Applications of the Definite Integral in Business and Economics

The mean value of a function on the interval $[a, b]$**:**

$$f(c) = \frac{1}{b-a} \cdot \int_a^b f(x)dx.$$

Example 1. Let us assume that the value of the UAH expressed in the US dollars from day 1 to day 30 of a certain month is represented by the function of time: $f(t) = t^2 + \frac{900}{t^3}$. Find the mean value of the UAH in this month.

Solution. First of all, let's find the definite integral:

$$\int_1^{30} \left(x^2 + \frac{900}{x^3} \right) dx = \int_1^{30} \left(x^2 + 900 \cdot x^{-3} \right) dx = \left(\frac{1}{3} x^3 - \frac{900}{2} x^{-2} \right) \Bigg|_1^{30} = \left(\frac{x^3}{3} - \frac{450}{x^2} \right) \Bigg|_{x=30} -$$

$$- \left(\frac{x^3}{3} - \frac{450}{x^2} \right) \Bigg|_{x=1} = \frac{30^3}{3} - \frac{450}{30^2} - \frac{1^3}{3} + \frac{450}{1^2} = 9000 - \frac{1}{2} - \frac{1}{3} + 450 \approx 9449.$$

According to the formula of the mean value of a function on the given interval $[1, 30]$, we have:

$$f(c) = \frac{1}{30-1} \int_1^{30} \left(x^2 + \frac{900}{x^3} \right) dx \approx \frac{1}{29} \cdot 9449 \approx 325.8.$$

Thus, the mean value of the UAH in this month is approximately 325.8.

Example 2. Let us assume that the production costs $f(x) = 10 + 20x + 0.03x^2$ The volume of manufacture changes from 100 UAH to 300 UAH per year. Find the mean value of the cost of production.

Solution. According to the formula of the mean value of a function on the given interval $[100, 300]$, we have:

$$f(c)=\frac{1}{300–100}\cdot\int\limits_{100}^{300}\left(10+20x+0,03x^2\right)dx=\frac{1}{200}\cdot\left(10x+20\cdot\frac{x^2}{2}+0,03\cdot\frac{x^3}{3}\right)\Bigg|_{100}^{300}=$$

$$=\frac{1}{200}\cdot\left[\left(10x+10x^2+0,01x^3\right)\Big|_{x=300}-\left(10x+10x^2+0,01x^3\right)\Big|_{x=100}\right]=$$

$$=\frac{1}{200}\cdot\left(3000+900000+270000-1000-100000-10000\right)=$$

$$=\frac{1}{200}\cdot\left(1173000-120000\right)=5865-600=5265.$$

Thus, the mean value of the cost of production is 5 265 UAH.

Example 3. The value of productivity of equipment at certain factory from the day of start (the first day) to the next day (the second day) is represented by the function of time $f(t)=2^t$. Find the average value of productivity of this equipment.

Solution. a) In order to find the average value of productivity of equipment we will calculate the average value of the given function $f(t)=2^t$ on the interval $[0, 2]$. Due to the formula of the mean value of a function on the interval:

$$f(c)=\frac{1}{2–1}\int\limits_{1}^{2}2^x dx=\frac{2^x}{\ln 2}\Bigg|_1^2=\frac{2^2}{\ln 2}-\frac{2^1}{\ln 2}=\frac{2}{\ln 2}\approx\frac{2}{0.6931472}\approx 2.88539.$$

Thus, the mean value of the cost of productivity of the given equipment is approximately 2.88539.

The income concentration index – the Gini coefficient:

$$G=2\int\limits_0^1\left(x-L(x)\right)dx,$$

where $L(x)$ is the Lorenz curve defined as follows. Let's assume that the vector of incomes $x=(x_1,\ldots,x_n)$ is arranged in a non-decreasing order:

$x_1 \le x_2 \le \ldots \le x_n$. The empirical Lorenz function is generated by points, whose first coordinates are numbers i/n, where $i = 0, 1, \ldots, n$; n is a fixed number, and second coordinates are determined as follows: $L(0) = 0$ and $L\left(\frac{i}{n}\right) = \frac{s_1}{s_n}$, where $s_1 = x_1 + x_2 + \ldots + x_i$. The Lorenz curve is defined at all points $p \in (0,1)$ through linear interpolation. One can show that $L'(x) > 0$ and $L''(x) > 0$, $L(0) = 0$ and $L(1) = 1$.

