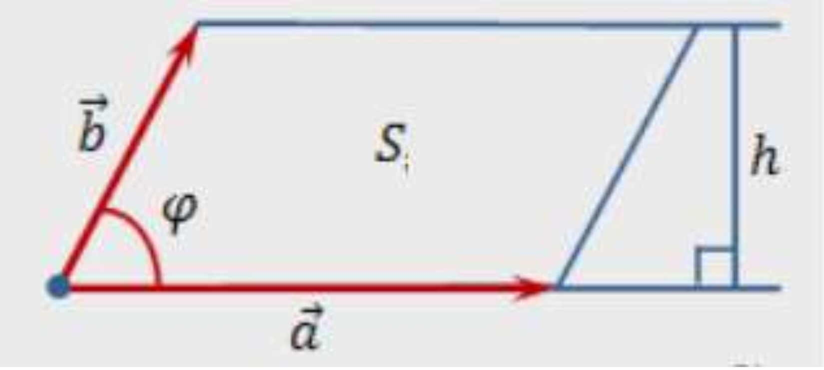


The vector product of the vectors $\vec{a}$ and $\vec{b}$ is designated by $\vec{c} = \vec{a} \times \vec{b}$ or $\vec{c} = [\vec{a}, \vec{b}]$.

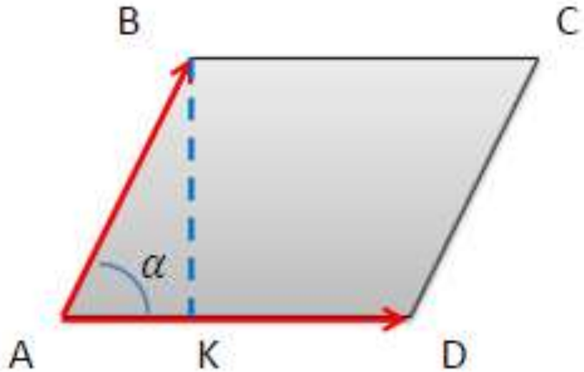


A module of a vector product is numerically equal to the area of a parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$:

$$S = \left|\vec{c}\right| = \left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin\varphi$$



PROPERTY



$$BK = AB \cdot sin(\alpha)$$

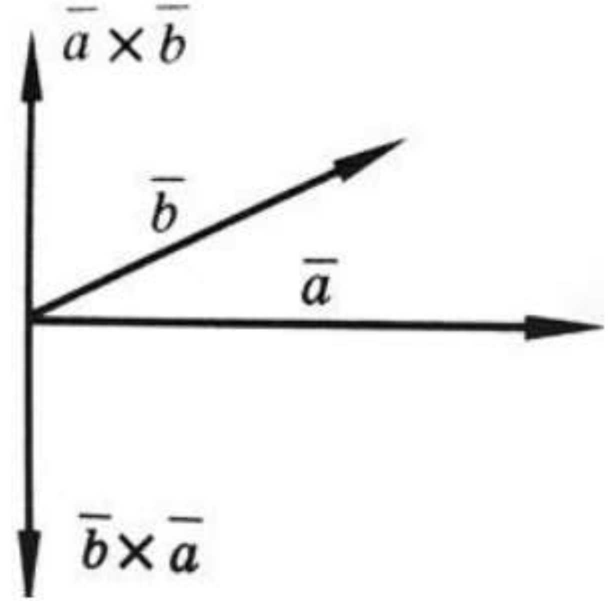
$$AB \cdot AD \cdot \sin(\alpha) = BK \cdot AD = S_{ABCD}$$

$$S_{\Delta ABD} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$$

The **basic properties** of a vector product:

1) . the sign of the vector product changes on opposite if the factors are transposed

$$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$$



2) $$\vec{a}\times\vec{a}=0$$

The **vector product** through the coordinates of the vector-multiplicands is expressed as it follows:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

If two vectors are collinear then their vector product is equal to zero, i.e. their coordinates are proportional or equal:

$$\vec{a}\times\vec{b}=0$$

$$\frac{b_x}{a_x}=\frac{b_y}{a_y}=\frac{b_z}{a_z}$$

Properties

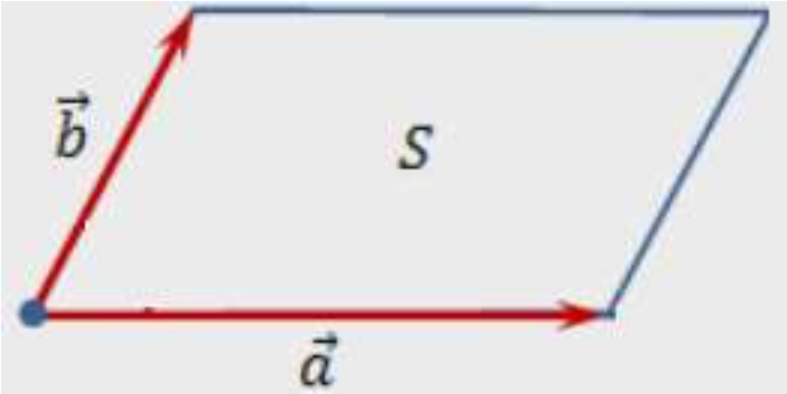
$$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a},$$

