Theme 9. The general theory of the system of linear algebraic equations

Lecture plan

1. Definition of system of linear algebraic equations, the augmented and matrix forms of its entry, a solution, consistent or inconsistent, determined or undetermined system
2. Solution of system of equations using Cramer method
3. Solution of system of equations using an inverse matrix
4. Solution of system of equations using Jordan–Gauss method
5. Investigation of the system compatibility with the help of . Kronecker-Capelli theorem

1. Definition of system of linear algebraic equations, the augmented and matrix forms of its entry, a solution, consistent or inconsistent, determined or undetermined system

A system of m *linear algebraic equations with* n *unknowns* has the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases}, \quad (1)$$

where $a_{11}, a_{12}, ..., a_{mn}$ are *the coefficients of the system*; $b_1, b_2, ..., b_m$ are *its free terms* and $x_1, x_2, ..., x_n$ are *the unknowns*.

A solution of the system (1) is a set of n numbers $x_1, x_2, ..., x_n$ satisfying every equation of system.

Every system of the form (1) has either no solution, one solution or infinitely many solutions.

A system is *consistent or compatible* if there exists at least one solution, otherwise it is *inconsistent or incompatible*.

A compatible system is called *definite or determined* if it has the only solution.

A compatible system is called *indefinite or undetermined* if it has more than one various solutions.

Systems are *equivalent* if they have the same solution set.

If $b_j = 0$ for all *j* the system (1) is *homogeneous*.

Note that the system (1) can be written in a matrix form as $AX = B$, where the $m \times n$-matrix $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ is *the matrix of coefficients or the basic matrix*, the matrix-column $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is *a matrix-column of free terms* and the matrix-column $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is *the unknown matrix-column*.

2. Solution of system of equations using Cramer method

Let the system (1) consist of n linear equations with n unknown quantities and its determinant $\det A \neq 0$ then unknown quantities can be found accordingly to the formulas by Cramer:

$$x_i = \frac{\Delta_i}{\Delta},\ i = \overline{1, n},$$

where Δ is the determinant of the system; Δ_i is determinant obtained from the determinant of the system by substituting the column i by the matrix-column B:

$$\Delta_i = \begin{vmatrix} a_{11} \dots a_{1\,i-1} & b_1 & a_{1\,i+1} \dots a_{1n} \\ a_{21} \dots a_{2\,i-i} & b_2 & a_{2\,i+1} \dots a_{2n} \\ \dots & \dots & \dots \\ a_{n1} \dots a_{n\,i-1} & b_n & a_{n\,i+1} \dots a_{nn} \end{vmatrix}.$$

For example, consider the system (1) which consists of 3 linear equations with 3 unknowns $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$. Then unknowns can be found accordingly to the formulas by Cramer:

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta},\ x_3 = \frac{\Delta_3}{\Delta},$$

where $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$,

$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$.

Let's illustrate this method by example.

Example 1. Solve the given system of equations using Cramer method:

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases}.$$

Solution. Find the determinant of the given system:

$$\Delta = \begin{vmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 5\cdot 2\cdot 2 + 4\cdot 3\cdot(-1) + 1\cdot 3\cdot(-1) - 4\cdot 2\cdot(-1) - 2\cdot 1\cdot(-1) -$$

$$-3\cdot 3\cdot 5 = 20 - 12 - 3 + 8 + 2 - 45 = -30 \neq 0.$$

Its determinant is non-zero. Apply the formulas by Cramer:

$$\Delta_1 = \begin{vmatrix} 0 & -1 & -1 \\ 14 & 2 & 3 \\ 16 & 3 & 2 \end{vmatrix} = 0 - 48 - 42 + 32 - 0 + 28 = -30,\ x_1 = \frac{\Delta_1}{\Delta} = \frac{-30}{-30} = 1,$$

$$\Delta_2 = \begin{vmatrix} 5 & 0 & -1 \\ 1 & 14 & 3 \\ 4 & 16 & 2 \end{vmatrix} = 140 + 0 - 16 + 56 - 0 - 240 = -60,\ x_2 = \frac{\Delta_2}{\Delta} = \frac{-60}{-30} = 2,$$

$$\Delta_3 = \begin{vmatrix} 5 & -1 & 0 \\ 1 & 2 & 14 \\ 4 & 3 & 16 \end{vmatrix} = 160 - 56 + 0 - 0 - 210 + 16 = -90,\ x_3 = \frac{\Delta_3}{\Delta} = \frac{-90}{-30} = 3.$$