Remark. When the value of the Gini coefficient is close to 0, the underlying distribution is almost uniform, whereas the value close to 1 indicates a maximal inequality, i.e., a total wealth of a population is concentrated in the hands of one man.

Example 4. The Lorenz curve of the income distribution within a certain group is given by the formula: $L(x) = 0.8x^2 + 0.28x$. Determine the degree of equality of the income distribution.

Solution. Let's find the Gini coefficient

$$G = 2\int_0^1 \left(x - (0.8x^2 + 0.28x)\right)dx = 2\int_0^1 \left(-0.8x^2 + 0.72x\right)dx =$$

$$= 2\left(-0.8 \cdot \frac{x^{2+1}}{2+1} + 0.72 \cdot \frac{x^{1+1}}{1+1}\right)\Bigg|_0^1 = 2\left(-\frac{4}{15} \cdot x + 0.36 \cdot x^2\right)\Bigg|_0^1 =$$

$$= 2\left(-\frac{4}{15} \cdot x + 0.36 \cdot x^2\right)\Bigg|_{x=1} - 2\left(-\frac{4}{15} \cdot x + 0.36 \cdot x^2\right)\Bigg|_{x=0} = 2\left(-\frac{4}{15} + 0.36\right) \approx 0.2.$$

As the Gini coefficient is relatively small, we conclude that the given income distribution is fairly uniform.

Example 5. The Lorenz functions L_1 and L_2 of an income in the population of civil servants and teachers of a certain country are given by the formulas: $L_1(x) = x^2$ and $L_2(x) = 0.3x^2 + 0.7x$ respectively. In which group of employees is income distributed more uniformly?

Solution. Let's find the Gini coefficients G_1 and G_2 by the formula

$$G_i = 2\int_0^1 \left(x - L_i(x)\right)dx,$$

where $L_i(x)$ are given.

Thus,

$$G_1 = 2\int_0^1 \left(x - x^2\right)dx = 2\cdot\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = 2\cdot\frac{1}{6} = 0.3(3)$$

and

$G_2 = 2\int_0^1 \left(x - (0.3x^2 + 0.7x)\right)dx = 2\int_0^1 \left(-0.3x^2 + 0.3x\right)dx = 0.1$, accordingly.

Obviously, that $0.1 < 0.3(3)$, hence we conclude that income is more uniformly distributed in the group of teachers.

The net change

In many practical applications, we are given the rate of change $Q'(x)$ of a quantity $Q(x)$ and required to compute the net change $NC = Q(b) - Q(a)$ in $Q(x)$ as x varies from $x = a$ to $x = b$. But since $Q(x)$ is an antiderivative of $Q'(x)$, the fundamental theorem of calculus tells us that the net change is given by the definite integral

$$NC = Q(b) - Q(a) = \int_a^b Q'(x)dx.$$

Here is an example illustrating the computation of the net change by definite integration.

Example 6. A firm's monthly marginal cost of a product is $MC(x)$ UAH per unit when the level of production is x units. Find the total manufacturing cost function $TC(x)$ of the given month if:

a) $MC(x) = x + 30$ and the level of production is raised from 4 units to 6 units;

b) $MC(x) = 3(x-5)^2$ and the level of production is raised from 6 units to 10 units.