$$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c},$$

$$(\lambda\cdot\vec{a})\times\vec{b}=\lambda\cdot(\vec{a}\times\vec{b}),$$

$$\vec{a}\times\vec{a}=0.$$

Example 6. Vectors $\overline{a}=(4,-5,0)$ and $\overline{b}=(0,4,-3)$ are given. Calculate the area of the parallelogram constructed on these vectors.



Solution. The area is calculated by the formula: $S=\left|\overline{a}\times\overline{b}\right|$

The vector product of the vectors $\overline{a}$ and $\overline{b}$ is equal to:

$$\overline{a}\times\overline{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -5 & 0\\ 0 & 4 & -3\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ 4 & -5\\ 0 & 4\end{matrix}=\vec{i}\cdot 15+\vec{j}\cdot 0+\vec{k}\cdot 16-\vec{k}\cdot 0-\vec{i}\cdot 0+\vec{j}\cdot 12=$$

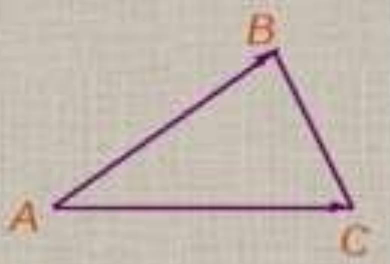
$$=15\vec{i}+12\vec{j}+16\vec{k}$$

Calculate the area of the parallelogram:

$$S = \left|\overline{a} \times \overline{b}\right| = \sqrt{15^2 + 12^2 + 16^2} = \sqrt{625} = 25$$ (sq. units)

TASK

$A(2;3;1)\quad B(5;6;3)\quad C(7;1;10)$



$$\overline{AB}=\{5-2;6-3;3-1\}=\{3;3;2\}$$

$$\overline{AC}=\{7-2;1-3;10-1\}=\{5;-2;9\}$$

$$S=\frac{1}{2}\left|\bar{a}\times\bar{b}\right|$$

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 3 & 3 & 2\\ 5 & -2 & 9\end{vmatrix}=31\bar{i}-17\bar{j}-21\bar{k}$$

$$S=\frac{1}{2}\sqrt{31^2+(-17)^2+(-21)^2}=\frac{1}{2}\sqrt{1691}\approx 20.6$$

Mixed product

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Місюра Є.Ю.)

Mixed product of vectors. *The mixed product* of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is equal to the value obtained after scalar-multiplying one vector by a vector product of two others:

$$\left(\vec{a},\vec{b},\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$$

If the vectors are given by their coordinates: $\vec{a}=\left(a_x,a_y,a_z\right)$, $\vec{b}=\left(b_x,b_y,b_z\right)$ and $\vec{c}=\left(c_x,c_y,c_z\right)$ then their mixed product can be found according to the formula:

$$\left(\vec{a},\vec{b},\vec{c}\right)=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Denotation of a mixed product

$$(\overline{a};\overline{b};\overline{c})$$

$$(\overline{a}\times\overline{b})\cdot\overline{c}$$

$$\overline{a}\cdot(\overline{b}\times\overline{c})$$

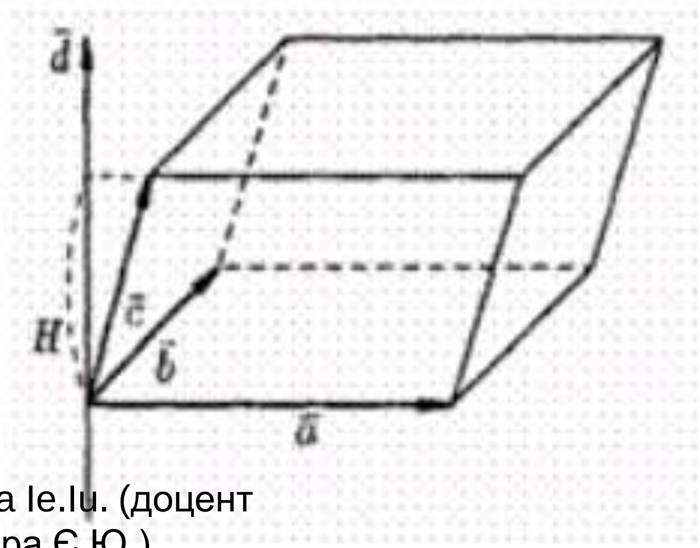
$$\overline{a}\,\overline{b}\,\overline{c}$$

