Example 1. Investigate and plot the graph of the function: $y=\dfrac{x^3}{x^2-1}$.

1. Let's determine the domain of the function definition: $(-\infty,-1)\cup(-1,1)\cup(1,+\infty)$.

2. Let's research the continuity of the function: $x=1$ and $x=-1$ are break points of the function, since $\lim\limits_{x\to 1\pm 0}\dfrac{x^3}{x^2-1}=\dfrac{(1\pm 0)^3}{(1\pm 0)^2-1}=\dfrac{1}{1-1}=\dfrac{1}{0}=+\infty$

and $\lim\limits_{x\to -1\pm 0}\dfrac{x^3}{x^2-1}=\dfrac{(-1\pm 0)^3}{(-1\pm 0)^2-1}=\dfrac{-1}{1-1}=-\dfrac{1}{0}=-\infty$,

and therefore $x=1$ and $x=-1$ are vertical asymptotes.

3. Investigate the function on parity:

If $f(-x)=f(x)$ then a function is parity. If $f(-x)=-f(x)$ then a function is oddness.

If $f(-x)\neq f(x)$ and $f(-x)\neq -f(x)$ then it's a function of a general view.

Let's check it: $f(-x)=\dfrac{(-x)^3}{(-x)^2-1}=\dfrac{-x^3}{x^2-1}=-\dfrac{x^3}{x^2-1}=-f(x)$

4. The function is nonperiodic, i.e. there is no such value T that the equality $f(x+T)=f(x),\quad \forall x\in D(f)$.

5. The inclined (oblique) asymptote has the equation $y=kx+b$, *where* $k=\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}$ *and* $b=\lim\limits_{x\to\pm\infty}\left(f(x)-k\cdot x\right)$.

If $k=0$ then the asymptote $y=b$ is called horizontal.

Let's find the inclined (oblique) asymptotes:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{x^3}{x\left(x^2-1\right)}=\lim_{x\to\pm\infty}\frac{x^2}{x^2-1}=\left|\frac{\infty}{\infty}\right|=$$

$$=\lim_{x\to\pm\infty}\frac{\left(x^2\right)'}{\left(x^2-1\right)'}=\lim_{x\to\pm\infty}\frac{2x}{2x-0}=\lim_{x\to\pm\infty}\frac{2x}{2x}=\lim_{x\to\pm\infty}1=1$$

$$b=\lim_{x\to\pm\infty}\left(f(x)-k\cdot x\right)=\lim_{x\to\pm\infty}\left(\frac{x^3}{x^2-1}-x\right)=\lim_{x\to\pm\infty}\left(\frac{x^3-x^3+x}{x^2-1}\right)=$$

$$=\lim_{x\to\infty\pm}\frac{x}{x^2-1}=\left|\frac{\infty}{\infty}\right|=\lim_{x\to\infty\pm}\frac{(x)'}{\left(x^2-1\right)'}=\lim_{x\to\infty\pm}\frac{1}{2x}=0.$$

Hence, this function has an inclined asymptote $y = 1 \cdot x + 0$ or $y = x$.

6. Determine the intervals of monotonicity and the extremums of the function. For this purpose it is necessary to find the first derivative of the function and to determine points, in which it is to zero or does not exist.

Let's find the first derivative of the function:

$$y' = \left(\frac{x^3}{x^2 - 1}\right)' = \frac{3x^2 \cdot (x^2 - 1) - 2x \cdot x^3}{(x^2 - 1)^2} = \frac{3x^4 - 3x^2 - 2x^4}{(x^2 - 1)^2} =$$

$$= \frac{x^4 - 3x^2}{(x^2 - 1)^2} = \frac{x^2 \cdot \left(x^2 - 3\right)}{(x^2 - 1)^2} = \frac{x^2 \cdot \left(x - \sqrt{3}\right) \cdot \left(x + \sqrt{3}\right)}{(x^2 - 1)^2}.$$

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x=sym("x")
Symbolic pkg v2.9.0: Python communication link active, SymPy v1.5.1.
x = (sym) x
octave:4> y=x^3/(x^2-1)
y = (sym)

    3
   x
 ──────
  2
 x  - 1

octave:6> df=diff(y,x)
df = (sym)

         4          2
      2⋅x        3⋅x
 - ────────── + ──────
            2    2
   ⎛ 2     ⎞    x  - 1
   ⎝x  - 1⎠
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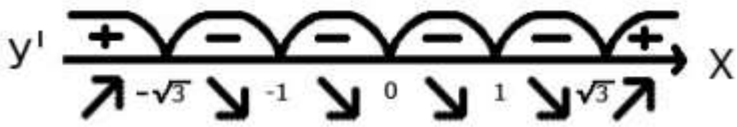
Let's find the critical points: $y' = 0$ or $y' = \dfrac{x^2 \cdot \left(x - \sqrt{3}\right) \cdot \left(x + \sqrt{3}\right)}{(x^2 - 1)^2}$

or $x^2\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right) = 0$ and $x^2 - 1 \neq 0$ or $x = 0$, $x = -\sqrt{3}$, $x = \sqrt{3}$, $x \neq -1$, $x \neq 1$.

