DIRECT INTEGRATION

Example 1.1.

$$\int x^5\,dx = \frac{x^6}{6} + C$$

Example 1.2.

1. $\int \frac{dx}{x^2} = \int x^{-2}dx = -x^{-1} + C = -\frac{1}{x} + C.$

2. $\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + C.$

3. $\int \frac{dx}{\sqrt{x}} = \int x^{-\frac{1}{2}}dx = 2x^{\frac{1}{2}} + C = 2\sqrt{x} + C.$

4. $\int \sqrt[12]{x^5}\,dx = \int x^{\frac{5}{12}}dx = \frac{12}{17}x^{\frac{17}{12}} + C.$

5. $\int \frac{dx}{\sqrt[5]{x^3}} = \int x^{-\frac{3}{5}}dx = \frac{5}{2}x^{\frac{2}{5}} + C.$

6. $\int \frac{dx}{x\sqrt{x}} = \int x^{-\frac{3}{2}}dx = -2x^{-\frac{1}{2}} + C.$

7. $\int \frac{dx}{x^2\cdot\sqrt[3]{x}} = \int x^{-\frac{7}{3}}dx = -\frac{3}{4}x^{-\frac{4}{3}} + C.$

8. $\int (x\cdot\sqrt[5]{x})^3\,dx = \int x^{\frac{18}{5}}\,dx = \frac{5}{23}x^{\frac{23}{5}} + C.$

9. $\int 5^x\,dx = \frac{5^x}{\ln 5} + C.$

10. $\int \frac{dx}{3^x} = \int \left(\frac{1}{3}\right)^x dx = \frac{\left(\frac{1}{3}\right)^x}{\ln\left(\frac{1}{3}\right)} + C = -\frac{1}{3^x\ln 3} + C.$

11. $\int 2^x\cdot 3^x\,dx = \int 6^x\,dx = \frac{6^x}{\ln 6} + C.$

12. $\int (2^x)^3\,dx = \int 8^x\,dx = \frac{8^x}{\ln 8} + C = \frac{(2^x)^3}{3\ln 2} + C.$

13. $\int 3^{2x}\,dx = \int 9^x\,dx = \frac{9^x}{\ln 9} + C = \frac{3^{2x}}{2\ln 3} + C.$

14. $\int \frac{dx}{x^2+16} = \frac{1}{4}\operatorname{arctg}\frac{x}{4} + C.$

15. $\int \frac{dx}{13+x^2} = \frac{1}{\sqrt{13}}\operatorname{arctg}\frac{x}{\sqrt{13}} + C.$

16. $\int \frac{dx}{\sqrt{14-x^2}} = \arcsin\frac{x}{\sqrt{14}} + C.$

17. $\int \frac{dx}{\sqrt{x^2-14}} = \ln\left|x+\sqrt{x^2-14}\right| + C$.

18. $\int \frac{dx}{\sqrt{x^2+14}} = \ln\left|x+\sqrt{x^2+14}\right| + C$.

19. $\int \frac{dx}{x^2-9} = \frac{1}{2\cdot 3}\ln\left|\frac{x-3}{x+3}\right| + C = \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$.

20. $\int \frac{dx}{x^2-10} = \frac{1}{2\cdot\sqrt{10}}\ln\left|\frac{x-\sqrt{10}}{x+\sqrt{10}}\right| + C$.

Example 1.3.

$$\int \frac{dx}{\sqrt{3-2x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{3}{2}-x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(\sqrt{\frac{3}{2}}\right)^2 - x^2}} =$$

$$= \frac{1}{\sqrt{2}}\arcsin\sqrt{\frac{2}{3}}x + C$$

Example 1.4.

$$\int (3x+15)^{17} dx = \frac{1}{3}\cdot\frac{(3x+15)^{18}}{18} + C = \frac{1}{54}(3x+15)^{18} + C.$$

Example 1.5.

$$\int \left(\sqrt{x}+\sqrt[3]{x}\right) dx = \int \sqrt{x}dx + \int \sqrt[3]{x}dx = \int x^{\frac{1}{2}}dx + \int x^{\frac{1}{3}}dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C$$

$$= \frac{2x^{\frac{3}{2}}}{3} + \frac{3x^{\frac{4}{3}}}{4} = \frac{2\sqrt{x^3}}{3} + \frac{3\sqrt[3]{x^4}}{4} + C.$$

Example 1.6.