Solution. It is well-known that the total cost function is the antiderivative of the marginal cost function:

$$TC(x) = \int MC(x)\, dx .$$

According to the condition of this task the level of production is raised from a units to b units. It means that the total cost function is given by the definite integral

$$TC(b) - TC(a) = \int_a^b MC(x)\, dx.$$

a) As x from $MC(x) = x + 30$ varies from $x = a = 4$ to $x = b = 6,$ we obtain

$$TC(6) - TC(4) = \int_4^6 MC(x)\, dx = \int_4^6 (x+30)\, dx = \left(\frac{x^2}{2} + 30x\right)\Bigg|_4^6 =$$

$$= \frac{6^2}{2} + 30 \cdot 6 - \left(\frac{4^2}{2} + 30 \cdot 4\right) = 18 + 180 - 8 - 120 = 70.$$

Thus, the total cost function in the given month is 70 UAH.

b) For $MC(x) = 3(x-5)^2$, according to the condition, x varies from $x = a = 6$ to $x = b = 10,$ we obtain

$$TC(10) - TC(6) = \int_6^{10} MC(x)\, dx = \int_6^{10} 3(x-5)^2 dx = (x-5)^3\Big|_6^{10} =$$

$$= (10-5)^3 - (6-5)^3 = 125 - 1 = 124.$$

Thus, the total cost function in the given month is 124 UAH.

The net excess profit

Suppose that t years from now, two investment plans will be generating profit $P_1(t)$ and $P_2(t)$, respectively. Assume also that the respective rates of profitability $P_1'(t)$ and $P_2'(t)$ satisfy an inequality $P_2'(t) \geq P_1'(t)$ for the first N years $(0 \leq t \leq N)$. Then

$$E(t) = P_2(t) - P_1(t)$$

represents the excess profit of plan 2 over plan 1 at the time t and the net excess profit

$$NE = E(N) - E(0)$$

over the time period $0 \leq t \leq N$ is given by the definite integral:

$$NE = E(N) - E(0) = \int_0^N E'(t)dt = \int_0^N \left[P_2'(t) - P_1'(t) \right] dt.$$

Remark. Note here, that the net excess profit NE which referred to above can be interpreted geometrically as the area between the curves $y = P_1'(t)$ and $y = P_2'(t)$ (Fig. 1).

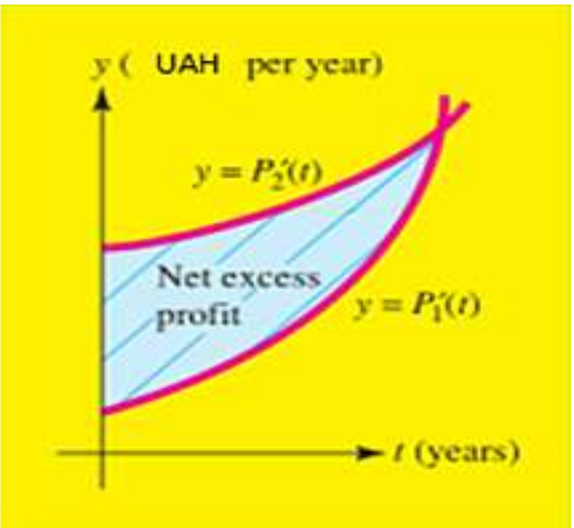

Fig. 1. The net excess profit as the area between two curves

Example 7. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t) = 5 + t^2$ UAH per year, while a second investment will be generating profit at the rate of $P_2'(t) = 10t + 80$ UAH per year.

a) For how many years does the rate of profitability of the second investment exceed that of the first?

b) Compute the net excess profit, in UAH, assuming that you invest in the second plan for the time period determined in part a).

c) Interpret the net excess profit as an area: sketch the rate of profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ and shade the region whose area represents the net excess profit computed in part b).

Solution. a) The rate of profitability of the second investment exceeds that of the first until

$$P_1'(t) = P_2'(t) \quad \text{or} \quad 5 + t^2 = 10t + 80.$$

Thus, we have a quadratic equation

$$t^2 - 10t - 75 = 0$$

from which it is easy to obtain

$$(t - 15)(t + 5) = 0,$$

hence

$$t_1 = 15 \text{ and } t_2 = -5.$$

Clearly, $t < 0$ cannot be the answer. So, we reject $t_2 = -5 < 0$ and accept the root of the quadric equation $t = 15$.