From the definition of a mixed product it follows that ***the condition of complanarity*** of vectors is equality to zero of their mixed product:

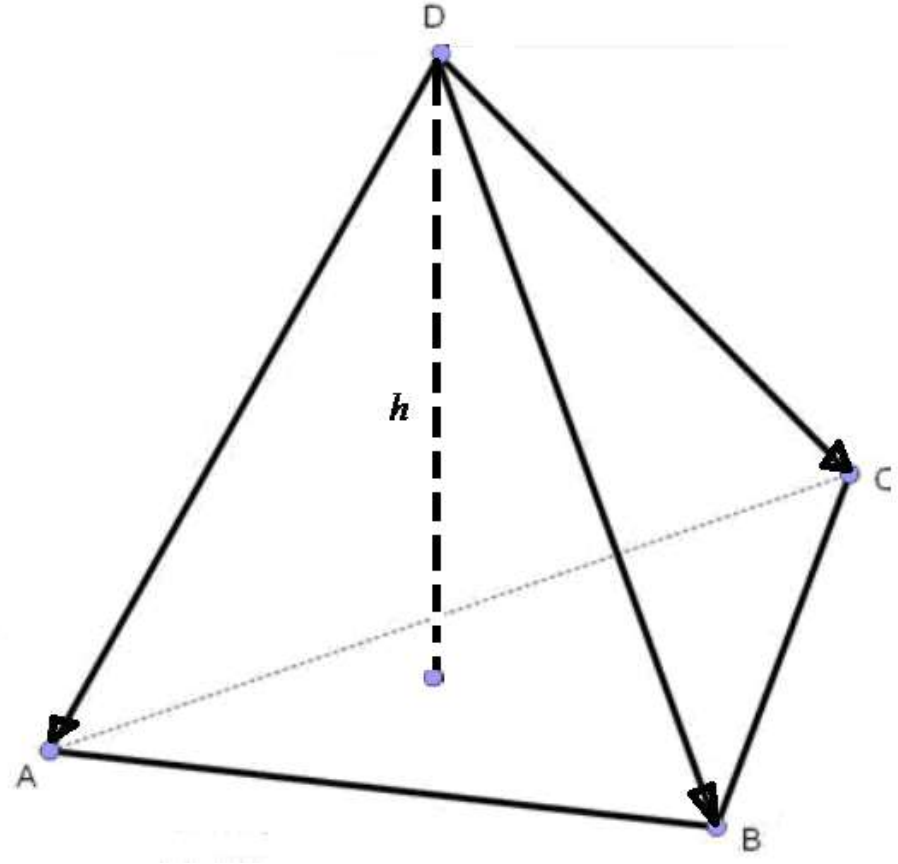
$$(\vec{a},\vec{b},\vec{c})=0 \qquad \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = 0$$

The module of the mixed product is equal to the volume of the parallelepiped constructed on three vectors, i.e.

$$V = \left|(\vec{a},\vec{b},\vec{c})\right|$$



The volume of a tetrahedron is equal to $V = \frac{1}{6}\left|\left(\vec{a}, \vec{b}, \vec{c}\right)\right|$

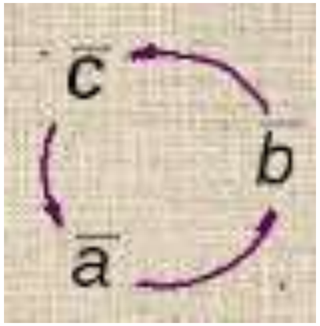


$$V = \frac{1}{6}\left|\left(\overrightarrow{DA}, \overrightarrow{DB}, \overrightarrow{DC}\right)\right|$$

Properties

The law of a circular permutation:

$$\overline{a}\,\overline{b}\,\overline{c} = \overline{b}\,\overline{c}\,\overline{a} = \overline{c}\,\overline{a}\,\overline{b} = -\overline{a}\,\overline{c}\,\overline{b} = -\overline{c}\,\overline{b}\,\overline{a} = -\overline{b}\,\overline{a}\,\overline{c}$$



To get the formula for the mixed product!!!

