Practice 1. Matrices and determinants

Exercise 1. Two matrices are given: $A=\begin{pmatrix}3 & -5 & 4\\ -1 & 2 & 0\\ 2 & -1 & 5\end{pmatrix}$, $B=\begin{pmatrix}2 & 0 & 1\\ 3 & -1 & 2\\ 1 & 1 & -1\end{pmatrix}$.

Find the matrix $C=2(B-A)-4A$.

Solution. $B-A=\begin{pmatrix}2-3 & 0+5 & 1-4\\ 3+1 & -1-2 & 2-0\\ 1-2 & 1+1 & -1-5\end{pmatrix}=\begin{pmatrix}-1 & 5 & -3\\ 4 & -3 & 2\\ -1 & 2 & -6\end{pmatrix}$.

$$2(B-A)=2\cdot\begin{pmatrix}-1 & 5 & -3\\ 4 & -3 & 2\\ -1 & 2 & -6\end{pmatrix}=\begin{pmatrix}2\cdot(-1) & 2\cdot 5 & 2\cdot(-3)\\ 2\cdot 4 & 2\cdot(-3) & 2\cdot 2\\ 2\cdot(-1) & 2\cdot 2 & 2\cdot(-6)\end{pmatrix}=\begin{pmatrix}-2 & 10 & -6\\ 8 & -6 & 4\\ -2 & 4 & -12\end{pmatrix}$$

$$C=\begin{pmatrix}-2 & 10 & -6\\ 8 & -6 & 4\\ -2 & 4 & -12\end{pmatrix}-4\cdot\begin{pmatrix}3 & -5 & 4\\ -1 & 2 & 0\\ 2 & -1 & 5\end{pmatrix}=\begin{pmatrix}-2 & 10 & -6\\ 8 & -6 & 4\\ -2 & 4 & -12\end{pmatrix}-\begin{pmatrix}12 & -20 & 16\\ -4 & 8 & 0\\ 8 & -4 & 20\end{pmatrix}=$$

$$=\begin{pmatrix}-2-12 & 10+20 & -6-16\\ 8+4 & -6-8 & 4-0\\ -2-8 & 4+4 & -12-20\end{pmatrix}=\begin{pmatrix}-14 & 30 & -22\\ 12 & -14 & 4\\ -10 & 8 & -32\end{pmatrix}.$$

Exercise 2. Consider the following matrices: $A=\begin{pmatrix}3 & -5 & 6\\ 2 & 0 & -1\end{pmatrix}$, $B=\begin{pmatrix}4 & 0 & -5\\ -1 & 2 & 0\\ 7 & -3 & 1\end{pmatrix}$

and $C=\begin{pmatrix}2 & 0\\ 3 & -1\\ 1 & 1\end{pmatrix}$. Define the size of the following products: AB, BA, AC, CA, BC and CB.

Solution. Define the size of the following products:

$A_{2\times3}\cdot B_{3\times3}=(AB)_{2\times3}$;

$B_{3\times3}\cdot A_{2\times3}$ - this product doesn't exist, because $3\neq 2$;

$A_{2\times3}\cdot C_{3\times2}=(AC)_{2\times2}$;

$C_{3\times2}\cdot A_{2\times3}=(CA)_{3\times3}$;

$B_{3\times3}\cdot C_{3\times2}=(BC)_{3\times2}$;

$C_{3\times 2}\cdot B_{3\times 3}$ - this product doesn't exist, because $2\neq 3$.

Find the following products:

$$CA=\begin{pmatrix}2 & 0\\ 3 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}3 & -5 & 6\\ 2 & 0 & -1\end{pmatrix}=$$

$$=\begin{pmatrix}2\cdot 3+0\cdot 2 & 2\cdot(-5)+0\cdot 0 & 2\cdot 6+0\cdot(-1)\\ 3\cdot 3+(-1)\cdot 2 & 3\cdot(-5)+(-1)\cdot 0 & 3\cdot 6+(-1)\cdot(-1)\\ 1\cdot 3+1\cdot 2 & 1\cdot(-5)+1\cdot 0 & 1\cdot 6+1\cdot(-1)\end{pmatrix}=\begin{pmatrix}6 & -10 & 12\\ 7 & -15 & 19\\ 5 & -5 & 5\end{pmatrix};$$

$$BC=\begin{pmatrix}4 & 0 & -5\\ -1 & 2 & 0\\ 7 & -3 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\ 3 & -1\\ 1 & 1\end{pmatrix}=$$

$$=\begin{pmatrix}4\cdot 2+0\cdot 3+(-5)\cdot 1 & 4\cdot 0+0\cdot(-1)+(-5)\cdot 1\\ (-1)\cdot 2+2\cdot 3+0\cdot 1 & (-1)\cdot 0+2\cdot(-1)+0\cdot 1\\ 7\cdot 2+(-3)\cdot 3+1\cdot 1 & 7\cdot 0+(-3)\cdot(-1)+1\cdot 1\end{pmatrix}=\begin{pmatrix}3 & -5\\ 4 & -2\\ 6 & 4\end{pmatrix}.$$

Exercise 3. A matrix is given: $A=\begin{pmatrix}4 & 1\\ -2 & 0\end{pmatrix}$. Find A^2 and A^3.