Conclusion: the rate of profitability of the second investment will exceed that of the first after 15 years.

b) The net excess profit for the time period $0 \le t \le 15$ is given by the definite integral

$$NE = \int_0^{15} \left[P_2'(t) - P_1'(t) \right] dt = \int_0^{15} \left[(80 + 10t) - (5 + t^2) \right] dt = \int_0^{15} (75 + 10t - t^2)\, dt =$$

$$= \left(75t + \frac{10}{2} \cdot t^2 - \frac{1}{3} \cdot t^3 \right) \Bigg|_0^{15} = 75 \cdot 15 + 5 \cdot 15^2 - \frac{1}{3} \cdot 15^3 = 1125 + 1125 - 1125 = 1125.$$

Thus, the net excess profit is 1125 UAH.

c) The rate of the profit curves for the two investments is shown in Fig. 2. The net excess profit is the area of the (shaded) region between the curves.

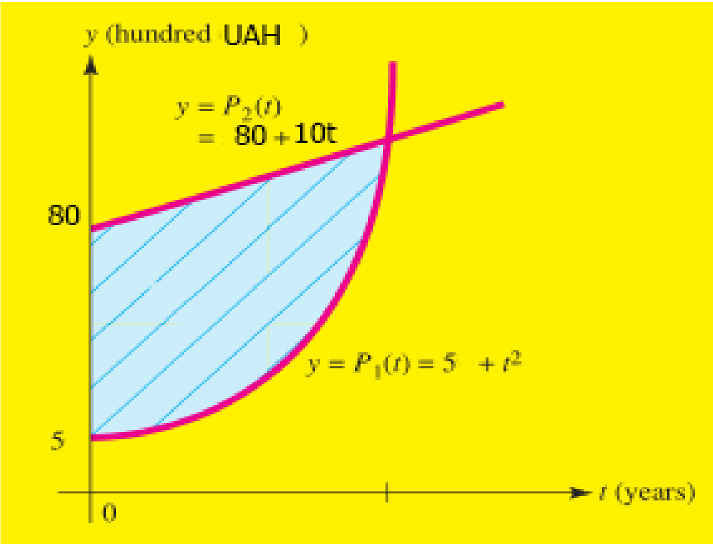

Fig. 2. The net excess profit for Example 5

The net earnings from industrial equipment

The net earnings generated by an industrial machine over a period of time is the difference between the total revenue generated by the machine and the total cost of operating and servicing the machine.

The following example shows how net earnings can be computed by definite integration.

Example 8. Suppose that when it is t years old, a particular industrial machine generates revenue at the rate $R'(t) = 500 - 2t^2$ UAH per year and that operating and servicing costs related to the machine accumulate at the rate $C'(t) = 200 + t^2$ UAH per year.

a) How many years will have pass before the profitability of the machine begins to decline?

b) Compute the net earnings generated by the machine over the time period determined in part a).

c) Interpret the net earnings as an area.

Solution. a) Let's remember the relationship between the revenue and the cost functions:

$$R(t) = P(t) + C(t),$$

where $R(t)$ is the revenue function, $P(t)$ is the profit function and $C(t)$ is the cost function. According to the condition of this task we have some information about the revenue and cost, but are asked about profit associated with the machine after t years of operation, so we will need to use the relationship

$$P(t) = R(t) - C(t).$$

Consequently, the rate of profitability is

$$P'(t) = R'(t) - C'(t) = (500 - 2t^2) - (200 + t^2) = 300 - 3t^2.$$

The profitability begins to decline when $P'(t) = 0$.

Thus, we have a quadric equation

$$300 - 3t^2 = 0,$$

from which it is easy to obtain

$$t^2 = 100,$$

hence

$$t_1 = 10 \text{ and } t_2 = -10.$$

Clearlly, $t < 0$ cannot be the answer. So, we reject $t_2 = -10$ and accept the root $t = 10$.