$$\bar{a} = x_1 \cdot \bar{i} + y_1 \cdot \bar{j} + z_1 \cdot \bar{k} \qquad \bar{b} = x_2 \cdot \bar{i} + y_2 \cdot \bar{j} + z_2 \cdot \bar{k}$$

$$\bar{c} = x_3 \cdot \bar{i} + y_3 \cdot \bar{j} + z_3 \cdot \bar{k}$$

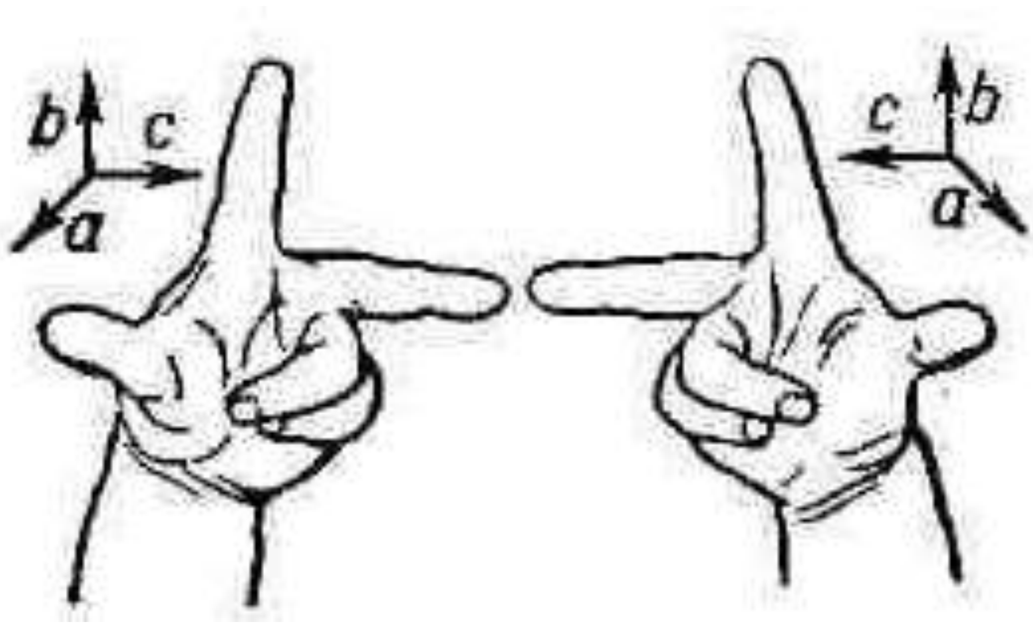
$$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} \cdot \bar{i} - \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} \cdot \bar{j} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \cdot \bar{k}$$

$$\bar{a} \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} \cdot x_1 - \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} \cdot y_1 + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \cdot z_1$$

