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TOPIC:

Elements of the theory of matrices and determinants. Matrix analysis of the international trade market

Application of matrices

Matrices are applied for various disciplines such as Business, Economics, Management, Tourism, statistics and so on.

Let’s discuss how various situations in business and economics can be presented using matrices.

This can be done using the following examples.

Example 1. Number of staff in the office can be represented as follows:

$$\begin{bmatrix} 2 \\ 4 \\ 3 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} \text{Peon} \\ \text{Clerk} \\ \text{Typist} \\ \text{Head clerk} \\ \text{Office superintendent} \end{bmatrix}$$

Application of matrices

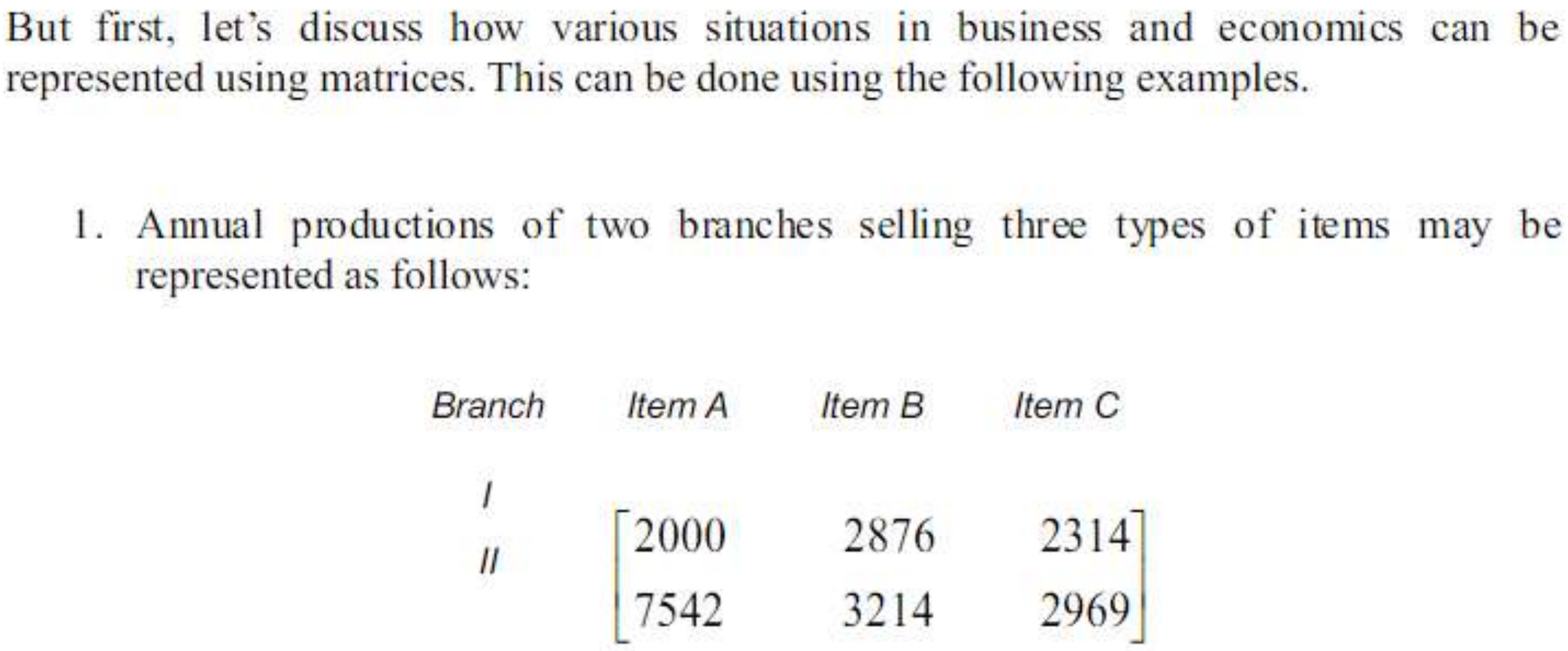
Example 2. An automobile company uses three types of steel c_1 , c_2 and c_3 for producing three types of cars S_1 , S_2 and S_3 . The steel requirements (in tons) for each type of car is given below:

	c_1	c_2	c_3
S_1	2	3	4
S_2	1	1	2
S_3	3	2	1

$$\begin{array}{cc} & \text{cars} \\ \text{steel} & \begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \end{array}$$

Application of matrices

Example 3.



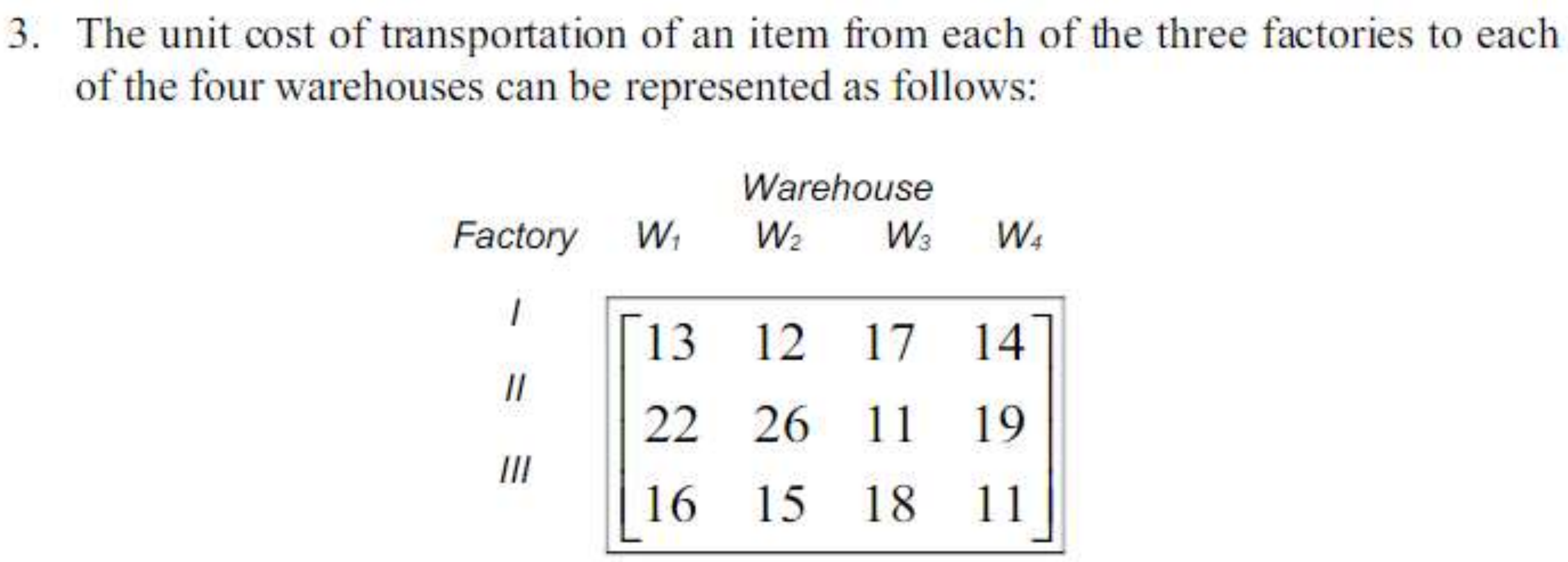
But first, let's discuss how various situations in business and economics can be represented using matrices. This can be done using the following examples.

1. Annual productions of two branches selling three types of items may be represented as follows:

$$\begin{array}{c|ccc} \text{Branch} & \text{Item A} & \text{Item B} & \text{Item C} \\ I & 2000 & 2876 & 2314 \\ II & 7542 & 3214 & 2969 \end{array}$$

Application of matrices

Example 4.



3. The unit cost of transportation of an item from each of the three factories to each of the four warehouses can be represented as follows:

$$\begin{array}{ccccc} & & \text{Warehouse} & & \\ \text{Factory} & W_1 & W_2 & W_3 & W_4 \\ I & 13 & 12 & 17 & 14 \\ II & 22 & 26 & 11 & 19 \\ III & 16 & 15 & 18 & 11 \end{array}$$

Lecture plan

1. A definition and types of basic matrices.

2. Basic operations with matrices and properties of these operations.

3. A definition of a determinant, rules of calculation and properties of determinants.

4. Minors and cofactors.

5. Laplace’ theorem of decomposition of a determinant in its rows or columns

6. Notion of an inverse matrix

7. Application Laplace’ theorem of the decomposition of the determinant and the 5-th property of determinants

1. A DEFINITION AND TYPES OF BASIC MATRICES

Definition

- Matrix / матриця

Definition. A ***matrix*** is a two-dimensional arrangement of numbers in rows and columns by a pair of square or round brackets in the form shown below.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \left[a_{ij}\right]_{m \times n}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \left(a_{ij}\right)_{m \times n}$$

Definition

- Matrix / матриця
- Row / рядок
- Column / стовпець

The above figure shows an ***m* × *n*** matrix of ***m*** rows and ***n*** columns.

Here

m is a number of rows; ***n*** is a number of columns; ***m* × *n*** is the size;

a_{ij} is an element of the matrix ***A*** and represents the entry in the matrix ***A*** on the ***i***-th row and ***j***-th column.

For example,

a_{11} - element in row 1, column 1;

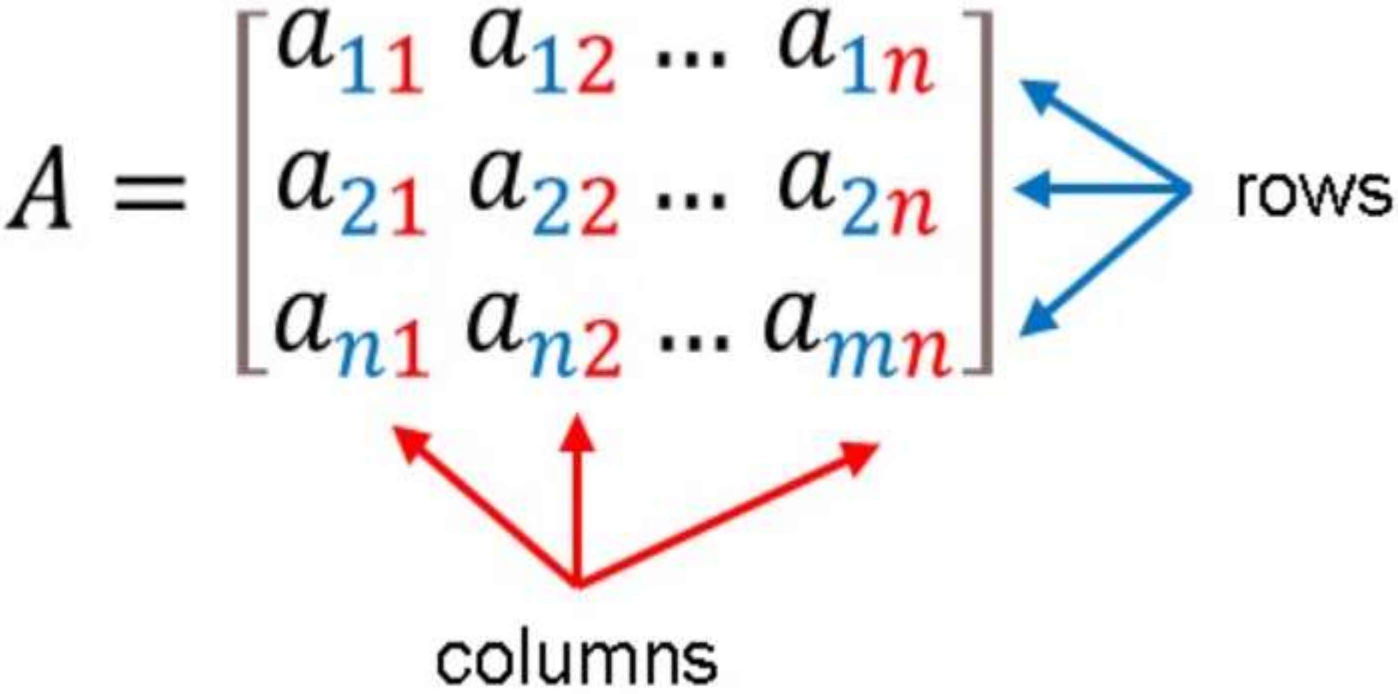
a_{12} - element in row 1, column 2;

a_{mn} - element in row *m*, column *n*.

a_{11} a_{12} a_{mn}

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

We usually denote matrices with capital letters, such as A, B, C, and so on.