Conclusion: 10 years will have passed before the profitability of the machine begins to decline.

b) The net earnings NE over the time period $0 \le t \le 10$ is given by the difference

$$NE = P(10) - P(0),$$

which can be computed by the definite integral:

$$NE = P(10) - P(0) = \int_0^{10} P'(t)dt = \int_0^{10} (300 - 3t^2)dt = (300t - t^3)\Big|_0^{10} = 2000.$$

So, the net earnings generated by the machine over the time period $0 \le t \le 10$ are 2 000 UAH.

c) The rate of revenue and the rate of cost curves are sketched in Fig. 3. The net earnings are the area of the (shaded) region between the curves.

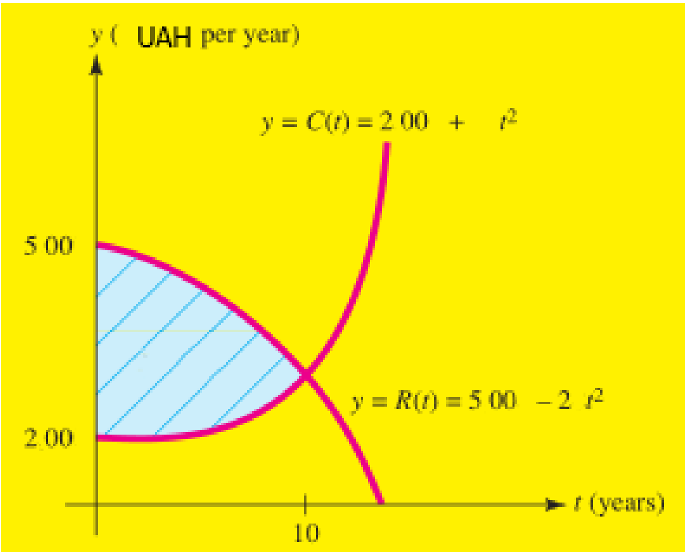

Fig. 3. The net earnings from an industrial machine

The consumers' demand curve and willingness to spend

One of the most fundamental economic models is the law of supply and demand for a certain product (bread, milk, fruit, coffee, chocolate etc.) or service (transportation, education, health care etc.) in a free-market environment. In this model the quantity of a certain item produced and sold is described by two curves called the demand and supply curves of the item.

Note here, that the consumers' demand function $p = D(q)$ can also be thought of as the rate of change with respect to q of the total amount $A(q)$ that consumers are willing to spend on q units; that is, $\frac{dA}{dq} = D(q)$. Integrating, you find that the total amount that consumers are willing to pay for q_0 units of the commodity is given by

$$A(q_0) - A(0) = \int_0^{q_0} \frac{dA}{dq} dq = \int_0^{q_0} D(q) dq.$$

Remark 1. In this context, economists call $A(q)$ the total willingness to spend and $D(q) = A'(q)$ the marginal willingness to spend.

Remark 2. In geometric terms, the total willingness to spend on q_0 units is the area under the demand curve $p = D(q)$ between $q = 0$ and $q = q_0$ (Fig. 4).

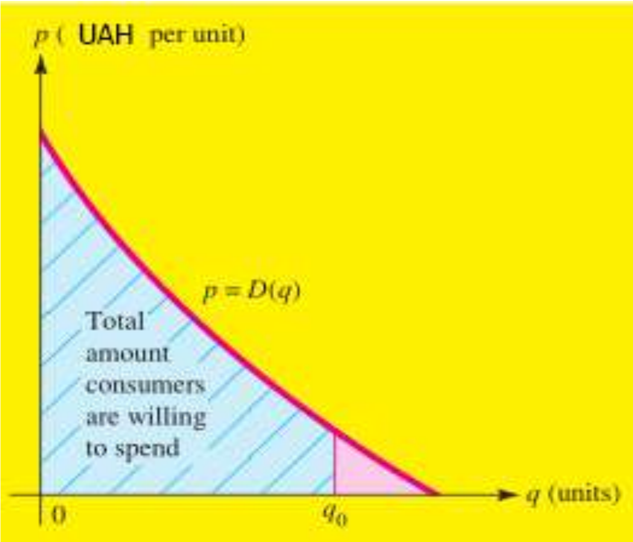

Fig. 4. The amount consumers are willing to spend is the area under the demand curve