$$\bar{a}\bar{b}\bar{c} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

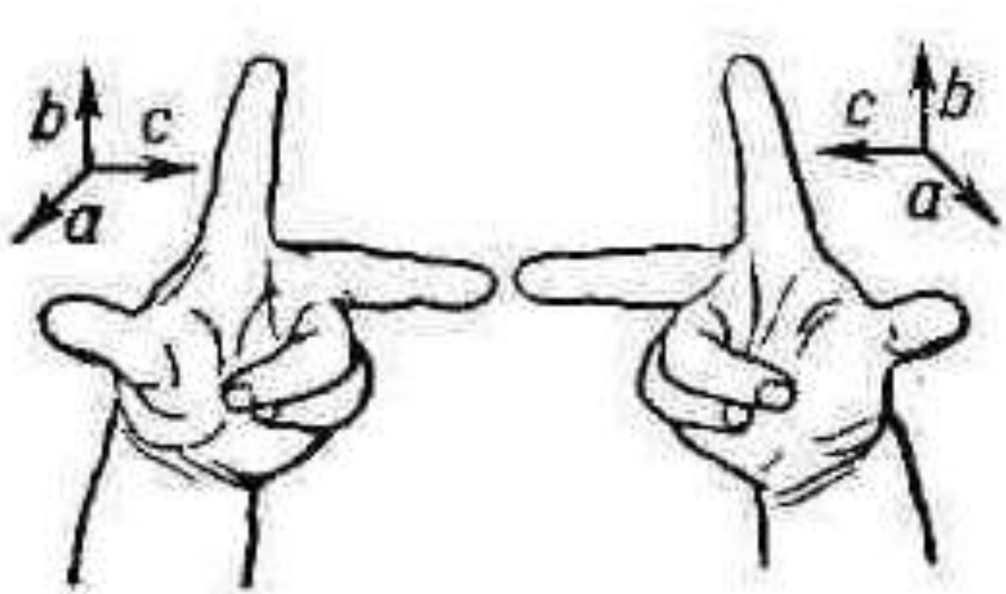
Right-hand triple of vectors if

$$(\bar{a}, \bar{b}, \bar{c}) > 0$$



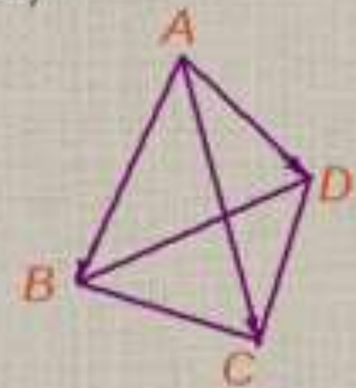
Left-hand triple of vectors if

$$(\bar{a}, \bar{b}, \bar{c}) < 0$$



TASK 1

$A(2;2;2)\quad B(4;3;3)\quad C(4;5;4)\quad D(5;5;6)$



$$\overline{AB}=\{4-2;3-2;3-2\}=\{2;1;1\}$$

$$\overline{AC}=\{4-2;5-2;4-2\}=\{2;3;2\}$$

$$\overline{AD}=\{5-2;5-2;6-2\}=\{3;3;4\}$$

$$\overline{AB}\,\overline{AC}\,\overline{AD}=\begin{vmatrix}2&1&1\\2&3&2\\3&3&4\end{vmatrix}=\begin{vmatrix}0&0&1\\-2&1&2\\-5&-1&4\end{vmatrix}=\begin{vmatrix}-2&1\\-5&-1\end{vmatrix}=7$$

$$V=\frac{1}{6}\left|\overline{a}\,\overline{b}\,\overline{c}\right|$$

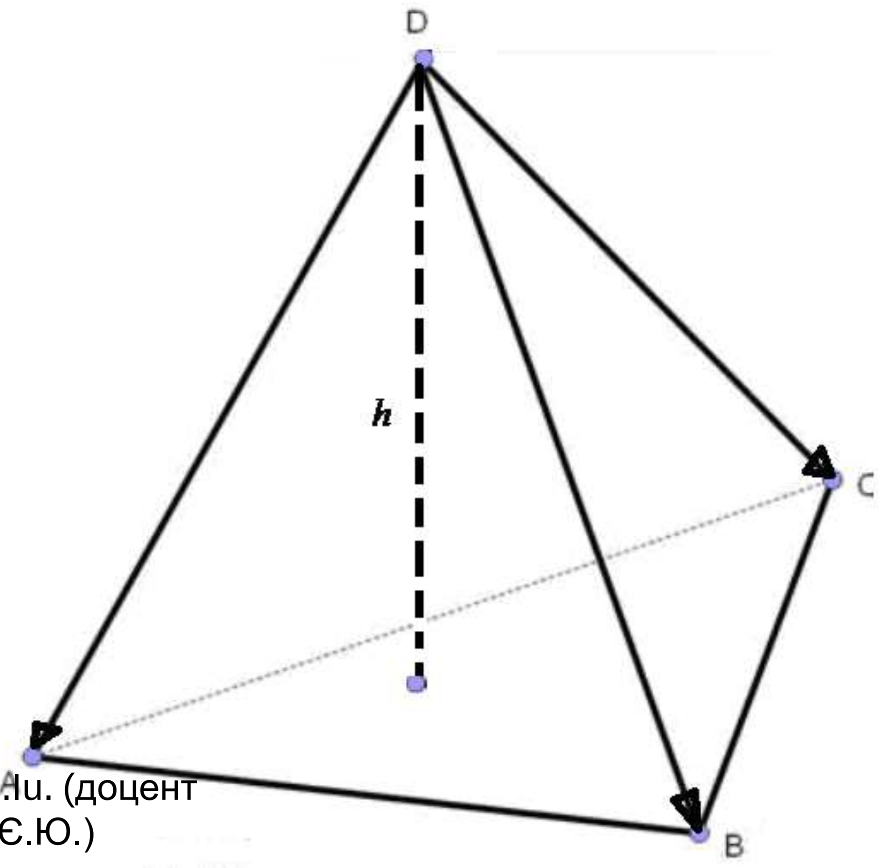
Example. Let the following apexes of a pyramid $A(2,3,1)$, $B(4,1,-2)$, $C(6,3,7)$, $D(-5,-4,2)$ be given. Calculate the pyramid volume and the length of the altitude put down from the apex ***D***.

Solution. The volume of the pyramid is equal to:

$$V = \frac{1}{6}\left|\left(\overrightarrow{DA}, \overrightarrow{DB}, \overrightarrow{DC}\right)\right|$$

or

$$V = \frac{1}{3} \cdot S \cdot h$$



Let's find $\overrightarrow{DA}$, $\overrightarrow{DB}$ and $\overrightarrow{DC}$:

$$\overrightarrow{DA}=\left(x_A-x_D,y_A-y_D,z_A-z_D\right)=\left(2-(-5),3-(-4),1-2\right)=\left(7,7,-1\right)$$

$$\overrightarrow{DB}=\left(x_B-x_D,y_B-y_D,z_B-z_D\right)=\left(4-(-5),1-(-4),-2-2\right)=\left(9,5,-4\right)$$

$$\overrightarrow{DC}=\left(x_C-x_D,y_C-y_D,z_C-z_D\right)=\left(6-(-5),3-(-4),7-2\right)=\left(11,7,5\right)$$