$$\int \left(\frac{3}{\sqrt[3]{x}} + \frac{2}{\sqrt{x}}\right) dx = \int \frac{3dx}{\sqrt[3]{x}} + \int \frac{2dx}{\sqrt{x}} = 3\int x^{-\frac{1}{3}}dx + 2\int x^{-\frac{1}{2}}dx$$

$$= 3\cdot\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + 2\cdot\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{9x^{\frac{2}{3}}}{2} + 4x^{\frac{1}{2}} + C = \frac{9\sqrt[3]{x^2}}{2} + 4\sqrt{x} + C.$$

Example 1.7.

$$\int \frac{4dx}{2+3x^2} = 4\int \frac{dx}{3\left(\frac{2}{3}+x^2\right)} = \frac{4}{3}\int \frac{dx}{\left(\sqrt{\frac{2}{3}}\right)^2 + x^2} = \frac{4}{3}\cdot\frac{1}{\sqrt{\frac{2}{3}}}\arctan\frac{x}{\sqrt{\frac{2}{3}}} + C$$

$$= \frac{4}{\sqrt{6}}\arctan\frac{\sqrt{3}x}{\sqrt{2}} + C.$$

Example 1.8.

$$\int\left(4x^5-\frac{1}{x}+\frac{2}{\sin^2 x}\right)dx=$$

$$=4\int x^5dx-\int\frac{dx}{x}+2\int\frac{dx}{\sin^2 x}=\frac{2}{3}x^6-\ln|x|-2\operatorname{ctg}x+C.$$

Example 1.9.

$$\int\frac{x^2}{x^2+1}dx=\int\frac{(x^2+1)-1}{x^2+1}dx=\int 1dx-\int\frac{1}{x^2+1}dx=x-\operatorname{arctg}x+C$$

Example 1.10.

$$\int\cos^2\frac{x}{2}dx=\int\frac{1+\cos x}{2}dx=\frac{1}{2}\left(\int dx+\int\cos x dx\right)=\frac{1}{2}(x+\sin x)+C$$

Example 1.11.

$$\int\frac{dx}{\cos^2 x\cdot\sin^2 x}=\int\frac{\cos^2 x+\sin^2 x}{\cos^2 x\cdot\sin^2 x}dx=\int\frac{dx}{\sin^2 x}+\int\frac{dx}{\cos^2 x}=\operatorname{tg}x-\operatorname{ctg}x+C$$

Example 1.12.

$$\int\frac{dx}{x^3\sqrt{x}}=\int x^{-\frac{7}{2}}dx=\frac{x^{-\frac{7}{2}+1}}{-\frac{7}{2}+1}+c=-\frac{2}{5}x^{-\frac{5}{2}}+c=-\frac{2}{5x^2\sqrt{x}}+c$$

Example 1.13.

$$\int\frac{5\cdot 3^x-3\cdot 5^x}{3^x}dx=\int\left(5-3\cdot\left(\frac{5}{3}\right)^x\right)dx=5\int dx-3\int\left(\frac{5}{3}\right)^x dx=$$

$$=5x-3\cdot\frac{\left(\frac{5}{3}\right)^x}{\ln\frac{5}{3}}+c=5x-\frac{3}{\ln\frac{5}{3}}\cdot\left(\frac{5}{3}\right)^x+c.$$

Example 1.14.

$$\int\frac{e^{2x}-1}{e^x+1}dx=\int\frac{(e^x)^2-1^2}{e^x+1}dx=\int\frac{(e^x-1)(e^x+1)}{e^x+1}dx=\int(e^x-1)dx=\int e^x dx-\int dx=$$

$$=e^x-x+c.$$

Example 1.15.

$$\int\operatorname{tg}^2 x\,dx=\int\left(\frac{1}{\cos^2 x}-1\right)dx=\operatorname{tg}x-x+C.$$

Example 1.16.

$$\int \frac{1+2x^2}{x^2(1+x^2)}dx = \int \frac{(1+x^2)+x^2}{x^2(1+x^2)}dx = \int \frac{dx}{x^2} + \int \frac{dx}{1+x^2} = -\frac{1}{x} + \text{arctg}\,x + C.$$

Example 1.17.