Example 9. Suppose that the consumers' demand function for a certain commodity is $D(q) = 25 - q^2$ UAH per unit.

a) Find the total amount of money consumers are willing to spend to get 3 units of the commodity.

b) Sketch the demand curve and interpret the answer to part a) as an area.

Solution. a) Since the demand function $D(q) = 25 - q^2$, measured in UAH per unit, is the rate of change with respect to q of consumers' willingness to spend, the total amount that consumers are willing to spend to get 3 units of the commodity is given by the definite integral

$$\int_0^3 D(q)dq = \int_0^3 \left(25 - q^2\right)dq = \left(25q - \frac{1}{3} \cdot q^3\right)\Bigg|_0^3 = 75 - 9 = 66.$$

So, the total amount of money consumers is willing to spend to get 3 units of the commodity is 66 UAH.

b) The consumers' demand curve is sketched in Fig. 5. In geometric terms, the total amount, 66 UAH, that consumers are willing to spend to get 3 units of the commodity is the area under the demand curve from $q = 0$ to $q = 3$.

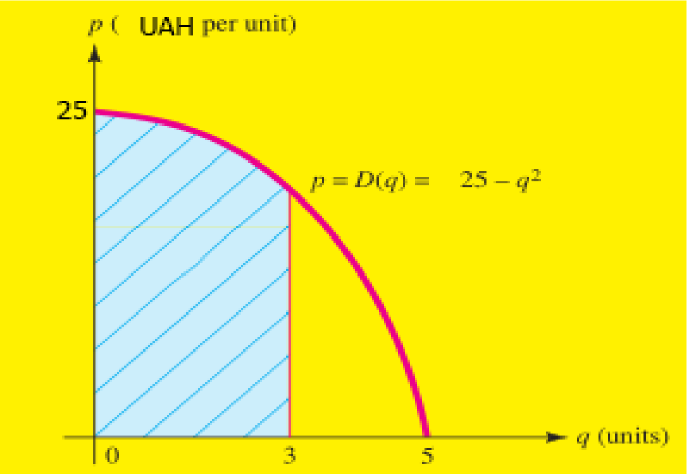

Fig. 5. Consumers' willingness to spend on 3 units when demand is given by $D(q) = 25 - q^2$

Consumers' Surplus

Clearly, that in a competitive economy, the total amount that consumers actually spend on a commodity is usually less than the total amount they would have been willing to spend. The difference between the two amounts can be thought of as savings realized by consumers and is known in economics as the consumers' surplus. That is,

$$\begin{bmatrix} Consumers' \\ surplus \end{bmatrix} = \begin{pmatrix} total\,amount\,consumers \\ would\,be\,willing\,to\,spend \end{pmatrix} - \begin{pmatrix} actual\,consumers' \\ expenditure \end{pmatrix}$$

Market conditions determine the price per unit at which a commodity is sold. Once the price, say p_0, is known, the demand equation $p = D(q)$ determines the number of units q_0 that consumers will buy. The actual consumer expenditure on q_0 units of the commodity at the price of p_0 UAH per unit is $p_0 q_0$ UAH. The consumers' surplus is calculated by subtracting this amount from the total amount consumers would have been willing to spend to get q_0 units of the commodity.