Then

$$\left(\overrightarrow{DA},\overrightarrow{DB},\overrightarrow{DC}\right)=\left|\begin{array}{ccc|cc} 7 & 7 & -1 & 7 & 7 \\ 9 & 5 & -4 & 9 & 5 \\ 11 & 7 & 5 & 11 & 7 \end{array}\right.=$$

$$=7\cdot5\cdot5+7\cdot(-4)\cdot11+(-1)\cdot9\cdot7-11\cdot5\cdot(-1)-7\cdot(-4)\cdot7-7\cdot9\cdot5=$$

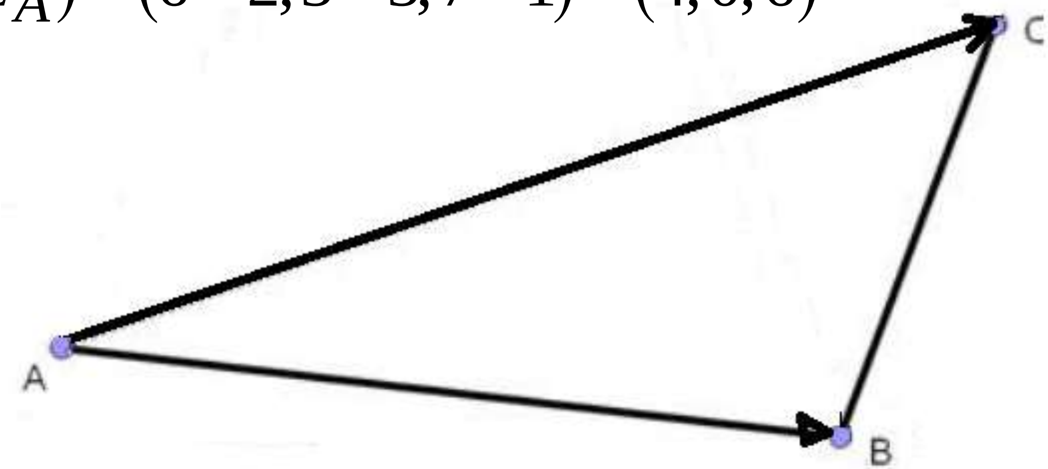
$$=175-308-63+55+196-315=-260$$

Thus $V=\frac{1}{6}\left|\left(\overrightarrow{DA},\overrightarrow{DB},\overrightarrow{DC}\right)\right|=\frac{1}{6}\cdot\left|-260\right|=\frac{260}{6}=\frac{130}{3}$ (cubed units)

Find the area of triangle *ABC*:

$$\overrightarrow{AB}=\left(x_B-x_A,y_B-y_A,z_B-z_A\right)=\left(4-2,1-3,-2-1\right)=\left(2,-2,-3\right)$$

$$\overrightarrow{AC}=\left(x_C-x_A,y_C-y_A,z_C-z_A\right)=\left(6-2,3-3,7-1\right)=\left(4,0,6\right)$$



$$\overrightarrow{AB}\times\overrightarrow{AC}=\left|\begin{matrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -2 & -3\\ 4 & 0 & 6\end{matrix}\right|\begin{matrix}\vec{i} & \vec{j}\\ 2 & -2\\ 4 & 0\end{matrix}=-12\cdot\vec{i}-12\cdot\vec{j}+0\cdot\vec{k}+8\cdot\vec{k}-0\cdot\vec{i}-12\cdot\vec{j}=$$

$$=-12\vec{i}-24\vec{j}+8\vec{k}$$

Let's find the area of triangle *ABC*:

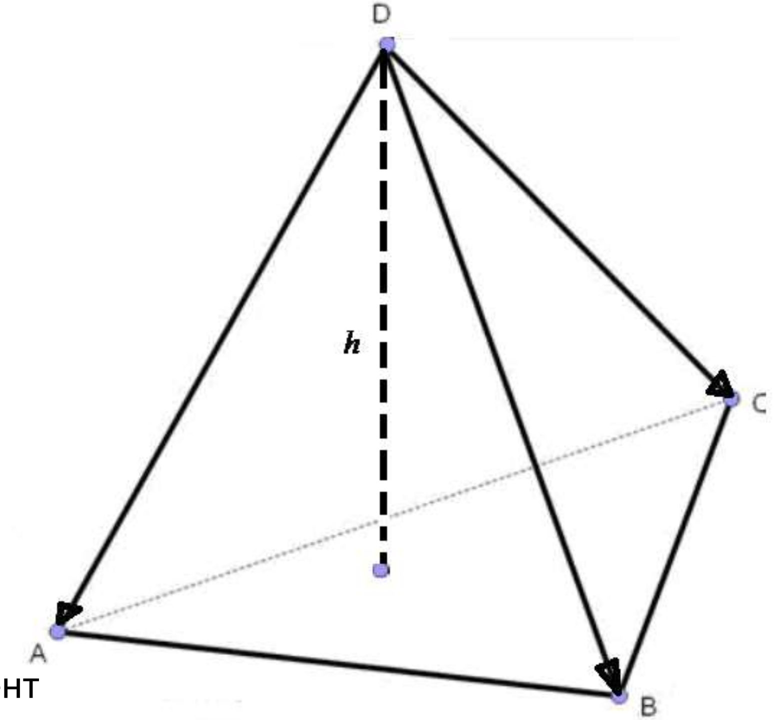
$$S_{ABC} = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\sqrt{(-12)^2+(-24)^2+8^2} =$$

$$= \frac{1}{2}\sqrt{784} = \frac{28}{2} = 14$$ (square units)