1. $\int 3^{x+2}dx = 9\int 3^x dx = 9\frac{3^x}{\ln 3} + C = \frac{3^{x+2}}{\ln 3} + C.$

2. $\int \frac{dx}{4x^2+9} = \frac{1}{4}\int \frac{dx}{x^2 + \left(\frac{3}{2}\right)^2} = \frac{1}{4}\cdot\frac{2}{3}\text{arctg}\frac{2x}{3} + C = \frac{1}{6}\text{arctg}\frac{2x}{3} + C.$

3. $\int \frac{dx}{\sqrt{16-9x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{4}{3}\right)^2 - x^2}} = \frac{1}{3}\arcsin\frac{3x}{4} + C.$

INTEGRATION BY SUBSTITUTION (CHANGE OF VARIABLE)

Example 2.1. $\int \frac{dx}{x\sqrt{4-\ln^2 x}} = \begin{vmatrix} t = \ln x \\ dt = (\ln x)' dx = \frac{1}{x}dx \end{vmatrix} = \int \frac{dt}{\sqrt{4-t^2}} =$

$$= \arcsin\frac{t}{2} + C = \arcsin\frac{\ln x}{2} + C$$

Example 2.2. $\int \frac{\sin x dx}{\cos^5 x} = \begin{vmatrix} t = \cos x \\ dt = -\sin x dx \\ -dt = \sin x dx \end{vmatrix} = \int \frac{-dt}{t^5} = -\int \frac{dt}{t^5} =$

$$= -\int t^{-5}dt = -\frac{t^{-5+1}}{-5+1} + C = \frac{1}{4t^4} + C = \frac{1}{4\cos^4 x} + C$$

Example 2.3.

$$\int \frac{x^5 dx}{\sqrt{x^{12}-5}} = \begin{vmatrix} t = x^6 \\ dt = 6x^5 dx \end{vmatrix} = \frac{1}{6}\int \frac{dt}{\sqrt{t^2-5}} = \frac{1}{6}\ln\left|t + \sqrt{t^2-5}\right| + c =$$

$$= \frac{1}{6}\ln\left|x^6 + \sqrt{x^{12}-5}\right| + c;$$

Example 2.4.

$$\int \frac{x^3dx}{6x^4+5} = \left| \begin{matrix} t = 6x^4+5 \\ dt = 24x^3dx \end{matrix} \right| = \frac{1}{24}\int \frac{dt}{t} = \frac{1}{24}\ln|t| + c = \frac{1}{24}\ln\left|6x^4+5\right| + c;$$

Example 2.5.

$$\int x \cdot 10^{x^2}\, dx = \left| \begin{matrix} t = x^2 \\ dt = 2xdx \end{matrix} \right| = \frac{1}{2}\int 10^t\, dt = \frac{1}{2}\cdot\frac{10^t}{\ln 10} + c = \frac{10^{x^2}}{2\ln 10} + c.$$

Example 2.6.

$$\int (1+\sin x)^3 \cos x dx \left\| \begin{matrix} 1+\sin x = t \\ \cos x dx = dt \end{matrix} \right\| = \int t^3 dx =$$

$$= \frac{t^{3+1}}{3+1} + C = \frac{t^4}{4} + C = \frac{(1+\sin x)^4}{4} + C$$

INTEGRATION BY PARTS

$$\int u \cdot dv = uv - \int v \cdot du$$

Example 3.1.

$$\int \ln x dx = \left| \begin{matrix} u = \ln x, & dv = dx, \\ du = \frac{1}{x}dx, & v = x \end{matrix} \right| = x\ln x - \int x \cdot \frac{1}{x} dx =$$

$$= x\ln x - x + C = x(\ln x - 1) + C.$$

Example 3.2.

$$\int xe^x dx = \left| \begin{matrix} u = x, & dv = e^x dx, \\ du = dx, & v = e^x \end{matrix} \right| = xe^x - \int e^x dx = xe^x - e^x + C.$$

Example 3.3.

$$\int x \cdot \cos 3x dx = \left| \begin{matrix} u = x,\ dv = \cos 3x dx \\ du = dx, \ v = \int \cos 3x dx = \frac{1}{3}\sin 3x \text{ (suppose that } C = 0) \end{matrix} \right| =$$

$$= x \cdot \frac{1}{3}\sin 3x - \int \frac{1}{3}\sin 3x dx = \frac{x}{3}\sin 3x - \frac{1}{3}\left(-\frac{1}{3}\cos 3x\right) + C =$$

$$= \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + C.$$

Example 3.4.

$$\int \arcsin x dx \; =$$

$$u = \arcsin x; \;\; du = \frac{dx}{\sqrt{1-x^2}};$$

$$dv = dx; \qquad v = \int dx = x.$$

$$= x\arcsin x - \int \frac{xdx}{\sqrt{1-x^2}} = x\arcsin x + \frac{1}{2}\int \frac{-2xdx}{\sqrt{1-x^2}} =$$

$$= x\arcsin x + \sqrt{1-x^2} + C.$$

Example 3.5.