$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{n1} & a_{n2} & \dots & a_{mn} \end{bmatrix}$$

A≔ A matrix is denoted by an uppercase letter.

a≔ A matrix entry is denoted by a lower case letter.

Definition

- Matrix / матриця
- Row / рядок
- Column / стовпець
- Order / порядок
- Size / розмір

Definition. The size ***m* × *n*** of a matrix is called its *order*.

The order is

(number of rows) × (number of columns).

Notation. Matrices are denoted by the capital letters ***A***, ***B***, ***C*** and so on.

The **size** of a matrix is

(number of rows) × (number of columns)

$$m \times n$$

$$3 \times 4$$

Example 1. Consider the matrix:

$$A = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$$

Question 1. Define the size.

Question 2. Define the 2-nd row.

Question 3. Define the 3-rd column.

Question 4. Define the element a_{23}

.

Example 1. Consider the matrix:

$$A=\begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$$

Answer 1. Its size ***3*** × ***4***.

Answer 2. The 2-nd row is $\begin{pmatrix} 3 & 1 & 5 & 2 \end{pmatrix}$

Answer 3. The 3-rd column is $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$

Answer 4. The element is $a_{23}=5$

Types of basic matrices

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Definition

- *Rectangular matrix /* прямокутна матриця
- *Square matrix / квадратна матриця*

Type 1. A matrix is called *rectangular matrix*, if $m \neq n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m \times n}$$

$$A = \begin{pmatrix} 1 & 7 & 2 & 9 \\ 9 & 2 & 7 & 1 \end{pmatrix}_{2 \times 4}$$

Type 2. A matrix is called *square matrix*, if $m = n$. The number of rows is considered to be *the order of this matrix*.

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}_{n \times n}$$

$$A = \begin{pmatrix} 0 & 3 \\ 4 & 5 \end{pmatrix}_{2 \times 2}$$

Diagonals of a square matrix

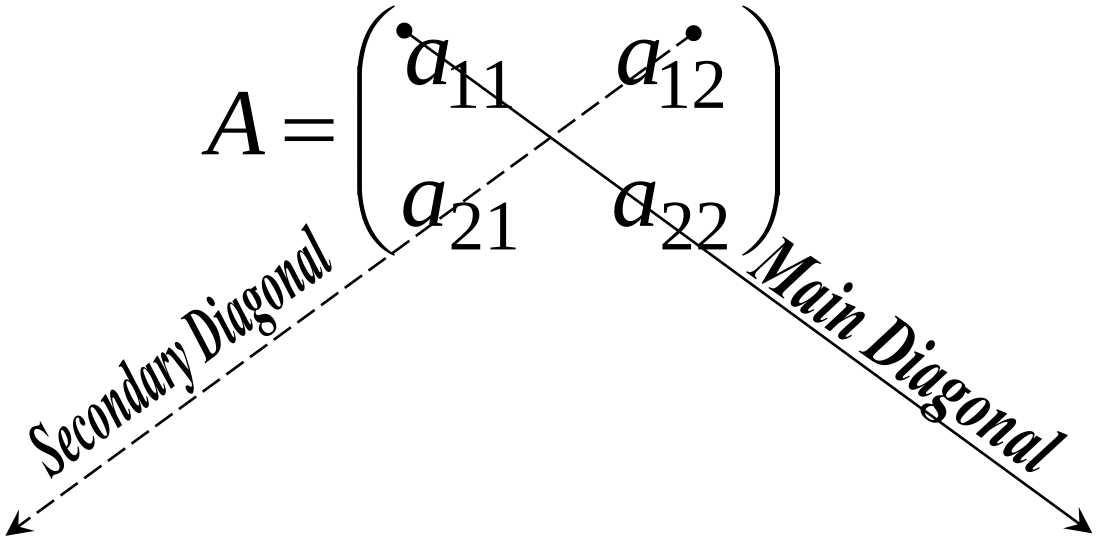
$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}_{n \times n}$$

$a_{11}, a_{22}, \ldots, a_{nn}$ are elements of *the main diagonal* of the matrix

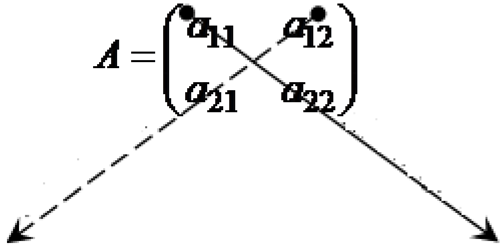
$a_{1\,n}, a_{2\,n-1}, \ldots, a_{n\,1}$ are elements of *the secondary diagonal* of the matrix

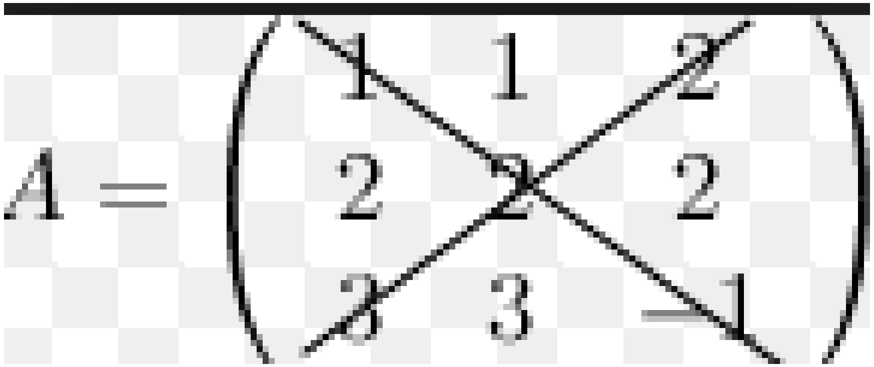
Definition

- *Main diagonal /* головна діагональ
- *Secondary diagonal / побічна* діагональ



$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$


$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$


$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 2 \\ 3 & 3 & -1 \end{pmatrix}$

Type 3. A matrix-row is called a matrix consisting of the only row:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix}_{1\times n}$$

$$A = \begin{pmatrix} -4 & 7 & 2 \end{pmatrix}_{1\times 3}$$

Type 4. *A matrix-column* is called a matrix consisting of the only column:

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{pmatrix}_{m\times 1} \qquad A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{3\times 1}$$

Definition

- *Diagonal matrix /* діагональна матриця
- *Unit (identity) matrix / одинична* матриця
- *Zero (or null) matrix / нульова матриця*

Type 5. A square matrix is called *diagonal* if all its elements except diagonal ones are equal to zero and denoted by:

$$D = \begin{pmatrix} d_{11} & 0 & ... & 0 \\ 0 & d_{22} & ... & 0 \\ ... & ... & ... & 0 \\ 0 & 0 & ... & d_{nn} \end{pmatrix}_{n\times n}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}_{3\times 3}$$

Type 6. If all the diagonal elements of the diagonal matrix are equal to 1 then the matrix is called a *unit (identity) matrix* and designated as *E*:

$$E = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{pmatrix}_{n \times n} \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3 \times 3}$$

Type 7. A matrix is called *zero (or null) matrix* if all its elements are equal to zero and designated by:

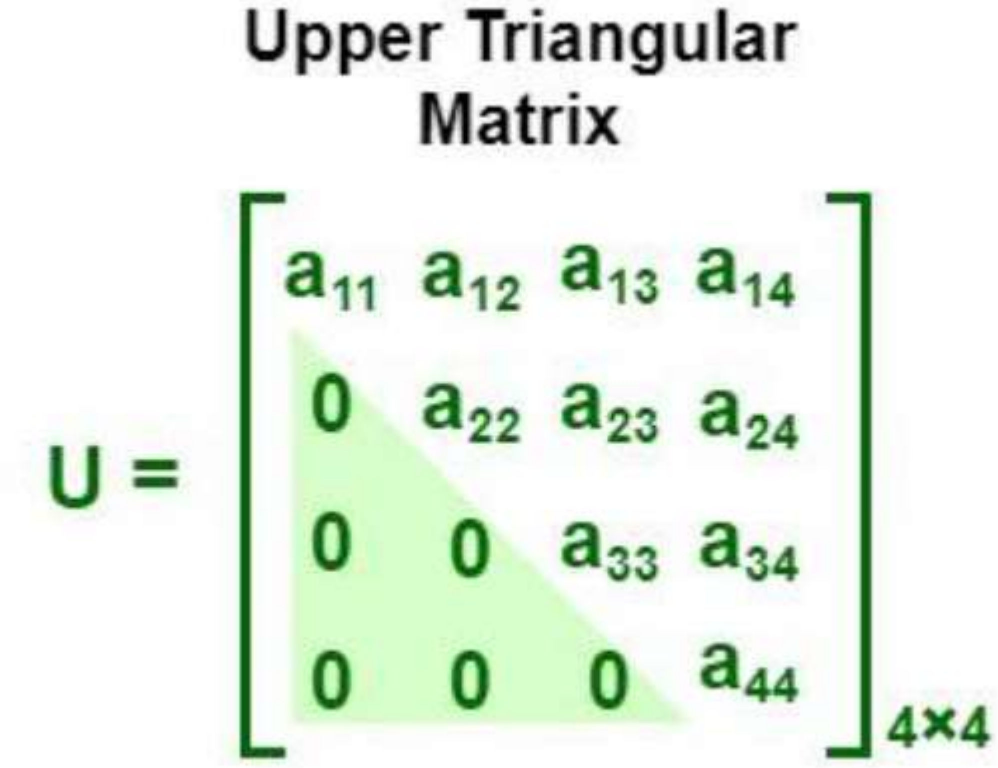
$$O = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2\times 2} \qquad O = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{3\times 4}$$

Definition

- *Upper triangular matrix* / верхня трикутна матриця
- *Lower triangular matrix* / нижня трикутна матриця