Thus, let's calculate the length of the altitude put down from the apex ***D***:

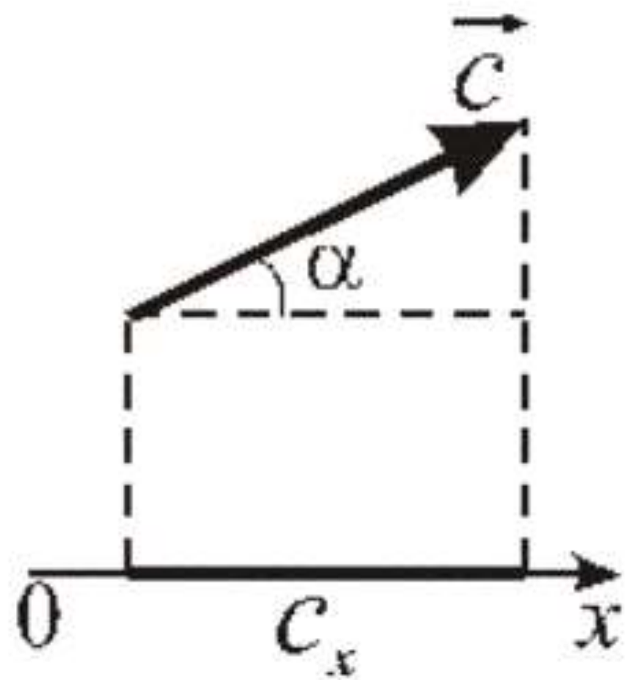
$$h = \frac{3\cdot V}{S} = \frac{3\cdot V}{S_{ABC}} = \frac{3\cdot\frac{130}{3}}{14} = \frac{65}{7}$$

(units of length)



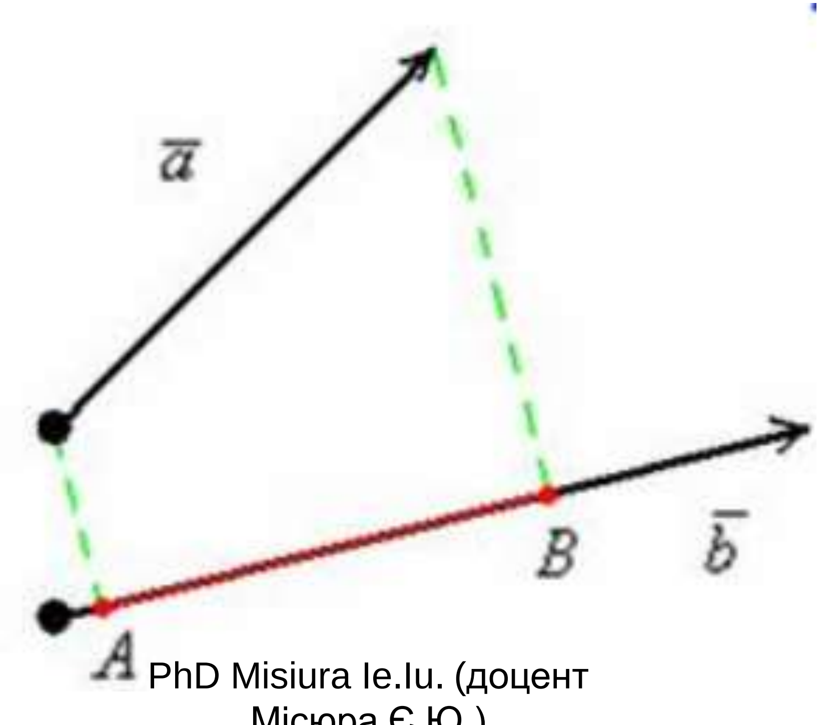
Projection of the vector $\vec{c}$ on the x-axis