$$\int \frac{\ln x}{\sqrt[3]{x}}dx =$$

$$u = ln\,x,\ dv = \frac{dx}{\sqrt[3]{x}}$$

$$du = (\ln x)' dx = \frac{1}{x}dx, \quad v = \int dv = \int \frac{dx}{\sqrt[3]{x}} = \int x^{-\frac{1}{3}}dx = \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} = \frac{3}{2}x^{\frac{2}{3}}$$

$$= \frac{3}{2}x^{\frac{2}{3}}\,ln\,x - \int \frac{3}{2}x^{\frac{2}{3}}\frac{dx}{x} = \frac{3}{2}x^{\frac{2}{3}}\,ln\,x - \frac{3}{2}\int x^{-\frac{1}{3}}dx =$$

$$= \frac{3}{2}x^{\frac{2}{3}}\,ln\,x - \frac{3}{2}\cdot\frac{3}{2}x^{\frac{2}{3}} + C = \frac{3}{2}x^{\frac{2}{3}}\,ln\,x - \frac{9}{4}x^{\frac{2}{3}} + C.$$

Example 3.6.

$$\int x\sin x dx =$$

$$u = x\ ,\ du = dx$$

$$dv = \sin x dx\ ,\ v = \int \sin x dx = -\cos x$$

$$= -x\cos x - \int (-\cos x)dx = -x\cos x + \int \cos x dx =$$

$$= -x\cos x + \sin x + C$$

Example 3.7.

$$\int \operatorname{arctg} x dx = \begin{Bmatrix} u = \operatorname{arctg} x, du = \dfrac{dx}{1+x^2} \\ dv = dx, v = x \end{Bmatrix} = x\operatorname{arctg} x - \int \frac{x\,dx}{1+x^2} =$$

$$= x\operatorname{arctg} x - \frac{1}{2}\ln(x^2+1) + C.$$

Example 3.8.

$$\int x^2 \sin xdx = \begin{Bmatrix} u = x^2; & dv = \sin xdx; \\ du = 2xdx; & v = -\cos x \end{Bmatrix} = -x^2\cos x + \int \cos x \cdot 2xdx =$$

$$= \begin{Bmatrix} u = x; & dv = \cos xdx; \\ du = dx; & v = \sin x \end{Bmatrix} = -x^2\cos x + 2\left[x\sin x - \int \sin xdx\right] =$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C.$$

INTEGRATION OF RATIONAL FUNCTIONS WITH QUADRATIC TRINOMIAL

Example 4.1.

$$\int \frac{(7-8x)dx}{x^2-6x+2} = \int \frac{(7-8x)dx}{(x-3)^2-7} = \begin{vmatrix} t = x-3 \\ dt = dx \\ x = t+3 \end{vmatrix} = \int \frac{(7-8(t+3))dt}{t^2-7} =$$

$$= \int \frac{(10-8t)dt}{t^2-7} = 10\int \frac{dt}{t^2-7} - 8\int \frac{tdt}{t^2-7} = \frac{10}{2\sqrt{7}} \ln\left|\frac{t-\sqrt{7}}{t+\sqrt{7}}\right| - \frac{8}{2}\ln\left|t^2-7\right| + C =$$

$$= \frac{5}{\sqrt{7}} \ln\left|\frac{x-3-\sqrt{7}}{x-3+\sqrt{7}}\right| - \frac{8}{2}\ln\left|x^2-6x+2\right| + C.$$

Example 4.2

$$\int \frac{dx}{x^2-4x+20} = \int \frac{dx}{(x-2)^2+16} = \int \frac{dx}{(x-2)^2+4^2} = \frac{1}{4} arctg \frac{x-2}{4} + c.$$

INTEGRATION OF RATIONAL FRACTIONS

Example 5.1.

$$\int \frac{x+1}{x-3}dx = \int \frac{(x-3)+4}{x-3}dx = \int\left(1+\frac{4}{x-3}\right)dx = \int dx + 4\int \frac{dx}{x-3} =$$

$$= x + 4\ln|x-3| + c.$$

Example 5.2.

$$\int \frac{dx}{x^3-1}.$$

Let us decompose the fraction into partial fractions:

$$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+x+1)}.$$

Since the denominators on the left and right are equal and the fractions are identically equal, then the numerators are equal too. Then $1 = A(x^2+x+1) + (Bx+C)(x-1)$.