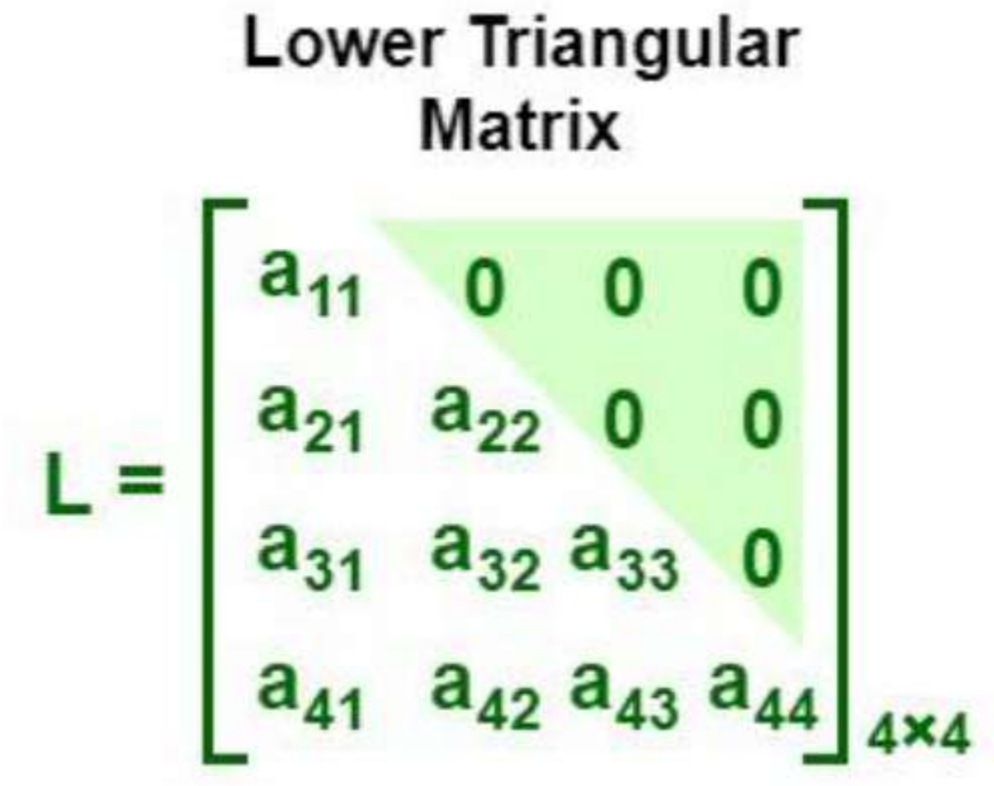
Type 8. If all the elements of a matrix located below (above) the main diagonal are equal to zero then the matrix is called *an upper (lower) triangular matrix*



Upper Triangular Matrix

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}_{4\times 4}$$

Type 8. If all the elements of a matrix located below (above) the main diagonal are equal to zero then the matrix is called *an upper (lower) triangular matrix*



Lower Triangular Matrix

$$L = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times4}$$

2. Basic operations with matrices and properties of these operations

Basic arithmetic operations with numbers

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Basic arithmetic operations with numbers

- Addition (+)
- Subtraction (-)
- Multiplication (*)
- Division(:)

Basic arithmetic operations with numbers

- Addition / додавання (+)
- Subtraction / віднімання (-)
- Multiplication / множення ((·) or (×))
- Division / ділення (:)

<u>Operation 1.</u> ***The addition and subtraction matrices.*** Matrices that have the same order can be added together, or subtracted. The addition, or subtraction, is performed on each of the corresponding elements. Suppose that both matrices *A and B* have *m* rows and *n* columns. Then we write and call this *the sum (or difference) of the two matrices A and B.*

$$C_{m\times n} = A_{m\times n} \pm B_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$$

Example 2. Matrices A and B are given by $A = \begin{pmatrix} 13 & 30 \\ 8 & 15 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 35 \\ 8 & 4 \end{pmatrix}$. Find $A + B$ and $A - B$.

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<u>*Solution.*</u> Matrices *A* and *B* have the same order 2×2, therefore, we can add them together, or subtract:

$$C = A + B = \begin{pmatrix} 13 & 30 \\ 8 & 15 \end{pmatrix} + \begin{pmatrix} 7 & 35 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 13+7 & 30+35 \\ 8+8 & 15+4 \end{pmatrix} = \begin{pmatrix} 20 & 65 \\ 16 & 19 \end{pmatrix}$$

$$C = A - B = \begin{pmatrix} 13 & 30 \\ 8 & 15 \end{pmatrix} - \begin{pmatrix} 7 & 35 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 13-7 & 30-35 \\ 8-8 & 15-4 \end{pmatrix} = \begin{pmatrix} 6 & -5 \\ 0 & 11 \end{pmatrix}$$

Example 3. We do not have a definition for “adding” the matrices ***A*** and ***B***, because matrices have the different order.

$$A = \begin{pmatrix} 5 & 4 & 12 & 7 \\ 10 & 12 & 9 & 14 \end{pmatrix}_{2\times 4} \qquad B = \begin{pmatrix} 13 & 30 \\ 8 & 15 \end{pmatrix}_{2\times 2}$$

The operations of matrix addition (or subtraction) satisfy the following laws:

$$A+B=B+A$$

$$\left(A+B\right)+C=A+\left(B+C\right)$$

$$A+O=A$$

Operation 2. ***The multiplication of a matrix by a scalar value.*** A matrix can be multiplied by a specific value, such as a number (scalar multiplication). Scalar multiplication simply involves the multiplication of each element in a matrix by the scalar value. Suppose that the matrix ***A*** has ***m*** rows and ***n*** columns and $\alpha \in R$ (is any non-zero number).

$$C_{m\times n} = \alpha \cdot A_{m\times n} = \begin{pmatrix} \alpha \cdot a_{11} & \cdots & \alpha \cdot a_{1n} \\ \vdots & & \vdots \\ \alpha \cdot a_{m1} & \cdots & \alpha \cdot a_{mn} \end{pmatrix}$$

The operation of matrix multiplication by some number satisfies the following laws:

$$(\alpha \cdot \beta)A = \alpha(\beta \cdot A)$$

$$(\alpha \pm \beta)A = \alpha A \pm \beta A$$

$$\alpha(A \pm B) = \alpha A \pm \alpha B$$

Example 4. Calculate the matrix $C = 3B - 2A$, if $A = \begin{pmatrix} 2 & -4 & 1 \\ 3 & 0 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 6 & -1 \end{pmatrix}$.

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<u>*Solution.*</u> Matrices ***A*** and ***B*** have the same order, therefore, we can obtain ***C***. The entry **2*A*** is multiplication the matrix ***A*** by **2**:

$$2A = \begin{pmatrix} 2\cdot 2 & 2\cdot(-4) & 2\cdot 1 \\ 2\cdot 3 & 2\cdot 0 & 2\cdot 7 \end{pmatrix} = \begin{pmatrix} 4 & -8 & 2 \\ 6 & 0 & 14 \end{pmatrix}$$

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Let's calculate **3*B*** like **2*A***:

$$3B = \begin{pmatrix} 3\cdot 5 & 3\cdot 1 & 3\cdot 2 \\ 3\cdot 3 & 3\cdot 6 & 3\cdot(-1) \end{pmatrix} = \begin{pmatrix} 15 & 3 & 6 \\ 9 & 18 & -3 \end{pmatrix}$$

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Then

$$C = 3B - 2A = \begin{pmatrix} 15 & 3 & 6 \\ 9 & 18 & -3 \end{pmatrix} - \begin{pmatrix} 4 & -8 & 2 \\ 6 & 0 & 14 \end{pmatrix} = \begin{pmatrix} 11 & 11 & 4 \\ 3 & 18 & -17 \end{pmatrix}$$

Operation 3. *Multiplication of a matrix by a matrix*

Let two matrices $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{p\times q}$ be given and the **number of columns at the first matrix be equal to the number of rows of the second one**, i.e. $n=p$. In this case we can define the operation of multiplication of the matrix ***A*** by the matrix ***B***. The matrix ***C*** of the size $m\times q$, which elements are calculated according to the following rule:

$$c_{ij}=\sum_{k=1}^{n}a_{ik}\cdot b_{kj}=a_{i1}\cdot b_{1j}+\ldots+a_{in}\cdot b_{nj} \qquad i=\overline{1,m} \qquad j=\overline{1,q}$$

$$C=A\cdot B=(a_{ij})_{m\times n}\cdot(b_{ij})_{p\times q}=(c_{ij})_{m\times q}$$

Operation 3. *Multiplication of a matrix by a matrix*

$$A=\left(a_{ij}\right)_{m\times n} \qquad B=\left(b_{ij}\right)_{p\times q} \qquad i=\overline{1,m}$$

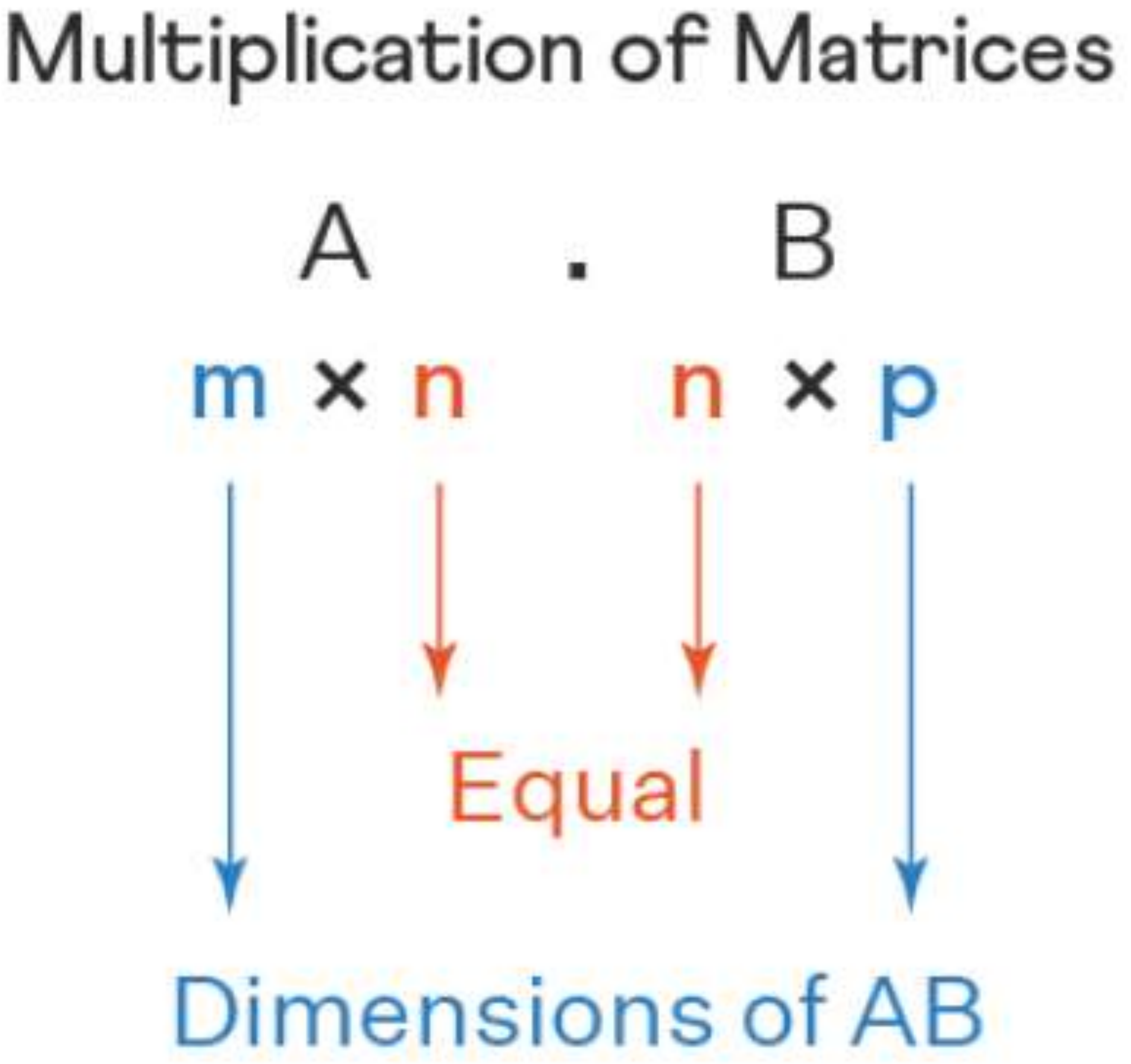
$$n=p \qquad j=\overline{1,q}$$

$$C=A\cdot B=\left(a_{ij}\right)_{m\times n}\cdot\left(b_{ij}\right)_{p\times q}=\left(c_{ij}\right)_{m\times q}$$

$$c_{ij}=\sum_{k=1}^{n}a_{ik}\cdot b_{kj}=a_{i1}\cdot b_{1j}+\ldots+a_{in}\cdot b_{nj}$$

The rule «row by column»: in order to get an element standing in the row ***i*** and the column ***j*** of the matrix ***C*** equal to the product of the matrices ***A*** and ***B*** it is necessary to multiply elements standing in the row ***i*** of the first matrix by the corresponding elements of the column ***j*** of the second matrix and then summarize the obtained products.

