CALCULATION OF THE AREA OF A PLANE FIGURE

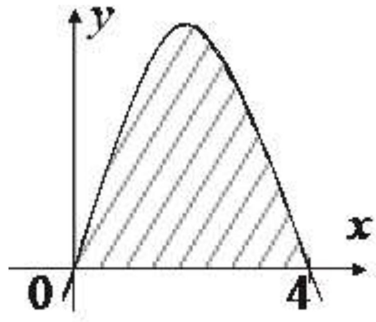
EXAMPLE 1. Calculate the area of the figure bounded by the parabola $y = 4x - x^2$ and the x-axis.

Solution. Find the limits of integration, i.e. the points of intersections of the given curve and x-axis ($y = 0$):

$$4x - x^2 = 0 \quad \text{or} \quad x(4 - x) = 0,$$

thus $x = 0,\ \ x = 4.$

Let's calculate the area of the figure by the **formula 1:**



$$S = \int_0^4 \left(4x - x^2\right)dx = 4\int_0^4 x dx - \int_0^4 x^2 dx = 2x^2\Big|_0^4 - \frac{x^3}{3}\Big|_0^4 =$$

$$= 32 - \frac{64}{3} = \frac{32}{3}.$$

EXAMPLE 2. Calculate the area of the figure bounded by curves $y = x^2 - 2,\ y = x.$

Solution. Find the limits of integration, i.e. the intersection points of given curves:

$$\begin{cases} y = x^2 - 2, \\ y = x \end{cases}$$

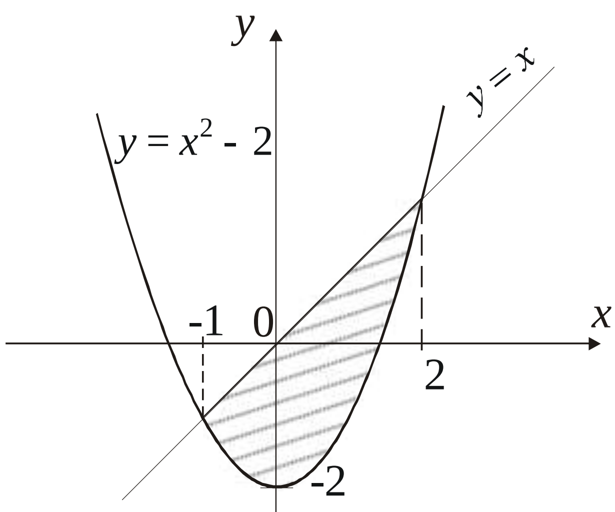
Thus $x^2 - 2 = x$ or $x^2 - x - 2 = 0.$

Find the roots: $x = \dfrac{1 \pm \sqrt{1+8}}{2}$,

$x_1 = -1,\ \ x_2 = 2; \qquad y_1 = -1,\ \ y_2 = 2.$

On this segment $[-1, 2]$ $x \geq x^2 - 2,$

thus $f_2(x) = x,\ f_1(x) = x^2 - 2.$



Since $f_2(x) \geq f_1(x)$ then the area, bounded by these curves is defined by the **formula 3**:

$$S = \int_{-1}^{2}\left(x - \left(x^2 - 2\right)\right)dx = \int_{-1}^{2}\left(x - x^2 + 2\right)dx = \frac{x^2}{2}\Big|_{-1}^{2} - \frac{x^3}{3}\Big|_{-1}^{2} +$$

$$+ 2x\Big|_{-1}^{2} = \frac{1}{2}\left(4 - (-1)^2\right) - \frac{1}{3}\left(2^3 - (-1)^3\right) + 2\left(2 - (-1)\right) =$$

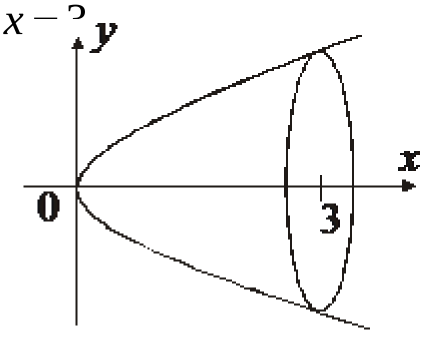
$$= \frac{3}{2} - 3 + 6 = \frac{9}{2}.$$