Let's consider the first method of finding of undetermined coefficients.

Let us collect: $1 = x^2(A+B) + x(A-B+C) + (A-C)$.

Equating the factors at the identical degrees of *x* we obtain the coefficients:

at x^2: $0 = A+B$,

at x^1: $0 = A-B+C$,

at x^0: $1 = A-C$,

we obtain the coefficients: from the 1-st equation: $B = -A$,

from the 2-nd equation: $C = -A+B = -A-A = -2A$, from the 3-rd equation: $1 = A - (-2A)$ or $1 = 3A$ or $A = \frac{1}{3}$, then $B = -A = -\frac{1}{3}$,

$C = -2A = -\frac{2}{3}$.

So, $\int \frac{dx}{x^3 - 1} = \int \left(\frac{A}{x-1} + \frac{Bx + C}{(x^2 + x + 1)} \right) dx = \int \left(\frac{1/3}{x-1} + \frac{-1/3\, x - 2/3}{(x^2 + x + 1)} \right) dx =$

$$= \frac{1}{3} \int \frac{dx}{x-1} - \frac{1}{3} \int \frac{x+2}{x^2 + x + 1} dx = \frac{1}{3} \ln|x-1| - \frac{1}{3} \int \frac{x+2}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} dx = \left| \begin{matrix} t = x + \frac{1}{2} \\ dt = dx \\ x = t - \frac{1}{2} \end{matrix} \right| =$$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{3} \int \frac{t - \frac{1}{2}}{t^2 + \frac{3}{4}} dt = \frac{1}{3} \ln|x-1| - \frac{1}{3} \int \frac{t}{t^2 + \frac{3}{4}} dt + \frac{1}{6} \int \frac{dt}{t^2 + \left(\frac{\sqrt{3}}{2} \right)^2} =$$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{3} \frac{1}{2} \ln\left| t^2 + \frac{3}{4} \right| + \frac{1}{6} \frac{1}{\sqrt{3}/2} arctg \frac{t}{\sqrt{3}/2} + C =$$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln\left| \left(x + \frac{1}{2} \right)^2 + \frac{3}{4} \right| + \frac{1}{6} \frac{1}{\sqrt{3}/2} arctg \frac{x + \frac{1}{2}}{\sqrt{3}/2} + C =$$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln\left| x^2 + x + 1 \right| + \frac{1}{3\sqrt{3}} arctg \frac{2x+1}{\sqrt{3}} + C.$$

Example 5.2. $\int \frac{15x^2 - 4x - 81}{(x-3)(x+4)(x-1)} dx$.

The integrand is a proper fraction, let us decompose it into partial fractions.

$$\frac{15x^2 - 4x - 81}{(x-3)(x+4)(x-1)} = \frac{A}{x-3} + \frac{B}{x+4} + \frac{C}{x-1}.$$

Since the denominators on the left and right are equal and the fractions are identically equal, then the numerators are equal too.

Let us reduce to the common denominator and equate the numerators:

$$15x^2 - 4x - 81 = A(x+4)(x-1) + B(x-3)(x-1) + C(x-3)(x+4).$$

Let’s consider the second method of finding of undetermined coefficients.

Let us substitute the first root of the denominator $x = 3$ into this expression:

$$15 \cdot 9 - 4 \cdot 3 - 81 = A(3+4)(3-1) + B \cdot 0 + C \cdot 0,$$

$$42 = 14A, \text{ or } A = 3.$$

Let us substitute the second root of the denominator $x=-4$ into this expression:

$$15\cdot 16+4\cdot 4-81=A\cdot 0+B(-4-3)(-4-1)+C\cdot 0,$$

$$175=35B, \text{ or } B=5.$$

Let us substitute the third root of the denominator $x=1$ into this expression:

$$15\cdot 1-4\cdot 1-81=A\cdot 0+B\cdot 0+C(1-3)(1+4),$$

$$-70=-10B, \text{ or } B=7.$$

So, $\int \frac{15x^2-4x-81}{(x-3)(x+4)(x-1)}dx=\int\left(\frac{A}{x-3}+\frac{B}{x+4}+\frac{C}{x-1}\right)dx=$

$$=\int\left(\frac{3}{x-3}+\frac{5}{x+4}+\frac{7}{x-1}\right)dx=3\int\frac{1}{x-3}dx+5\int\frac{1}{x+4}dx+7\int\frac{1}{x-1}dx=$$

$$=3\ln|x-3|+5\ln|x+4|+7\ln|x-1|+C.$$