Operation 3. ***Multiplication of a matrix by a matrix***



The operation of multiplication of a matrix by a matrix satisfies the following laws:

$$\alpha(AB) = (\alpha A)B = A(\alpha B)$$

$$(A+B)C = AC + BC$$

$$C(A+B) = CA + CB$$

$$AO = OA = O$$

$$A(BC) = A(BC)$$

$$AE = EA = A$$

Example 5. Multiply the following matrices: $A = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$

and $B = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & -2 \\ 3 & 1 \end{pmatrix}$.

Example 5. Multiply the following matrices: $A = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$

and $B = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & -2 \\ 3 & 1 \end{pmatrix}$.

Can you multiply these matrices?

Example 5. Multiply the following matrices: $A = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$

and $B = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & -2 \\ 3 & 1 \end{pmatrix}$.

<u>Solution.</u> Note that ***A*** is a **3×4** matrix and ***B*** is a **4×2** matrix and the number of columns of the matrix ***A*** is equal to the number of rows of the matrix ***B***, so that the product ***C*** is a **3×2** matrix.

Let us calculate the product

$$C = A \cdot B = \left(a_{ij}\right)_{3\times 4} \cdot \left(b_{ij}\right)_{4\times 2} = \left(c_{ij}\right)_{3\times 2} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}$$

Consider first of all c_{11}. To calculate this, we need the 1-st row of ***A*** and the 1-st column of ***B***.

$$C = \begin{pmatrix} 2 & 4 & 3 & -1 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \cdot \begin{pmatrix} 1 & \times \\ 2 & \times \\ 0 & \times \\ 3 & \times \end{pmatrix} = \begin{pmatrix} c_{11} & \times \\ \times & \times \\ \times & \times \end{pmatrix}$$

From the definition of the product of the matrix ***A*** by the matrix ***B***, we have

$$c_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} + a_{14} \cdot b_{41} =$$

$$= 2 \cdot 1 + 4 \cdot 2 + 3 \cdot 0 + (-1) \cdot 3 = 2 + 8 + 0 - 3 = 7$$

(multiply elements standing in the row 1 of ***A*** by the corresponding elements of the column 1 of ***B*** and then summarize the obtained products).

Consider next c_{12} *.* To calculate this, we need the 1-st row of ***A*** and the 2-nd column of ***B***.

$$C=\begin{pmatrix}2 & 4 & 3 & -1\\ \times & \times & \times & \times\\ \times & \times & \times & \times\end{pmatrix}\cdot\begin{pmatrix}\times & 4\\ \times & 3\\ \times & -2\\ \times & 1\end{pmatrix}=\begin{pmatrix}\times & c_{12}\\ \times & \times\\ \times & \times\end{pmatrix}$$

From the definition of the product of the matrix ***A*** by the matrix ***B***, we have

$$c_{12}=a_{11}\cdot b_{12}+a_{12}\cdot b_{22}+a_{13}\cdot b_{32}+a_{14}\cdot b_{42}=$$

$$=2\cdot 4+4\cdot 3+3\cdot(-2)+(-1)\cdot 1=8+12-6-1=13$$

(multiply elements standing in the row 1 of ***A*** by the corresponding elements of the column 2 of ***B*** and then summarize the obtained products).

Consider next c_{21}. To calculate this, we need the 2-nd row of ***A*** and the 1-st column of ***B***.

$$C = \begin{pmatrix} \times & \times & \times & \times \\ 3 & 1 & 5 & 2 \\ \times & \times & \times & \times \end{pmatrix} \cdot \begin{pmatrix} 1 & \times \\ 2 & \times \\ 0 & \times \\ 3 & \times \end{pmatrix} = \begin{pmatrix} \times & \times \\ c_{21} & \times \\ \times & \times \end{pmatrix}$$

From the definition of the product of the matrix ***A*** by the matrix ***B***, we have

$$c_{21} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} + a_{24} \cdot b_{41} =$$

$$= 3 \cdot 1 + 1 \cdot 2 + 5 \cdot 0 + 2 \cdot 3 = 3 + 2 + 0 + 6 = 11$$

(multiply elements standing in the row 2 of ***A*** by the corresponding elements of the column 1 of ***B*** and then summarize the obtained products).

Consider next c_{22} . To calculate this, we need the 2-nd row of ***A*** and the 2-nd column of ***B***.

$$C = \begin{pmatrix} \times & \times & \times & \times \\ 3 & 1 & 5 & 2 \\ \times & \times & \times & \times \end{pmatrix} \cdot \begin{pmatrix} \times & 4 \\ \times & 3 \\ \times & -2 \\ \times & 1 \end{pmatrix} = \begin{pmatrix} \times & \times \\ \times & c_{22} \\ \times & \times \end{pmatrix}$$

From the definition of the product of the matrix ***A*** by the matrix ***B***, we have

$$c_{22} = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} + a_{24} \cdot b_{42} =$$

$$= 3 \cdot 4 + 1 \cdot 3 + 5 \cdot (-2) + 2 \cdot 1 = 12 + 3 - 10 + 2 = 7$$

(multiply elements standing in the row 2 of ***A*** by the corresponding elements of the column 2 of ***B*** and then summarize the obtained products).

Consider next c_{31}. To calculate this, we need the 3-rd row of ***A*** and the 1-st column of ***B***.

$$C = \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ -1 & 0 & 7 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & \times \\ 2 & \times \\ 0 & \times \\ 3 & \times \end{pmatrix} = \begin{pmatrix} \times & \times \\ \times & \times \\ c_{31} & \times \end{pmatrix}$$

From the definition of the product of the matrix ***A*** by the matrix ***B***, we have

$$c_{31} = a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} + a_{34} \cdot b_{41} =$$

$$= (-1) \cdot 1 + 0 \cdot 2 + 7 \cdot 0 + 6 \cdot 3 = -1 + 0 + 0 + 18 = 17$$

(multiply elements standing in the row 3 of ***A*** by the corresponding elements of the column 1 of ***B*** and then summarize the obtained products).

Consider next c_{32} *.* To calculate this, we need the 3-rd row of ***A*** and the 2-nd column of ***B***.

$$C = \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ -1 & 0 & 7 & 6 \end{pmatrix} \cdot \begin{pmatrix} \times & 4 \\ \times & 3 \\ \times & -2 \\ \times & 1 \end{pmatrix} = \begin{pmatrix} \times & \times \\ \times & \times \\ \times & c_{32} \end{pmatrix}$$

From the definition of the product of the matrix ***A*** by the matrix ***B***, we have

$$c_{32} = a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} + a_{34} \cdot b_{42} =$$

$$= (-1) \cdot 4 + 0 \cdot 3 + 7 \cdot (-2) + 6 \cdot 1 = -4 + 0 - 14 + 6 = -12$$

(multiply elements standing in the row 3 of ***A*** by the corresponding elements of the column 2 of ***B*** and then summarize the obtained products).

Therefore we conclude that

$$C = A \cdot B = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} = \begin{pmatrix} 7 & 13 \\ 11 & 7 \\ 17 & -12 \end{pmatrix}$$

Example 6. Consider the same matrices in example 5:

$$A=\begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix} \qquad B=\begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & -2 \\ 3 & 1 \end{pmatrix}$$

Can you multiply these matrices?

Example 6. Consider the same matrices in example 5:

$$B=\begin{pmatrix}1 & 4\\ 2 & 3\\ 0 & -2\\ 3 & 1\end{pmatrix} \qquad A=\begin{pmatrix}2 & 4 & 3 & -1\\ 3 & 1 & 5 & 2\\ -1 & 0 & 7 & 6\end{pmatrix}$$

Note that ***B*** is a **4×2** matrix and ***A*** is a **3×4** matrix, so that we do not have a definition for the product $B\cdot A$, because the number of columns of the matrix ***A*** is not equal to the number of rows of the matrix ***B***.

$$B\cdot A=\left(b_{ij}\right)_{4\times 2}\cdot\left(a_{ij}\right)_{3\times 4}\neq A\cdot B=\left(a_{ij}\right)_{3\times 4}\cdot\left(b_{ij}\right)_{4\times 2}=\left(c_{ij}\right)_{3\times 2}$$

Operation 4. ***Transposition of a matrix.*** Consider the $m \times n$ matrix ***A***. By the transposed A^T of ***A***, we mean *the transposed matrix* A^T

obtained from ***A*** by transposing rows and columns.

$$A = \left(a_{ij}\right)_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad A^T = \left(a_{ji}\right)_{n \times m} = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{pmatrix}$$

To get a transposition of a matrix, usually written as A^T , the rows and columns are swapped around, i. e. row 1 becomes column 1 and column 1 becomes row 1, etc.

Example 7. Consider the matrix $A=\begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$. Then

$$A^T =$$

Example 7. Consider the matrix $A = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$. Then

$$A^T = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 1 & 0 \\ 3 & 5 & 7 \\ -1 & 2 & 6 \end{pmatrix}$$

Example 7. Consider the matrix $A = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$. Then

$$A^T = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 1 & 0 \\ 3 & 5 & 7 \\ -1 & 2 & 6 \end{pmatrix}$$

Note that ***A*** is a **3×4** matrix and A^T is a **4×3** matrix.

Operation 5. **Raising to power.** For the $n \times n$ matrix and a positive integer ***m***, *the **m**-th power of* is

$$A^m = \underbrace{A \cdot A \cdot \ldots \cdot A}_{m \text{ copies of } A}$$

$$A^0 = E$$

Operation 5. **Raising to power.** For the $n \times n$ matrix and a positive integer ***m***, *the **m**-th power of* is

$$A^m = \underbrace{A \cdot A \cdot \ldots \cdot A}_{m \text{ copies of } A}$$

$$A^0 = E$$

$$A^2 =$$

Operation 5. **Raising to power.** For the $n \times n$ matrix and a positive integer ***m***, *the* ***m****-th power of* is

$$A^m = \underbrace{A \cdot A \cdot \ldots \cdot A}_{m \text{ copies of } A}$$

$$A^0 = E$$

$$A^2 = \underbrace{A \cdot A}_{2 \text{ copies of } A}$$

Operation 5. **Raising to power.** For the $n \times n$ matrix and a positive integer ***m***, *the* ***m****-th power of* is

$$A^m = \underbrace{A \cdot A \cdot \ldots \cdot A}_{m \text{ copies of } A}$$

$$A^0 = E$$

$$A^3 = \underbrace{A \cdot A \cdot A}_{3 \text{ copies of } A}$$

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$$A^0 = E$$

$$A^3 = \underbrace{A \cdot A \cdot A}_{3 \text{ copies of } A}$$

$$A^3 = A^2 \cdot A = A \cdot A^2$$

Example. A matrix is given: $A = \begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix}$

Find A^2 and A^3 .

Can you find?

Example. A matrix is given: $A = \begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix}$

Find A^2 and A^3 .

This operation can be applied to the square matrix!!!