$$pr_x \vec{c} = c_x = |\vec{c}| \cdot \cos\alpha$$

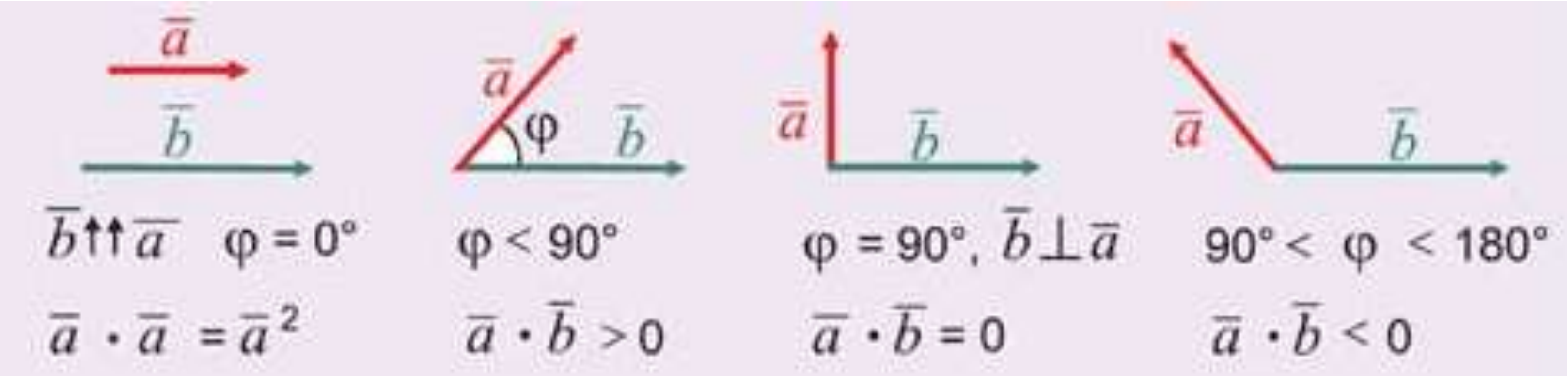


Projection of the vector $\vec{a}$ on the vector $\vec{b}$:

$$pr_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|}$$



Conclusions



Economic EXAMPLE

An Economic Example: Two Plants

Consider a firm operating two plants in two different locations. They both produce the same output (say, 10 units) using the same type of inputs. Although the amounts of inputs vary between the plants the output level is the same.

The firm management suspects that the production cost in Plant 2 is higher than in Plant 1. The following information was collected from the managers of these plants.

PLANT 1

Input	Price	Amount used
Input 1	3	9
Input 2	5	10
Input 3	7	8

PLANT 2

Input	Price	Amount used
Input 1	4	8
Input 2	7	12
Input 3	3	9

Question 1. Does this information confirm the suspicion of the firm management?

Answer. In order to answer this question one needs to calculate the cost function. Let w_{ij} denote the price of the ith input at the jth plant and x_{ij} denote the quantity of ith input used in production jth plant ($i = 1, 2, 3$ and $j = 1, 2$). Suppose both of these magnitudes are perfectly divisible, therefore can be represented by real numbers. The cost of production can be calculated by multiplying the amount of each input by its price and summing over all inputs.

This means price and quantity vectors (**p** and **q**) are defined on real space and inner product of these vectors are defined. In other words, both **p** and **q** are in the space $\mathbb{R}^3$. The cost function in this case can be written as an inner product of price and quantity vectors as

$$c = (\mathbf{w}, \mathbf{q}), \tag{2.8}$$

where c is the cost, a scalar. Using the data in the above tables cost of production can be calculated by using (2.8) as:

In Plant 1 the total cost is 133, which implies that unit cost is 13.3.

In Plant 2, on the other hand, cost of production is 143, which gives unit cost as 14.3 which is higher than the first plant.

That is, the suspicion is reasonable.

Question 2. The manager of the Plant 2 claims that the reason of the cost differences is the higher input prices in her region than in the other. Is the available information supports her claim?

Answer. Let the input price vectors for Plant 1 and 2 be denoted as $\mathbf{p}_1$ and $\mathbf{p}_2$. Suppose that the latter is a multiple λ of the former, i.e.,

$$\mathbf{p}_2 = \lambda \mathbf{p}_1.$$

Since both vectors are in the space $\mathbb{R}^3$, length is defined for both. From the definition of length one can obtain that

$$|\mathbf{p}_2| = \lambda |\mathbf{p}_1|.$$

In this case, however as can be seen from the tables this is not the case. Plant I enjoys lower prices for inputs 2 and 3, whereas Plant 2 enjoys lower price for input 3. For a rough guess, one can still compare the lengths of the input price vectors which are

$$|\mathbf{p}_1| = 9.11, |\mathbf{p}_2| = 8.60,$$

which indicates that price data does not support the claim of the manager of the Plant 2. When examined more closely, one can see that the Plant 2 uses the most expensive input (input 2) intensely. In contrast, Plant 2 managed to save from using the most expensive input (in this case input 3). Therefore, the manager needs to explain the reasons behind the choice mixture of inputs in her plant.