Checking by substitution x_1, x_2, x_3 into the initial system:

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases} \Rightarrow \begin{cases} 5 \cdot 1 - 2 - 3 = 0 \\ 1 + 2 \cdot 2 + 3 \cdot 3 = 14 \\ 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 = 16 \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ 14 = 14. \\ 16 = 16 \end{cases}$$

3. Solution of system of equations using an inverse matrix

Let the system (1) consist of n linear equations with n unknowns and its determinant $\det A \neq 0$. Write this system in a matrix form as

$$AX = B. \qquad (2)$$

Let us multiply both parts (2) by the inverse matrix A^{-1} on the left. Then we obtain

$$A^{-1} \cdot AX = A^{-1} \cdot B.$$

Since $A^{-1} \cdot A = E$ and $E \cdot X = X$ we can get a solution by the formula $X = A^{-1} \cdot B.$

Let's illustrate this method by example.

Example Let's find a solution of the system from example 1 by the matrix method.

Solution. Here $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 14 \\ 16 \end{pmatrix}$, $A = \begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$.

Let's calculate the determinant of the given system:

$$\Delta = \begin{vmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 5 \cdot 2 \cdot 2 + 4 \cdot 3 \cdot (-1) + 1 \cdot 3 \cdot (-1) - 4 \cdot 2 \cdot (-1) - 2 \cdot 1 \cdot (-1) -$$
$$- 3 \cdot 3 \cdot 5 = 20 - 12 - 3 + 8 + 2 - 45 = -30 \neq 0.$$

Its determinant is non-zero. Let's find the inverse matrix by cofactors:

$$A_{11} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5, \qquad A_{21} = -\begin{vmatrix} -1 & -1 \\ 3 & 2 \end{vmatrix} = -1, \qquad A_{31} = \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} = -1,$$

$A_{12}=-\begin{vmatrix}1 & 3\\ 4 & 2\end{vmatrix}=10$, $A_{22}=\begin{vmatrix}5 & -1\\ 4 & 2\end{vmatrix}=14$, $A_{32}=-\begin{vmatrix}5 & -1\\ 1 & 3\end{vmatrix}=-16$,

$A_{13}=\begin{vmatrix}1 & 2\\ 4 & 3\end{vmatrix}=-5$, $A_{23}=-\begin{vmatrix}5 & -1\\ 4 & 3\end{vmatrix}=-19$, $A_{33}=\begin{vmatrix}5 & -1\\ 1 & 2\end{vmatrix}=11$,

$$A^{-1}=\frac{1}{-30}\cdot\begin{pmatrix}-5 & -1 & -1\\ 10 & 14 & -16\\ -5 & -19 & 11\end{pmatrix}=\begin{pmatrix}\frac{5}{30} & \frac{1}{30} & \frac{1}{30}\\ -\frac{10}{30} & -\frac{14}{30} & \frac{16}{30}\\ \frac{5}{30} & \frac{19}{30} & -\frac{11}{30}\end{pmatrix}.$$