Example. A matrix is given: $A=\begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix}$

Find A^2 and A^3 .

$$A^2 = \underbrace{A\cdot A}_{\text{2 copies of } A}$$

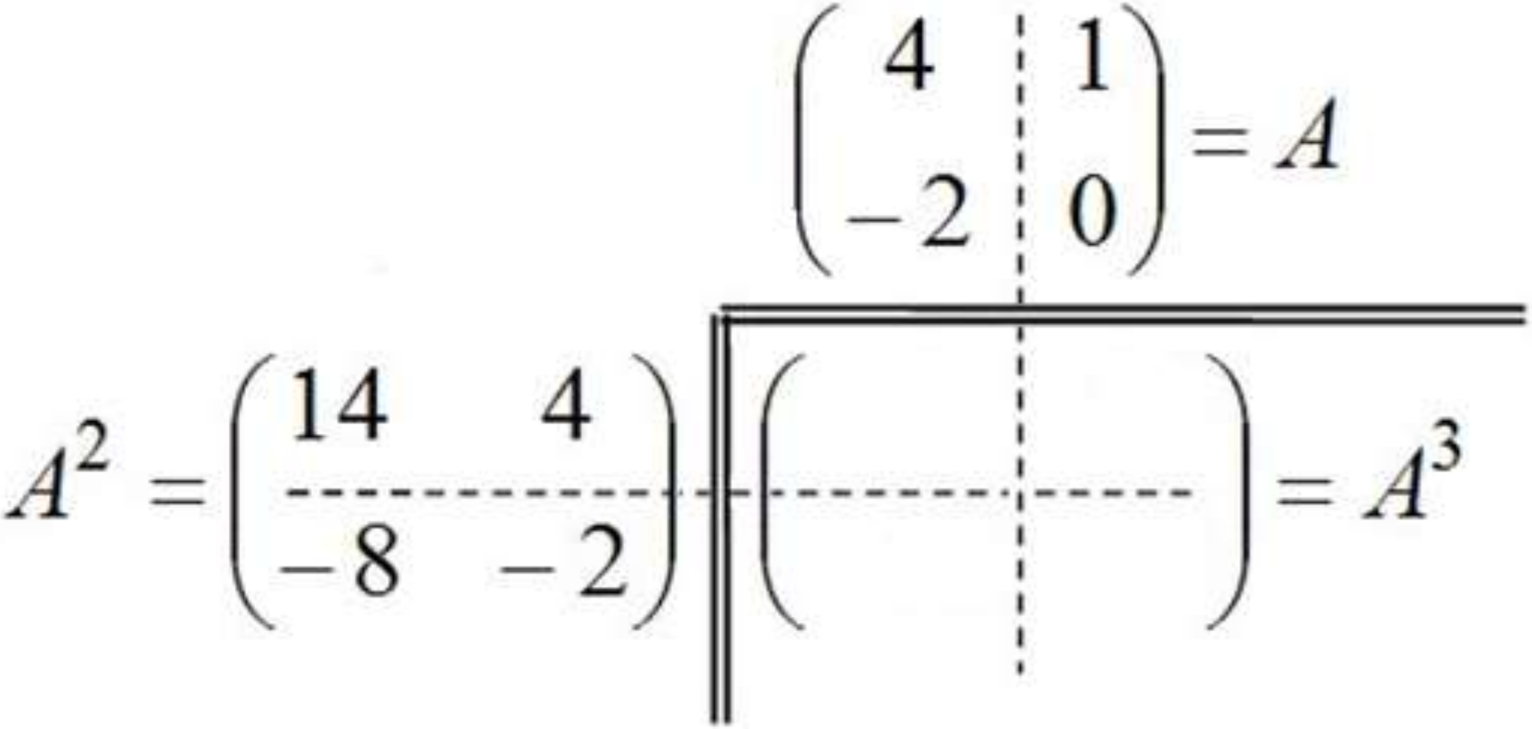
$$A^2 = A\cdot A = \begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix}\cdot\begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 4\cdot 4+1\cdot(-2) & 4\cdot 1+1\cdot 0 \\ -2\cdot 4+0\cdot(-2) & -2\cdot 1+0\cdot 0 \end{pmatrix} = \begin{pmatrix} 14 & 4 \\ -8 & -2 \end{pmatrix}$$

$$A^3 = \underbrace{A \cdot A \cdot A}_{3 \text{ copies of } A}$$

$$A^3 = A^2 \cdot A = A \cdot A^2$$

The matrix A^3 can be obtained by another way:

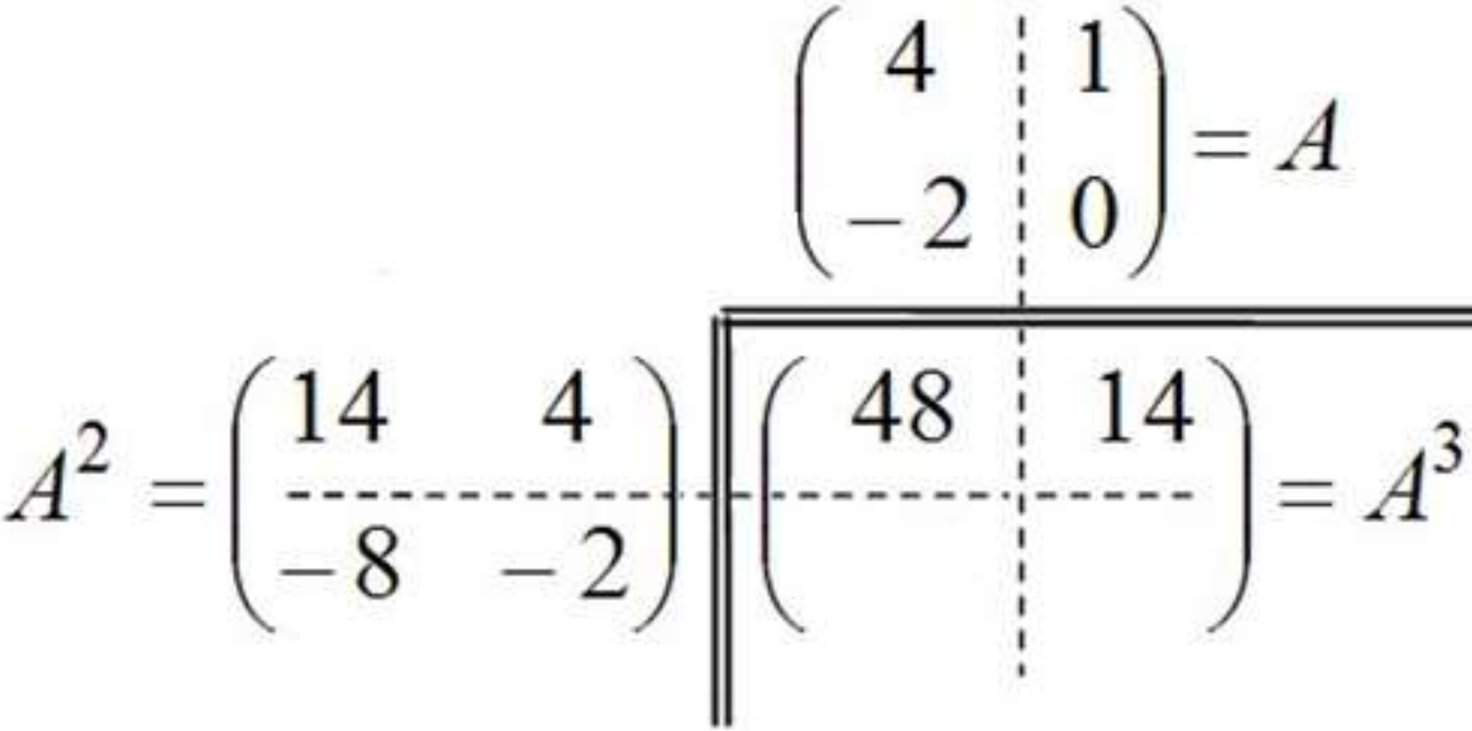


$$\begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix} = A$$

$$A^2 = \begin{pmatrix} 14 & 4 \\ -8 & -2 \end{pmatrix} \qquad \begin{pmatrix} & \\ & \end{pmatrix} = A^3$$

$$\begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix} = A$$

$$A^2 = \begin{pmatrix} 14 & 4 \\ -8 & -2 \end{pmatrix} \begin{pmatrix} 48 & \\ & \end{pmatrix} = A^3$$



$$\begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix} = A$$

$$A^2 = \begin{pmatrix} 14 & 4 \\ -8 & -2 \end{pmatrix} \quad \begin{pmatrix} 48 & 14 \\ & \end{pmatrix} = A^3$$

$$\begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix} = A$$

$$A^2 = \begin{pmatrix} 14 & 4 \\ -8 & -2 \end{pmatrix} \quad \begin{pmatrix} 48 & 14 \\ -28 & \end{pmatrix} = A^3$$

$$\begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix} = A$$

$$A^2 = \begin{pmatrix} 14 & 4 \\ -8 & -2 \end{pmatrix} \quad \begin{pmatrix} 48 & 14 \\ -28 & -8 \end{pmatrix} = A^3$$

$$\begin{array}{c} \textbf{co} \\ \textbf{lu} \\ \textbf{mn} \end{array}\begin{pmatrix}4 & 1\\ -2 & 0\end{pmatrix} = A$$

rows

$$A^2 = \begin{pmatrix}14 & 4\\ -8 & -2\end{pmatrix} \quad \begin{pmatrix}48 & 14\\ -28 & -8\end{pmatrix} = A^3$$

Question

Do you have operation of division?

3. A definition of a determinant, rules of calculation and properties of determinants

Definition

- *Determinant /* детермінант, визначник
- Property / властивість

Any square matrix can be associated with some value (number) called its ***determinant*** and designated as $\det A$, $|A|$ or Δ (delta).

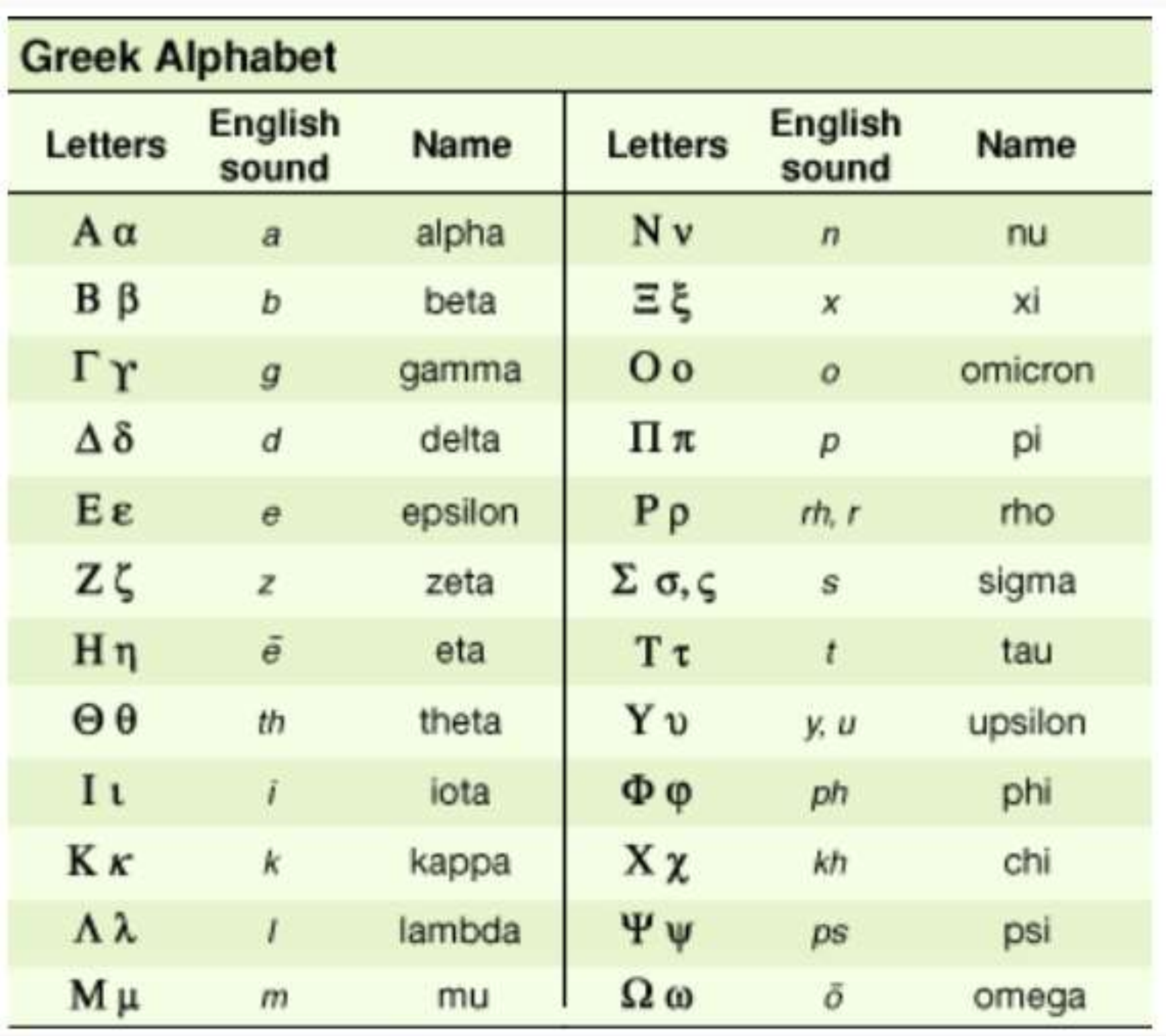
For example, a determinant of a matrix of ***the 2-nd order*** is calculated according to the following formula:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