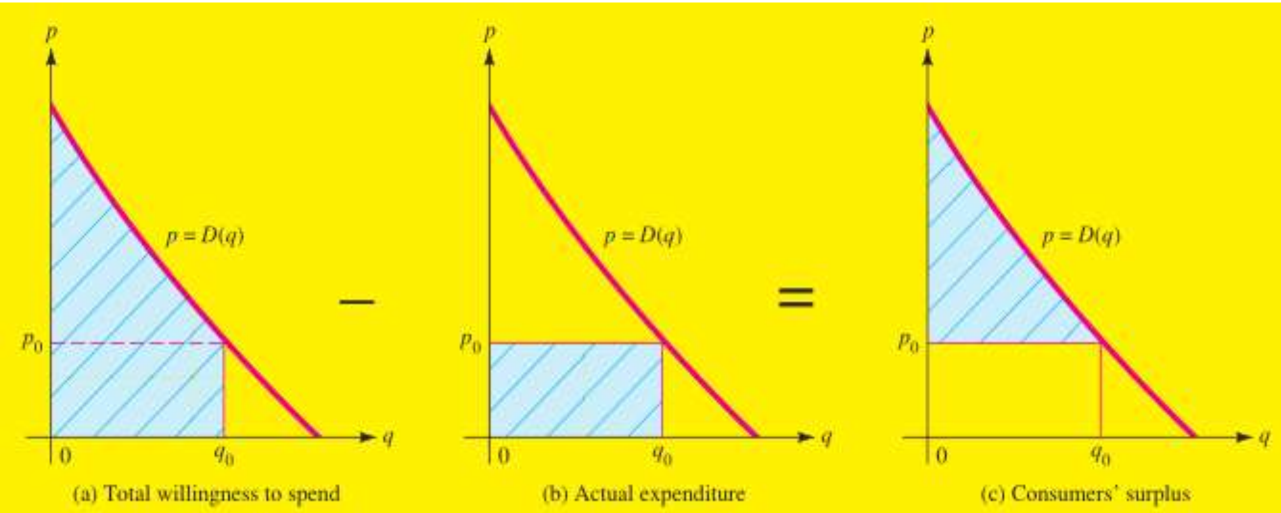

Fig. 6. Geometric interpretation of consumers' surplus

Consumers' surplus has a simple geometric interpretation, which is illustrated in Fig. 6, a, b, c. The symbols p_0 and q_0 denote the market price and corresponding demand, respectively. Fig. 6, a shows the region under the demand curve from $q = 0$ to $q = q_0$. Its area, as we have seen, represents the total amount that consumers are willing to spend to get q_0 units of the commodity.

The rectangle in Fig. 6, b has an area of $p_0 q_0$ and hence represents the actual consumer expenditure on q_0 units at p_0 UAH per unit. The difference between these two areas (Fig. 6, c) represents the consumers' surplus. That is, consumers' surplus CS is the area of the region between the demand curve $p = D(q)$ and the horizontal line $p = p_0$. Hence, if q_0 units of a commodity are sold at a price of p_0 per unit and if $p = D(q)$is the consumers' demand function for the commodity, then

$$CS = \int_0^{q_0} \left[D(q) - p_0 \right] dq = \int_0^{q_0} D(q) dq - \int_0^{q_0} p_0 dq = \int_0^{q_0} D(q) dq - p_0 q_0 .$$

Thus,

$$CS = \int_0^{q_0} D(q) dq - p_0 q_0 .$$

Producers*' *Surplus

Producers' surplus is the other side of the coin of consumers' surplus. In particular, the supply function $p = S(q)$ gives the price per unit that producers are willing to accept in order to supply q_0 units to the marketplace. However, any producer who is willing to accept less than $p_0 = S(q_0)$ dollars for q_0 units gains from the fact that the price is p_0. Then producers' surplus is the difference between what producers would be willing to accept for supplying q_0 units and the price they actually receive. Assume that q_0 units of a commodity are sold at a price of p_0 UAH per unit and $p = S(q)$ is the producers' supply function for the commodity, then

$$PS = p_0 q_0 - \int_0^{q_0} \mathrm{S}(q) dq.$$

Consumers' surplus has a simple geometric interpretation, which is illustrated in Fig. 7.