TASKS. FIND AREAS OF FIGURES, BOUNDED BY LINES

1 $y=2x-x^2,\ x+y=0.$	*Answer*: 4,5.
2 $y=\sqrt{x},\ x=1,\ x=4$	*Answer*: 14/3.
3 $y=x^2+2,\ y=0,\ x=-2,\ x=1$	*Answer*: 9.
4 $y=x^2,\ x=1,\ x=3$	*Answer*: 26/3.
5 $y=\frac{5}{x},\ x+y=6$	*Answer*: 12-5ln5.
6 $y=-x^2+4x,\ y=x.$	*Answer*: 4,5.
7 $y=-x^2+4x,\ y=0.$	*Answer*: 32/3.
8 $y=-x^2+4x,\ y=3x.$	*Answer*: 20 5/6.
9 $y=\frac{1}{x},\ x=1,\ x=3$	*Answer*: ln3.
15 $y=6x-x^2,\ y=0.$	*Answer*: 36.
16 $y=x^2+4x,\ x-y+4=0.$	*Answer*: $\frac{125}{6}.$
17 $y=3+2x-x^2, y=x+1.$	*Answer*: $4,5.$
18 $y=ln\,x, y=0, x=e.$	*Answer*: 1.
19 $y=\frac{1}{x}, y=x, x=2, y=0.$	*Answer*: $\frac{1}{2}+ln\,2.$
20 $y=x^2+2, y=1-x^2, x=0, x=1.$	*Answer*: $\frac{5}{3}.$
21 $y=\sqrt{x}, y=\sqrt{4-3x}, y=0.$	*Answer*: $\frac{8}{9}.$
22 $y=x^2+3, xy=4, y=2, x=0.$	*Answer*: $4\,ln\,2-\frac{2}{3}.$
23 $y^2=x^3, x=0, y=4.$	*Answer*: $\frac{24}{5}\cdot\sqrt[3]{2}.$
24 $xy=6, y=7-x.$	*Answer*: $6,76.$
25 $y=x^3, y=x, y=2x.$	*Answer*: $1,5.$
26 $y=sin\,x, y=0, x\in[0;\pi].$	*Answer*: 2.
27 $y=x^3, y=8, x=0.$	*Answer*: 12.

VOLUME OF ROTATION BODY

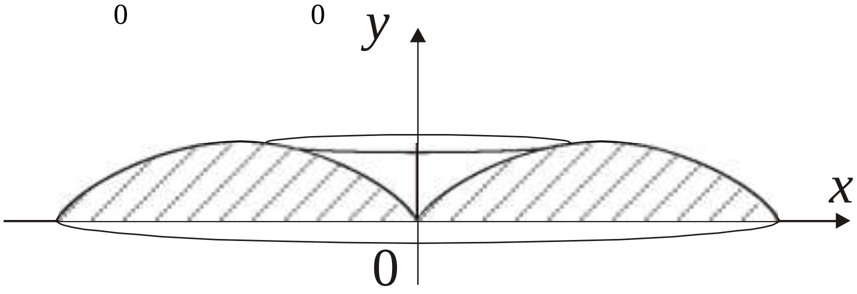
EXAMPLE 1. Find the rotation body, formed by rotation round the axis Ox the curvilinear trapezoid, bounded by the parabola $y^2 = 2x$ and the straight line $x = 3$.



Solution.
Let's find limits of integration: $a = 0$, $b = 3$.
Let's calculate:

$$V_x = \pi \int_0^3 y^2 dx = \pi \int_0^3 2x dx = \pi \cdot 2 \frac{x^2}{2} \Big|_0^3 = 9\pi .$$

EXAMPLE 2. Find the rotation body, formed by rotation round the axis Oy of the figure, bounded by $y = \sin x$, $y = 0$, $0 \le x \le \pi$.
Solution.
Let's calculate: $V_y = 2\pi \int_0^\pi xy dx = 2\pi \int_0^\pi x \sin x dx$.



Let's use the method of integration by parts:
Let $u = x$, $dv = \sin x dx$, then $du = dx$, $v = -\cos x$.
Let's get:

$$V_y = 2\pi \left(x(-\cos x)\Big|_0^\pi - \int_0^\pi (-\cos x) dx \right) =$$

$$= 2\pi \left(\pi + \sin x \Big|_0^\pi \right) = 2\pi^2 .$$

TASKS

Find volumes of rotation bodies, formed by rotation of the figure, bounded by lines:

1	$y = \sin x,\ \ y = 0,\ \ 0 \le x \le \pi$ round the axis Ox	*Answer*: $\frac{\pi^2}{2}$.
2	$y^2 + x - 4 = 0,\ \ x = 0$ round the axis Oy	*Answer*: $\frac{512}{15}\pi$.
3	$xy = 4,\ \ y = 0,\ \ x = 1,\ \ x = 4$ round the axis Ox.	*Answer*: 12π.
4	$y = -x^2 + 2x,\ \ y = 0$ round the axis Oy.	*Answer*: $\frac{8}{3}\pi$.
5	$y = x^2 + 1,\ y = 0,\ x = 1\ \ x = 2$ round the axis Ox, round the axis Oy.	*Answer*: $\frac{178}{15}\pi$, $\frac{21\pi}{2}$.
6	$y = x^2,\ x = y^2$ round the axis Ox, round the axis Oy	*Answer*: $\frac{3}{10}\pi$, $\frac{3}{10}\pi$.