i. e. the product of elements of the main diagonal minus the product of elements of the secondary diagonal.

Example 8. Calculate the determinant of the matrix: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Solution. $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$



Greek Alphabet

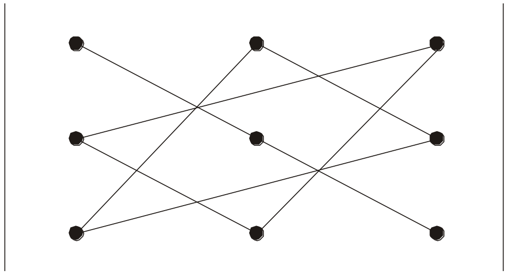
Letters	English sound	Name	Letters	English sound	Name
Α α	*a*	alpha	Ν ν	*n*	nu
Β β	*b*	beta	Ξ ξ	*x*	xi
Γ γ	*g*	gamma	Ο ο	*o*	omicron
Δ δ	*d*	delta	Π π	*p*	pi
Ε ε	*e*	epsilon	Ρ ρ	*rh, r*	rho
Ζ ζ	*z*	zeta	Σ σ, ς	*s*	sigma
Η η	*ē*	eta	Τ τ	*t*	tau
Θ θ	*th*	theta	Υ υ	*y, u*	upsilon
Ι ι	*i*	iota	Φ φ	*ph*	phi
Κ κ	*k*	kappa	Χ χ	*kh*	chi
Λ λ	*l*	lambda	Ψ ψ	*ps*	psi
Μ μ	*m*	mu	Ω ω	*ō*	omega

A determinant of the 3-rd order is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} + a_{21} \cdot a_{32} \cdot a_{13} + a_{31} \cdot a_{12} \cdot a_{23} -$$

$$- a_{13} \cdot a_{22} \cdot a_{31} - a_{21} \cdot a_{12} \cdot a_{33} - a_{32} \cdot a_{23} \cdot a_{11}$$

To memorize the last formula ***the rule of triangle (or Sarrus formula)*** is often used. It says: product of elements from the main diagonal and 2 products of elements forming in a matrix isosceles triangles with their bases parallel to the main diagonal are taken with the sign *plus* and a product of elements from the secondary diagonal and 2 products of elements forming triangles with their bases parallel to the secondary diagonal are taken with the sign *minus:*



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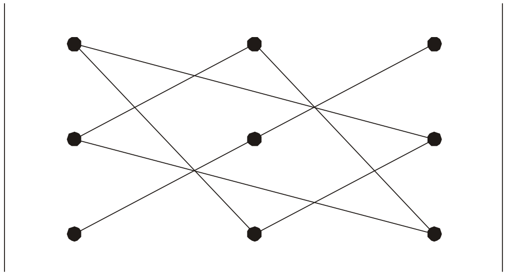

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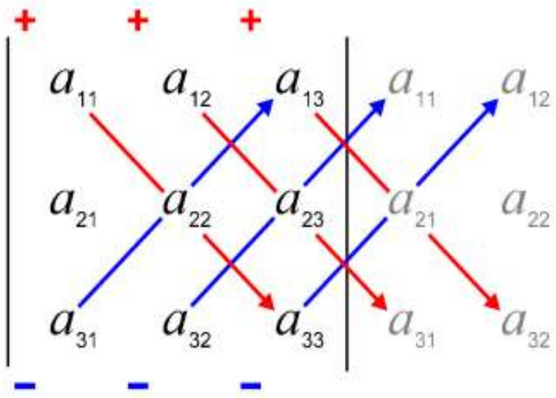
Another way of memorizing of **the rule of triangle:**

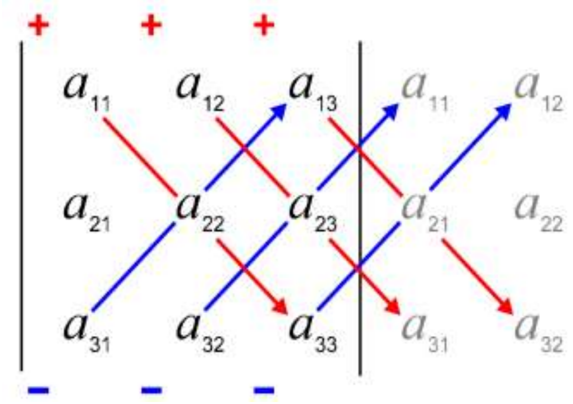


$$|A| = \begin{array}{|ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} = a_{11}\cdot a_{22}\cdot a_{33} + a_{21}\cdot a_{32}\cdot a_{13} + a_{31}\cdot a_{12}\cdot a_{23} -$$

$$- a_{13}\cdot a_{22}\cdot a_{31} - a_{21}\cdot a_{12}\cdot a_{33} - a_{32}\cdot a_{23}\cdot a_{11}$$

We take the main diagonal and 2 parallel to it with plus sign and the secondary diagonal with and 2 parallel to it with minus sign

Another way of memorizing of **the rule of triangle:**

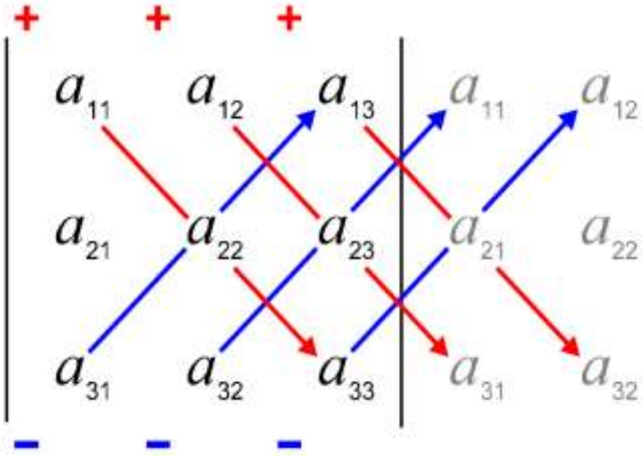


https://www.youtube.com/watch?app=desktop&v=x2vWqtYwZ1g

Example 9. Calculate the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

Solution. Calculate the determinant, using the rule of triangle:



$$\begin{array}{|ccc|cc} 1 & 2 & 1 & 1 & 2 \\ 0 & -2 & 3 & 0 & -2 \\ 3 & 1 & 1 & 3 & 1 \end{array} = 1\cdot(-2)\cdot 1 + 2\cdot 3\cdot 3 + 0\cdot 1\cdot 1 - 1\cdot(-2)\cdot 3 - 0\cdot 1\cdot 2 - 3\cdot 1\cdot 1 =$$

$$= -2 + 18 + 6 - 3 = 19$$

Example 9. Calculate the determinant of the matrix

$$A=\begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

Solution. Calculate the determinant, using the rule of triangle:

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{vmatrix}=1\cdot(-2)\cdot1+2\cdot3\cdot3+0\cdot1\cdot1-1\cdot(-2)\cdot3-0\cdot1\cdot2-3\cdot1\cdot1=$$

$$=-2+18+6-3=19$$

Other way of application of **the rule of triangle:**

$$\left|\begin{matrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{matrix}\right|\begin{matrix} 1 & 2 \\ 0 & -2 \\ 3 & 1 \end{matrix}=1\cdot(-2)\cdot1+2\cdot3\cdot3+0\cdot1\cdot1-1\cdot(-2)\cdot3-0\cdot1\cdot2-3\cdot1\cdot1=$$

$$=-2+18+6-3=19$$

Next way of application of **the rule of triangle:**

With the three elements the determinant can be written as a sum of 2x2 determinants.

$$\det \mathrm{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

It is important to consider that the sign of the elements alternate in the following manner.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

3.1. Basic properties of determinants

1. A determinant does not change its value at the transposition of the matrix, i. e.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

or

$$\det A = \det A^T$$

Example 10. Check the property:

$$\begin{vmatrix} 3 & 2 & 5 \\ 4 & 1 & 8 \\ -1 & 2 & 4 \end{vmatrix} = 12+40-16+5-32-48=-39$$

Let’s transpose rows and columns and obtain:

$$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 1 & 2 \\ 5 & 8 & 4 \end{vmatrix} = 12-16+40+5-32-48=-39$$

2. Transposing of two any rows (columns) the determinant changes its sign on opposite one. For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example 11*.* Check the property:

$$\begin{vmatrix} 5 & 1 & 0 \\ 2 & 5 & 6 \\ 3 & 2 & -1 \end{vmatrix} = -25 + 18 + 0 - 0 + 2 - 60 = -65$$

Let’s transpose the first row and the second one and obtain:

$$\begin{vmatrix} 2 & 5 & 6 \\ 5 & 1 & 0 \\ 3 & 2 & -1 \end{vmatrix} = -2 + 60 + 0 - 18 + 25 - 0 = 65$$

3. If any row (column) of the determinant completely consists of zeros then the determinant is equal to zero.

Example 12. Check the property:

$$\begin{vmatrix} 3 & 5 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 0 \end{vmatrix} = 0+0+0-0-0-0=0$$

4. A common factor of all elements of a row (column) can be taken out of the determinant. For example,

$$\begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

where $k \in R$

Example 13. Check the property.

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ 6 & 5 & 4 \end{vmatrix} = 16 + 60 - 6 - 36 + 16 - 10 = 40$$

The first column has a common factor 2. We take it out of the determinant and obtain

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ 6 & 5 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & -1 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 2\left(8 + 30 - 3 - 18 + 8 - 5\right) = 2 \cdot 20 = 40$$

5. If we add to all elements of a row (column) of the determinant the corresponding elements of other row (column) multiplied by some number then the value of the determinant will not change, i. e.

$$\begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{where } k \in R$$

Example 14. Check the property.

$$\begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 1 & 2 & 3 \end{vmatrix} = 9+20+4-5+6+24=58$$

For example, calculate this determinant by adding to all elements of row 3 of the determinant the corresponding elements of row 1 multiplied by 2. Thus

$$\begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 1+3\cdot 2 & 2+\left(-1\right)\cdot 2 & 3+5\cdot 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 7 & 0 & 13 \end{vmatrix} =$$

$$=39+28-35+26=58.$$

6. The determinant possessing two identical or proportional rows (columns) is equal to zero.

Example 15. Check the property.