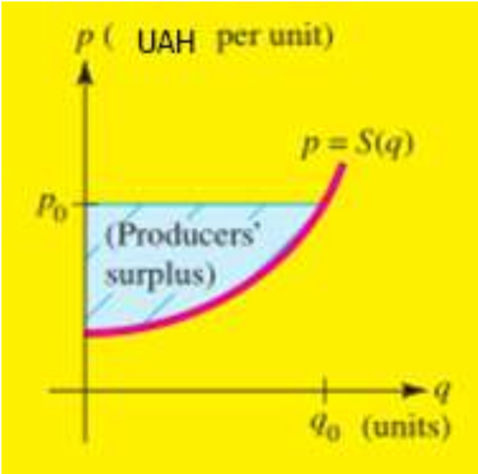

Fig. 7. Geometric interpretation of producers' surplus

Example 10. A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is

$$p = D(q) = -q^2 + 116 \text{ UAH per tire}$$

and the same number of tires will be supplied when the price is

$$p = S(q) = 20 + \frac{5}{2} \cdot q \text{ UAH per tire.}$$

a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price.

b) Determine the consumers' surplus at the equilibrium price.

c) Determine the producers' surplus at the equilibrium price.

Solution. a) First of all we will find the equilibrium price. Supply equals demand when

$$-q^2 + 116 = 20 + \frac{5}{3} \cdot q$$

or

$$3q^2 + 5q - 288 = 0.$$

Let's find roots of the quadric equation obtained above: $q_1 = 9$ and $q_2 = -\frac{32}{3}$. Underline here, that $q_2 < 0$ cannot be the answer. Hence we reject $q_2 = -\frac{32}{3}$ and accept the root $q_1 = 9$.

So, under $q_0 = 9$ we have $p_0 = D(9) = 116 - 9^2 = 116 - 81 = 35$. Thus, equilibrium occurs at a price of 35 UAH per tire.

b) Using $p_0 = 35$ and $q_0 = 9$, we find that the consumers' surplus is

$$CS = \int_0^{q_0} D(q)dq - p_0 q_0 = \int_0^9 \left(-q^2 + 116\right)dq - 35 \cdot 9 = \left(-\frac{q^3}{3} + 116q\right)\Bigg|_0^9 - 315 =$$

$$= -\frac{9^3}{3} + 116 \cdot 9 - 315 = -\frac{729}{3} + 1044 - 315 = -243 + 729 = 486.$$

So, the consumers' surplus at the equilibrium price ($p_0 = 35$) is 486 UAH.

c) According to the producers' surplus formula we have

$$PS = p_0 q_0 - \int_0^{q_0} S(q)dq = 35 \cdot 9 - \int_0^9 \left(20 + \frac{5}{2} \cdot q\right)dq = 315 - \left(20 \cdot q + \frac{5}{6} \cdot q^2\right)\Bigg|_0^9 =$$

$$= 315 - 20 \cdot 9 - \frac{5}{6} \cdot 9^2 = 315 - 180 - \frac{405}{6} = 315 - 180 - 67.5 = 67.5.$$

Thus, the producers' surplus at the equilibrium price $p_0 = 35$ is 67.5 UAH.

Volume of production

Let the function $z = f(t)$ describe the changing of the productivity of an enterprise under the time t. Then the volume of production V produced by the time $[t_1, t_2]$ is defined by the formula

$$V = \int_{t_1}^{t_2} f(t)\,dt.$$

Example 11. Find the volume of production V produced by an employee over the second working hour if the productivity is defined by the function:

$$f(t) = \frac{2}{3t+4} + 3 \text{ (kg)}.$$

Solution. Obviously the second working hour is the time from the first to the second hour. So, due to the formula of the volume of production V produced by the time $[1, 2]$, we have:

$$V = \int_1^2 \left(\frac{2}{3t+4} + 3 \right) dt = \left(\frac{2}{3} \ln|3t+4| + 3t \right)\Bigg|_1^2 = \frac{2}{3} \ln\frac{10}{7} + 3 \approx 3.24.$$

Thus, the volume of production is 3.24 kg.