Let's determine the intervals of monotonicity.

These points divide a range into 6 intervals, on each of which the first derivative keeps its sign.



If $x \in \left(-\infty; -\sqrt{3}\right) \cup \left(\sqrt{3}; \infty\right)$ then $y' > 0$, i.e. the function increases.

If $x \in \left(-\sqrt{3}; -1\right) \cup (-1; 0) \cup (0; 1) \cup \left(1; \sqrt{3}\right)$ then $y' < 0$, i.e. the function decreases.

Hence, the point $x = -\sqrt{3}$ is maximum and the point $x = \sqrt{3}$ is minimum. The values of the functions at these points are equal to

$$y_{\max}\left(-\sqrt{3}\right) = \frac{\left(-\sqrt{3}\right)^3}{\left(-\sqrt{3}\right)^2 - 1} = -\frac{3\sqrt{3}}{3-1} = -\frac{3\sqrt{3}}{2} \text{ and } y_{\min}\left(\sqrt{3}\right) = \frac{\left(\sqrt{3}\right)^3}{\left(\sqrt{3}\right)^2 - 1} = \frac{3\sqrt{3}}{3-1} = \frac{3\sqrt{3}}{2}.$$

7. To define the intervals of concavity up (or down) and the inflection points we find the second derivative:

$$y'' = \left(y'\right)' = \left(\frac{x^4 - 3x^2}{(x^2-1)^2}\right)' = \frac{(4x^3 - 6x)(x^2-1)^2 - (x^4 - 3x^2)4x(x^2-1)}{(x^2-1)^4} =$$

$$= \frac{\left(x^2-1\right)\left((4x^3 - 6x)(x^2-1) - (x^4 - 3x^2)4x\right)}{(x^2-1)^4} =$$

$$= \frac{4x^5 - 4x^3 - 6x^3 + 6x - 4x^5 + 12x^3}{(x^2-1)^3} = \frac{2x^3 + 6x}{\left(x^2-1\right)^3} = \frac{2x\left(x^2+3\right)}{\left(x^2-1\right)^3}.$$

If $y'' = 0$ then $y'' = \dfrac{2x(x^2+3)}{(x^2-1)^3} = 0$ or $x = 0$, $x^2 + 3 \neq 0$ $x \neq -1$, $x \neq 1$.

We obtain that at $x \in (-1; 0) \cup (1; +\infty)$ the graph is concave up, at $x \in (-\infty; -1) \cup (0; 1)$ the graph is concave down.

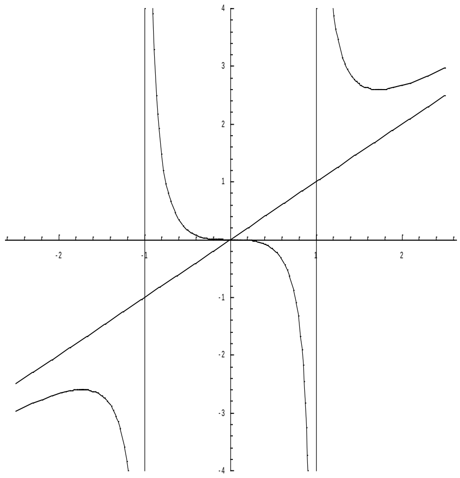
The point with coordinates $x = 0$ and $y = \dfrac{0^3}{0^2 - 1} = 0$, i.e. the origin, is a point of the inflection.

8. Determine the points of intersection of the graph with the coordinate axes:

with the x-axis: $y=0$ and $\frac{x^3}{x^2-1}=0$ or $x^3=0$ or $x=0$; with the y-axis: $x=0$ and $y=\frac{0^3}{0^2-1}=0$. This point is the origin $O(0;0)$.