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 5 & 1 & 3 \end{vmatrix} = 18 + 8 + 120 - 120 - 18 - 8 = 0$$

The determinant is equal to zero, because the first row and the second one are proportional rows:

$$\frac{a_{21}}{a_{11}} = \frac{a_{22}}{a_{12}} = \frac{a_{23}}{a_{13}}$$

or

$$\frac{2}{1} = \frac{6}{3} = \frac{8}{4} = 2$$

7. The determinant of the upper (lower) triangular matrix is equal to **a product of elements of the main diagonal.**

Example 16. Check the property.

$$\begin{vmatrix} 1 & 3 & 4 \\ 0 & 6 & 8 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 6 \cdot 3 = 18$$

4. Minors and algebraic cofactors

Definition

- *Minor* / мінор
- Algebraic cofactors / алгебраїчне доповнення

Definition. *The minor* M_{ij} of the element a_{ij} of the determinant of *n*-th order is called the determinant of the (*n* - 1)-th order obtained from the given one by crossing the row and the column on which intersection the element a_{ij} is located.

Definition. *The algebraic cofactor* A_{ij} (or *the cofactor*) of the element a_{ij} of the determinant is called the following value

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Example. Find the minor M_{11} and the algebraic cofactor A_{11} for the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$$

Solution.

Example. Find the minor M_{11} and the algebraic cofactor A_{11} for the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$$

Solution.

$$A = \begin{pmatrix} 1_{11} & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$$

Example. Find the minor M_{11} and the algebraic cofactor A_{11} for the matrix

$$A=\begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$$

Solution.

$$A=\begin{pmatrix} 1_{11} & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$$

$$M_{11}=\begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix}=0-9=-9$$

Example. Find the minor M_{11} and the algebraic cofactor A_{11} for the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$$

Solution.

$$A = \begin{pmatrix} 1_{11} & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$$

$$M_{11} = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = 0 - 9 = -9$$

$$A_{11} = (-1)^{1+1} \cdot M_{11} = (-1)^2 \cdot (-9) = 1 \cdot (-9) = -9$$

M_{23} A_{23}

Solution.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3_{23} \\ -2 & 3 & 0 \end{pmatrix}$$

M_{23} A_{23}

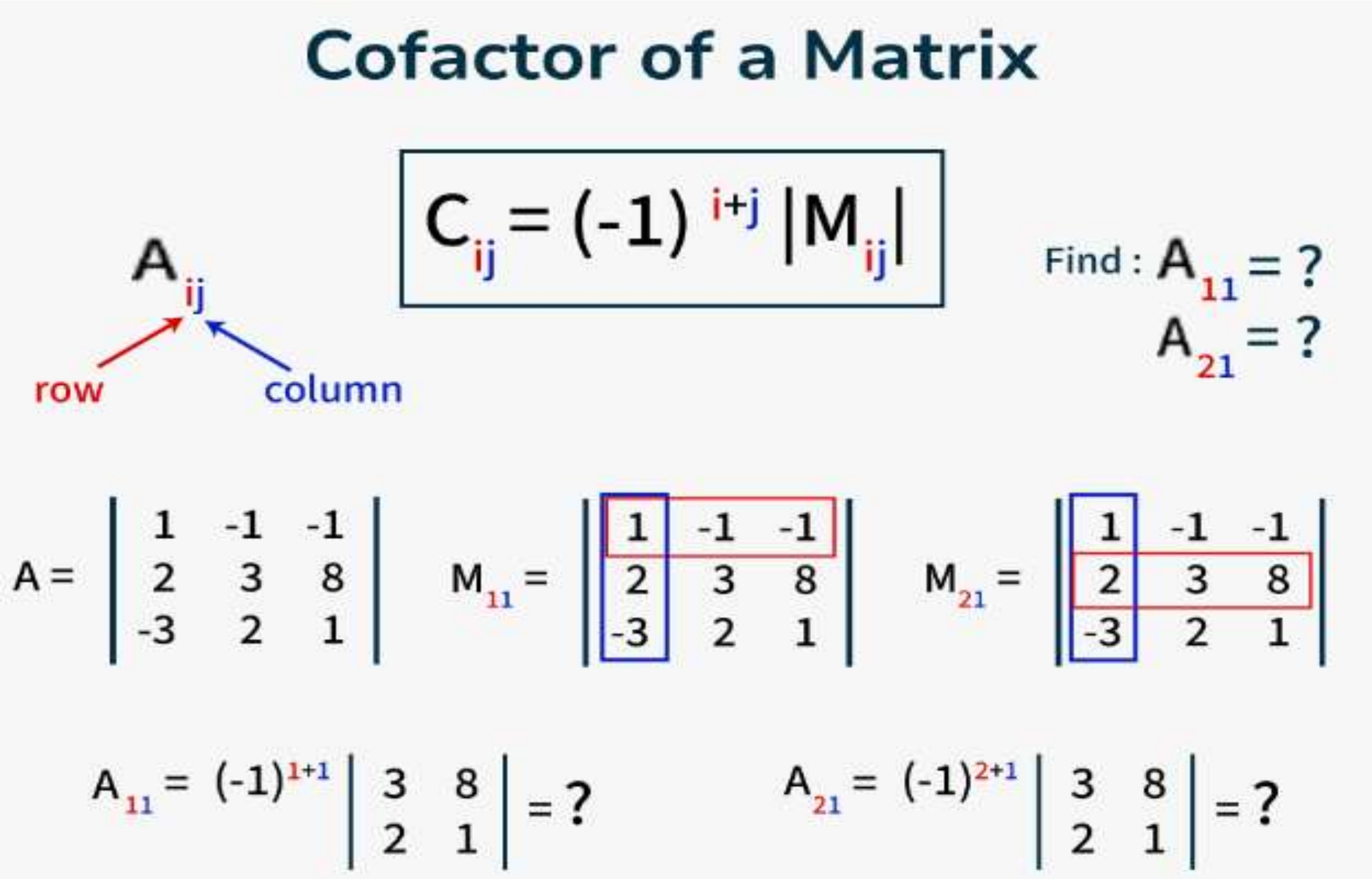
Solution.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3_{23} \\ -2 & 3 & 0 \end{pmatrix}$$

$$M_{23} = \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} = 3 - \left(-2\right) = 5$$

$$A_{23} = (-1)^{2+3} \cdot M_{23} = (-1)^{5} \cdot 5 = -5$$

EXAMPLE



Cofactor of a Matrix

$$\boxed{C_{ij} = (-1)^{i+j} |M_{ij}|}$$

A_{ij}

row

column

Find : $A_{11} = ?$

$A_{21} = ?$

$$A = \begin{vmatrix} 1 & -1 & -1 \\ 2 & 3 & 8 \\ -3 & 2 & 1 \end{vmatrix} \qquad M_{11} = \begin{vmatrix} 1 & -1 & -1 \\ 2 & 3 & 8 \\ -3 & 2 & 1 \end{vmatrix} \qquad M_{21} = \begin{vmatrix} 1 & -1 & -1 \\ 2 & 3 & 8 \\ -3 & 2 & 1 \end{vmatrix}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 8 \\ 2 & 1 \end{vmatrix} = ? \qquad A_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 8 \\ 2 & 1 \end{vmatrix} = ?$$

5. Laplace’ theorem (concerning decomposition of a determinant in its rows or columns)

Laplace' theorem *(concerning decomposition of a determinant in its rows or columns).* The sum of products of elements of any row (column) by their cofactors is equal to this determinant, i. e.

$$|A| = \sum_{k=1}^{n} a_{kj} \cdot A_{kj}$$

$$j = \overline{1,n}$$

Questions

1. The first order determinant.
2. The second order determinant.
3. The third order determinant.
4. The forth order determinant and so on.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Let’s decompose the determinant in the 1-st row:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Let’s decompose the determinant in the 1-st row:

$$|A| = a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + a_{13} \cdot A_{13}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Let’s decompose the determinant in the 3-rd column:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Let’s decompose the determinant in the 3-rd column:

$$|A| = a_{13} \cdot A_{13} + a_{23} \cdot A_{23} + a_{33} \cdot A_{33}$$

Example. Calculate the determinant on the base of the rule of triangle and check the result using the theorem concerning decomposition of the determinant in its 1-st row.

Solution. Calculate the determinant, using the rule of triangle:

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = 1\cdot(-2)\cdot 1 + 2\cdot 3\cdot 3 + 0\cdot 1\cdot 1 - 1\cdot(-2)\cdot 3 - 0\cdot 1\cdot 2 - 3\cdot 1\cdot 1 =$$

$$= -2 + 18 + 6 - 3 = 19$$

Example. Calculate the determinant on the base of the rule of triangle and check the result using the theorem concerning decomposition of the determinant in its 1-st row.

Solution. Calculate the determinant, using the rule of triangle:

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 0 & -2 \\ 3 & 1 \end{matrix} =$$

$$= 1\cdot(-2)\cdot 1 + 2\cdot 3\cdot 3 + 0\cdot 1\cdot 1 - 1\cdot(-2)\cdot 3 - 0\cdot 1\cdot 2 - 3\cdot 1\cdot 1 =$$

$$= -2 + 18 + 6 - 3 = 19$$

Let’s use ***Laplace’ theorem*** and check the determinant value by decomposing in the 1-st row:

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + a_{13} \cdot A_{13} = 1 \cdot (-1)^{1+1} \cdot M_{11} + 2 \cdot (-1)^{1+2} \cdot M_{12} +$$

$$+1 \cdot (-1)^{1+3} \cdot M_{13} = 1 \cdot 1 \cdot \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} + 2 \cdot (-1) \cdot \begin{vmatrix} 0 & 3 \\ 3 & 1 \end{vmatrix} + 1 \cdot 1 \cdot \begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} = \left(-2-3\right) -$$

$$-2 \cdot \left(0-9\right) + \left(0 - \left(-6\right)\right) = -5 + 18 + 6 = 19$$

Let’s use ***Laplace’ theorem*** and check the determinant value by decomposing in the 3-rd column:

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = a_{13} \cdot A_{13} + a_{23} \cdot A_{23} + a_{33} \cdot A_{33} = 1 \cdot (-1)^{1+3} \cdot M_{13} + 3 \cdot (-1)^{2+3} \cdot M_{23} +$$

$$+1 \cdot (-1)^{3+3} \cdot M_{33} = 1 \cdot 1 \cdot \begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + 3 \cdot (-1) \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 1 \cdot 1 \cdot \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = \left(0 - (-6)\right) -$$

$$-3 \cdot \left(1 - 6\right) + \left(-2 - 0\right) = 6 + 15 - 2 = 19$$

6. Notion of an inverse matrix

Definition

- *Inverse matrix /* обернена матриця

Definition. The square matrix ***A*** is called ***nonsingular***,
if $\det A \neq 0$, otherwise it is called singular.

Definition. The matrix A^{-1} is called ***inverse*** relatively to the square nonsingular matrix ***A*** if

$$A \cdot A^{-1} = A^{-1} \cdot A = E$$

where ***E*** is the unit matrix.