Solution. This operation can be applied to the square matrix, i.e. for the $n\times n$ matrix. Find A^2 and A^3. By the definition of the m-th power of A we have $A^2=\underbrace{A\cdot A}_{2\text{ copies of }A}$ and $A^3=\underbrace{A\cdot A\cdot A}_{3\text{ copies of }A}$ or $A^3=A^2\cdot A=A\cdot A^2$.

We obtain:

$$A^2=A\cdot A=\begin{pmatrix}4 & 1\\ -2 & 0\end{pmatrix}\cdot\begin{pmatrix}4 & 1\\ -2 & 0\end{pmatrix}=\begin{pmatrix}4\cdot 4+1\cdot(-2) & 4\cdot 1+1\cdot 0\\ -2\cdot 4+0\cdot(-2) & -2\cdot 1+0\cdot 0\end{pmatrix}=\begin{pmatrix}14 & 4\\ -8 & -2\end{pmatrix}.$$

The matrix A^3 can be obtained by another way:

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$$A^2=\begin{pmatrix}14 & 4\\ -8 & -2\end{pmatrix}\quad\begin{pmatrix}48 & 14\\ -28 & -8\end{pmatrix}=A^3$$

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Exercise 3. Consider the matrix $A=\begin{pmatrix} -2 & 0 \\ 3 & -1 \\ 6 & 4 \end{pmatrix}$. We obtain $A^T=\begin{pmatrix} -2 & 3 & 6 \\ 0 & -1 & 4 \end{pmatrix}$. The initial matrix has the size 3×2, then the matrix A^T is a 2×3 matrix.

Calculation of an inverse matrix by cofactors

Exercise 9. Find the inverse matrix for the following matrix $A=\begin{pmatrix} 3 & -5 & 4 \\ -1 & 2 & 0 \\ 2 & -1 & 5 \end{pmatrix}$.

Solution. Let us find the determinant of the given matrix:

$$|A|=\begin{vmatrix} 3 & -5 & 4 \\ -1 & 2 & 0 \\ 2 & -1 & 5 \end{vmatrix}=3\cdot 2\cdot 5+(-5)\cdot 2\cdot 0+(-1)\cdot(-1)\cdot 4-2\cdot 2\cdot 4-3\cdot(-1)\cdot 0-$$
$$-(-1)\cdot(-5)\cdot 5=30+0+4-16-0-25=-7\neq 0.$$

Its determinant is non-zero.

Finding inverses by cofactors.

Find the cofactors for the elements of the matrix A:

$A_{11}=(-1)^{1+1}\cdot\begin{vmatrix} 2 & 0 \\ -1 & 5 \end{vmatrix}=1\cdot(10-0)=10$, $A_{12}=(-1)^{1+2}\cdot\begin{vmatrix} -1 & 0 \\ 2 & 5 \end{vmatrix}=(-1)\cdot(-5-0)=5$,

$A_{13}=(-1)^{1+3}\cdot\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix}=1\cdot(1-4)=-3$,

$A_{21}=(-1)^{2+1}\cdot\begin{vmatrix} -5 & 4 \\ -1 & 5 \end{vmatrix}=(-1)\cdot(-25+4)=21$,

$A_{22}=(-1)^{2+2}\cdot\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}=1\cdot(15-8)=7$,

$A_{23}=(-1)^{2+3}\cdot\begin{vmatrix} 3 & -5 \\ 2 & -1 \end{vmatrix}=(-1)\cdot(-3+10)=-7$,

$A_{31}=(-1)^{3+1}\cdot\begin{vmatrix} -5 & 4 \\ 2 & 0 \end{vmatrix}=1\cdot(0-8)=-8$,

$A_{32}=(-1)^{3+2}\cdot\begin{vmatrix} 3 & 4 \\ -1 & 0 \end{vmatrix}=(-1)\cdot(0+4)=-4$,

$$A_{33}=(-1)^{3+3}\cdot\begin{vmatrix}3 & -5\\ -1 & 2\end{vmatrix}=1\cdot(6-5)=1.$$

We obtain the inverse matrix:

$$A^{-1}=\frac{1}{-7}\begin{pmatrix}10 & 21 & -8\\ 5 & 7 & -4\\ -3 & -7 & 1\end{pmatrix}=\begin{pmatrix}-\frac{10}{7} & -3 & \frac{8}{7}\\ -\frac{5}{7} & -1 & \frac{4}{7}\\ \frac{3}{7} & 1 & -\frac{1}{7}\end{pmatrix}.$$

Checking the condition $A\cdot A^{-1}=E$:

$$A\cdot A^{-1}=\begin{pmatrix}3 & -5 & 4\\ -1 & 2 & 0\\ 2 & -1 & 5\end{pmatrix}\cdot\begin{pmatrix}-\frac{10}{7} & -3 & \frac{8}{7}\\ -\frac{5}{7} & -1 & \frac{4}{7}\\ \frac{3}{7} & 1 & -\frac{1}{7}\end{pmatrix}=$$

$$=\begin{pmatrix}-\frac{30}{7}+\frac{25}{7}+\frac{12}{7} & -9+5+4 & \frac{24}{7}-\frac{20}{7}-\frac{4}{7}\\ \frac{10}{7}-\frac{10}{7}+0 & 3-2+0 & -\frac{8}{7}+\frac{8}{7}+0\\ -\frac{20}{7}+\frac{5}{7}+\frac{15}{7} & -6+1+5 & \frac{16}{7}-\frac{4}{7}-\frac{5}{7}\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E.$$