9. Test the function on infinity: $\lim\limits_{x\to\pm\infty}\frac{x^3}{x^2-1}=\pm\infty$.

10. Let's plot a graph of the function.



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octave:2> L1=ezplot('x^3/(x^2-1)');set(L1,'LineWidth',3,'Color','k')
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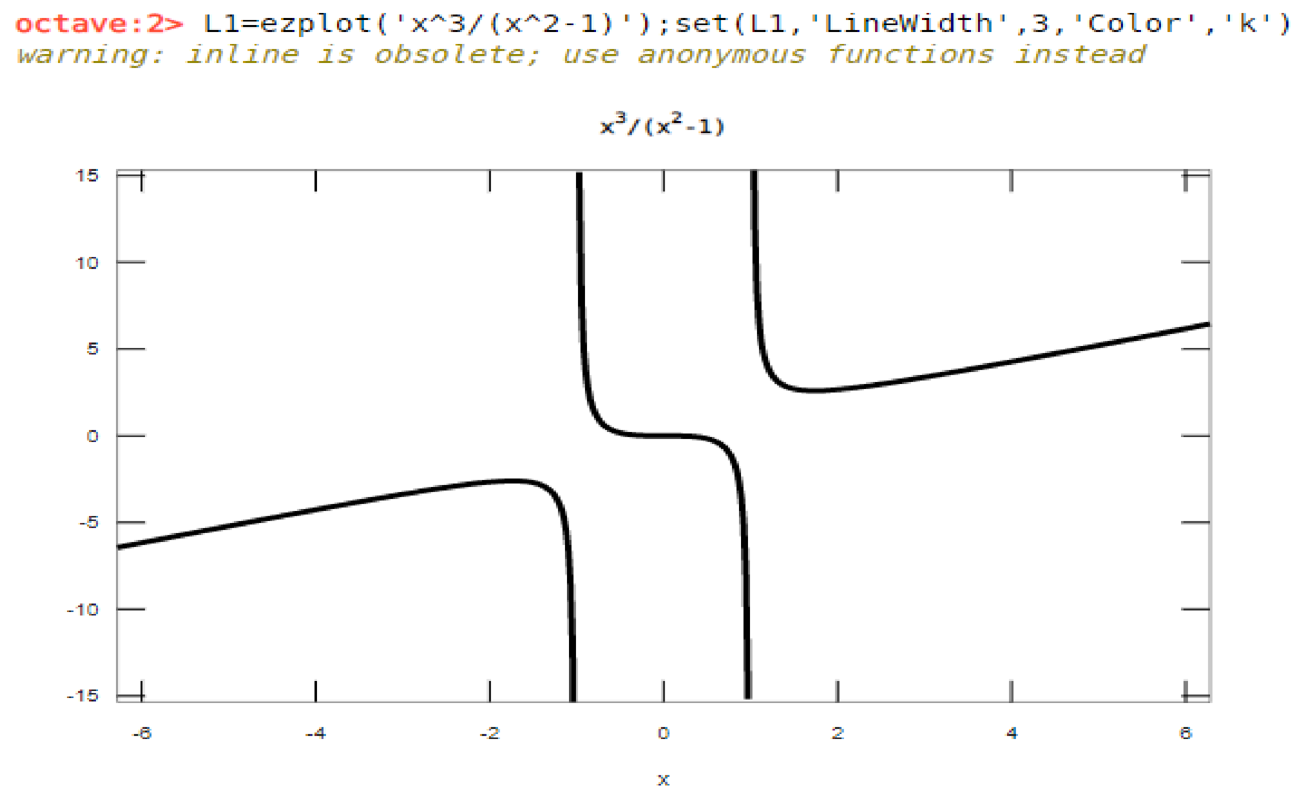

Example 2. Find the greatest and the least values of the function $f(x)=x^3-3x^2+1$ on the interval $[-1,4]$.

Solution. Let's find all the critical points using the condition $f'(x)=0$.

We have the derivative: $f'(x)=\left(x^3-3x^2+1\right)'=3x^2-6x$.

Let's equate it to 0 and find the critical points:

$$f'(x)=0 \text{ or } 3x^2-6x=0 \text{ or } 3x(x-2)=0.$$

We obtain $3x(x-2)=0$ at $x=0$ and $x=2$.

Let's define all the critical points, belonging to the given interval $[-1,4]$.

Thus, the given function has two stationary points $x_1=0$ and $x_2=2$ inside the interval $[-1,4]$.

Let's calculate the function values at these points and on the borders of the interval:

$f(0)=0^3-3\cdot 0^2+1=1$, $f(2)=2^3-3\cdot 2^2+1=8-12+1=-3$,

$f(-1)=(-1)^3-3\cdot(-1)^2+1=-1-3+1=-3$, $f(4)=4^3-3\cdot 4^2+1=64-48+1=17$.

As we see, the function takes the greatest value on the right border of the interval $[-1,4]$ and the least value is taken at the internal point $x=2$ and on the left border of the interval, i.e. $f_{least}=f(2)=f(-1)=-3$, $f_{greatest}=f(4)=17$.