An inverse matrix can be obtained according to the following formula:

$$A^{-1}=\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n1} & \cdots & A_{nn} \end{pmatrix}^{T}=\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

where A_{ij} is cofactor of the element a_{ij} of the matrix ***A***

An inverse matrix can be obtained according to the following formula:

$$A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^{T} = \frac{1}{|A|} \cdot \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

where A_{ij} is cofactor of the element a_{ij} of the matrix ***A***

Example. Find the inverse matrix for the following matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$$

Solution. Let's find the determinant of the given matrix:

$$|A| = \left| \begin{matrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{matrix} \right| \begin{matrix} 1 & 1 \\ 3 & 0 \\ -2 & 3 \end{matrix} = 1\cdot 0\cdot 0 + 1\cdot 3\cdot(-2) + 3\cdot 3\cdot 2 -$$

$$-0\cdot(-2)\cdot 2 - 3\cdot 3\cdot 1 - 3\cdot 1\cdot 0 = 0 - 6 + 18 - 0 - 9 - 0 = 3 \neq 0$$

Its determinant is non-zero. Find the cofactors for the elements of the matrix ***A***:

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{vmatrix} = 3 \neq 0$$

$$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = 1 \cdot (0-9) = -9$$

$$A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 3 & 3 \\ -2 & 0 \end{vmatrix} = (-1) \cdot (0-(-6)) = -6$$

$$A_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix} = 1 \cdot (9-0) = 9$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{vmatrix} = 3 \neq 0$$

$$A_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} = (-1) \cdot (0-6) = 6$$

$$A_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 1 \cdot (0-(-4)) = 4$$

$$A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} = (-1) \cdot (3-(-2)) = -5$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{vmatrix} = 3 \neq 0$$

$$A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 1 \cdot (3-0) = 3$$

$$A_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = (-1) \cdot (3-6) = 3$$

$$A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = 1 \cdot (0-3) = -3$$

An inverse matrix can be obtained according to the following formula:

$$A^{-1}=\frac{1}{|A|}\cdot\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix}^T=\frac{1}{|A|}\cdot\begin{pmatrix}A_{11}&A_{21}&A_{31}\\A_{12}&A_{22}&A_{32}\\A_{13}&A_{23}&A_{33}\end{pmatrix}$$

We obtain the inverse matrix:

$$A^{-1}=\frac{1}{\Delta}\cdot\begin{pmatrix}A_{11}&A_{21}&A_{31}\\A_{12}&A_{22}&A_{32}\\A_{13}&A_{23}&A_{33}\end{pmatrix}=\frac{1}{3}\cdot\begin{pmatrix}-9&6&3\\-6&4&3\\9&-5&-3\end{pmatrix}=\begin{pmatrix}-3&2&1\\-2&4/3&1\\3&-5/3&-1\end{pmatrix}$$

We obtain the inverse matrix:

$$A^{-1}=\frac{1}{\Delta}\cdot\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}=\frac{1}{3}\cdot\begin{pmatrix} -9 & 6 & 3 \\ -6 & 4 & 3 \\ 9 & -5 & -3 \end{pmatrix}=\begin{pmatrix} -3 & 2 & 1 \\ -2 & 4/3 & 1 \\ 3 & -5/3 & -1 \end{pmatrix}$$

Checking the condition $A\cdot A^{-1}=E$

$$A\cdot A^{-1}=\begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}\cdot\begin{pmatrix} -3 & 2 & 1 \\ -2 & 4/3 & 1 \\ 3 & -5/3 & -1 \end{pmatrix}=$$

$$=\begin{pmatrix} -3-2+6 & 2+4/3-10/3 & 1+1-2 \\ -9+0+9 & 6+0-5 & 3+0-3 \\ 6-6+0 & -4+4+0 & -2+3+0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=E$$

Example 2. Let’s find the inverse matrix:

$$A = \begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$$

Let’s calculate its determinant:

$$\Delta = \begin{vmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 5\cdot 2\cdot 2 + 4\cdot 3\cdot (-1) + 1\cdot 3\cdot \left(-1\right) - 4\cdot 2\cdot (-1) - 2\cdot 1\cdot (-1) -$$

$$-3\cdot 3\cdot 5 = 20 - 12 - 3 + 8 + 2 - 45 = -30 \neq 0$$

Its determinant is non-zero. Find the inverse matrix by cofactors:

Its determinant is non-zero. Find the inverse matrix by cofactors:

$$A = \begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 10$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = -5$$

Its determinant is non-zero. Find the inverse matrix by cofactors:

$$A=\begin{pmatrix}5 & -1 & -1\\ 1 & 2 & 3\\ 4 & 3 & 2\end{pmatrix}$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}5 & -1\\ 4 & 2\end{vmatrix}=14$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}-1 & -1\\ 3 & 2\end{vmatrix}=-1$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}5 & -1\\ 4 & 3\end{vmatrix}=-19$$

Its determinant is non-zero. Find the inverse matrix by cofactors:

$$A = \begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} = -1 \qquad A_{32} = (-1)^{3+2} \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} = -16$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 11$$

Let’s substitute:

$$A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \frac{1}{|A|} \cdot \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{-30} \cdot \begin{pmatrix} -5 & -1 & -1 \\ 10 & 14 & -16 \\ -5 & -19 & 11 \end{pmatrix} = \begin{pmatrix} \frac{5}{30} & \frac{1}{30} & \frac{1}{30} \\ -\frac{10}{30} & -\frac{14}{30} & \frac{16}{30} \\ \frac{5}{30} & \frac{19}{30} & -\frac{11}{30} \end{pmatrix}$$

Let's check the condition $A\cdot A^{-1}=E$

$$A\cdot A^{-1}=\begin{pmatrix}5 & -1 & -1\\ 1 & 2 & 3\\ 4 & 3 & 2\end{pmatrix}\cdot\begin{pmatrix}\frac{5}{30} & \frac{1}{30} & \frac{1}{30}\\ -\frac{10}{30} & -\frac{14}{30} & \frac{16}{30}\\ \frac{5}{30} & \frac{19}{30} & -\frac{11}{30}\end{pmatrix}=$$

$$=\frac{1}{30}\cdot\begin{pmatrix}25+10-5 & 5+14-19 & 5-16+11\\ 5-20+15 & 1-28+57 & 1+32-33\\ 20-30+10 & 4-42+38 & 4+48-22\end{pmatrix}=$$

$$=\frac{1}{30}\cdot\begin{pmatrix}30 & 0 & 0\\ 0 & 30 & 0\\ 0 & 0 & 30\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E$$

7. Application Laplace’ theorem of the decomposition of the determinant and the 5-th property of determinants

Exercise 18. Calculate the determinant using the theorem concerning the decomposition of the determinant and the 5-th property.

Solution.

$$\Delta = \begin{vmatrix} 1 & 0 & 1 & 2 \\ 2 & 4 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ 3 & 2 & -1 & 1 \end{vmatrix} = \begin{bmatrix} \textit{let's factorize} \\ \textit{by the third} \\ \textit{column} \end{bmatrix} = \begin{bmatrix} \\ [1]\cdot(-3)+[2] \\ [1]\cdot(-2)+[3] \\ [1]+[4] \end{bmatrix} = \begin{vmatrix} 1 & 0 & 1 & 2 \\ -1 & 4 & 0 & -2 \\ -3 & 1 & 0 & -1 \\ 4 & 2 & 0 & 3 \end{vmatrix} =$$

$$= \begin{bmatrix} let's\ \ factorize \\ by\ the\ third \\ column \end{bmatrix} = a_{13} \cdot A_{13} + a_{23} \cdot A_{23} + a_{33} \cdot A_{33} + a_{43} \cdot A_{43} =$$

$$= 1 \cdot (-1)^{1+3} \cdot \begin{vmatrix} -1 & 4 & -2 \\ -3 & 1 & -1 \\ 4 & 2 & 3 \end{vmatrix} + 0 + 0 + 0 = \begin{vmatrix} -1 & 4 & -2 \\ -3 & 1 & -1 \\ 4 & 2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & -4 & 2 \\ -3 & 1 & -1 \\ 4 & 2 & 3 \end{vmatrix} =$$

$$=\begin{bmatrix} let's\ \ factorize \\ by\ the\ \ first \\ row \end{bmatrix} = \begin{bmatrix} \\ [1]\cdot 4+[2] \\ [1]\cdot\left(-2\right)+[3] \end{bmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 3 & -11 & 5 \\ -4 & 18 & -5 \end{vmatrix} =$$

$$=\left(-1\right)\cdot\left(-1\right)^{1+1}\begin{vmatrix} -11 & 5 \\ 18 & -5 \end{vmatrix} = -\left(\left(-11\right)\cdot\left(-5\right)-18\cdot 5\right)=35$$

TASK 1

Matrix A is given: $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

Indicate 4A:

☐ $\begin{pmatrix} 2 & -4 \\ 3 & -8 \end{pmatrix}$	☐ $\begin{pmatrix} 8 & -4 \\ 3 & -2 \end{pmatrix}$
☐ $\begin{pmatrix} 8 & -4 \\ 12 & -8 \end{pmatrix}$	☐ $\begin{pmatrix} 2 & -1 \\ 12 & -8 \end{pmatrix}$
☐ $\begin{pmatrix} 8 & -1 \\ 12 & -2 \end{pmatrix}$	

TASK 2

Matrices A and B are given: $A = \begin{pmatrix} 1 & 2 \\ -4 & 2 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} -2 & 1 \\ 3 & -2 \\ 2 & 0 \end{pmatrix}$

Indicate 3A-2B:

☐ $\begin{pmatrix} -1 & 0 \\ -6 & 2 \\ 4 & 1 \end{pmatrix}$	☐ $\begin{pmatrix} -1 & 4 \\ -6 & 2 \\ 4 & 3 \end{pmatrix}$
☐ $\begin{pmatrix} -7 & -4 \\ -6 & 2 \\ -4 & 1 \end{pmatrix}$	☐ $\begin{pmatrix} -7 & -4 \\ 18 & -10 \\ -4 & -3 \end{pmatrix}$
☐ $\begin{pmatrix} 7 & 4 \\ -18 & 10 \\ -4 & 3 \end{pmatrix}$	

TASK 3

Matrices A and B are given: $A = \begin{pmatrix} 1 & 2 \\ -4 & 2 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$

Indicate possible operations:

☐ $B \cdot A$

☐ $B \cdot A^T$

☐ $B^T \cdot A$

☐ $B^T \cdot A^T$

☐ $A \cdot B$

☐ $A^T \cdot B$

☐ $A \cdot B^T$

☐ $A^T \cdot B^T$

TASK 4

Matrix A and B are given:

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & -2 \\ 0 & 2 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -5 & 3 \\ 0 & -2 & 1 \\ 2 & -3 & 0 \end{pmatrix}$$

$$C = B \cdot A.$$

Calculate the element c_{23}

TASK 5

Indicate correspondence for $A \cdot B$:

$A=\begin{pmatrix}1 & -1\\ 3 & -2\end{pmatrix},\ B=\begin{pmatrix}2 & 4\\ 3 & 2\end{pmatrix}$	▼
$A=\begin{pmatrix}2 & -1 & 3\\ 1 & 0 & 2\end{pmatrix},\ B=\begin{pmatrix}-1 & 1\\ 3 & -2\\ 1 & -2\end{pmatrix}$	▼
$A=\begin{pmatrix}2\\ -1\end{pmatrix},\ B=\begin{pmatrix}-2 & 3\end{pmatrix}$	▼
$A=\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix},\ B=\begin{pmatrix}3 & 0\\ 0 & -2\end{pmatrix}$	▼

1	$\begin{pmatrix}-4 & 6\\ 2 & -3\end{pmatrix}$
2	$\begin{pmatrix}6 & 0\\ 0 & 4\end{pmatrix}$
3	$\begin{pmatrix}-1 & 2\\ 0 & 8\end{pmatrix}$
4	$\begin{pmatrix}1 & -1\\ 3 & -2\end{pmatrix}$
5	$\begin{pmatrix}-2 & -2\\ 1 & -3\end{pmatrix}$

HOMEWORK

Solve this system and find unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$