Let’s check the condition $A\cdot A^{-1}=E$:

$$A\cdot A^{-1}=\begin{pmatrix}5 & -1 & -1\\ 1 & 2 & 3\\ 4 & 3 & 2\end{pmatrix}\cdot\begin{pmatrix}\frac{5}{30} & \frac{1}{30} & \frac{1}{30}\\ -\frac{10}{30} & -\frac{14}{30} & \frac{16}{30}\\ \frac{5}{30} & \frac{19}{30} & -\frac{11}{30}\end{pmatrix}=$$

$$=\begin{pmatrix}\frac{25+10-5}{30} & \frac{5+14-19}{30} & \frac{5-16+11}{30}\\ \frac{5-20+15}{30} & \frac{1-28+57}{30} & \frac{1+32-33}{30}\\ \frac{20-30+10}{30} & \frac{4-42+38}{30} & \frac{4+48-22}{30}\end{pmatrix}=\begin{pmatrix}\frac{30}{30} & 0 & 0\\ 0 & \frac{30}{30} & 0\\ 0 & 0 & \frac{30}{30}\end{pmatrix}=$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E.$$

The solution of given system is $X=A^{-1}\cdot B$. Then

$$X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=A^{-1}\cdot B=\begin{pmatrix}\frac{5}{30}&\frac{1}{30}&\frac{1}{30}\\-\frac{10}{30}&-\frac{14}{30}&\frac{16}{30}\\\frac{5}{30}&\frac{19}{30}&-\frac{11}{30}\end{pmatrix}\cdot\begin{pmatrix}0\\14\\16\end{pmatrix}=$$

$$=\begin{pmatrix}\frac{5}{30}\cdot 0+\frac{1}{30}\cdot 14+\frac{1}{30}\cdot 16\\-\frac{10}{30}\cdot 0-\frac{14}{30}\cdot 14+\frac{16}{30}\cdot 16\\\frac{5}{30}\cdot 0+\frac{19}{30}\cdot 14-\frac{11}{30}\cdot 16\end{pmatrix}=\begin{pmatrix}\frac{0+14+16}{30}\\\frac{0-196+256}{30}\\\frac{0+266-176}{30}\end{pmatrix}=\begin{pmatrix}\frac{30}{30}\\\frac{60}{30}\\\frac{90}{30}\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}.$$

Thus $x_1=1$, $x_2=2$, $x_3=3$.

4. Solution of system of equations using Jordan–Gauss method

Definition. Matrices obtained one from another by elementary row operations are called *equivalent*. The equivalence of matrices is marked by the sign $\sim$.

Jordan–Gauss method is used to solve the system (1), which consists of *m* linear equations with *n* unknowns. This method includes sequential elimination of unknowns to following scheme.

1. Create an augmented matrix of the given system $A|B=\left(\begin{array}{cccc|c}a_{11}&a_{12}&\ldots&a_{1n}&b_1\\a_{21}&a_{22}&\ldots&a_{2n}&b_2\\\vdots&\vdots&\ldots&\vdots&\vdots\\a_{m1}&a_{m2}&\ldots&a_{mn}&b_m\end{array}\right)$. *The augmented matrix* is called an array with the matrix A on the left and the matrix-column B of free terms on the right and denoted by $A|B$. The vertical line separates the matrix-column B.

The leading row and the leading element in $A|B$ that corresponds to the choice of the leading equation and the leading unknown in the system (1) are chosen. The system should be transformed in order to let the leading equation be the first one.

The leading unknown by means of the leading equation is eliminated from the other equations. For this the certain *elementary row operations* of the matrix $A|B$ are performed it is possible:

1) to change the order of rows (that corresponds to change of the order of the equations' sequence);

2) to multiply rows by any non-zero numbers (that corresponds to multiplying the corresponding equations by these numbers);

3) to add to any row of the matrix $A|B$ its any other row multiplied by any number (that corresponds to addition to one equation of the system another equation multiplied by this number).

Due to such transformations we obtain an augmented matrix, *equivalent to the initial one* (i. e. having the same solutions).

On the second step a new leading unknown and a corresponding leading equation are chosen and then this variable is eliminated from all the other equations. The leading row in the matrix $A|B$ remains without change. After such actions the initial matrix *A* will be reduced to the triangular (1.1) or truncated-triangular form (1.2) with the elements of the main diagonal equal to 1.

Let's illustrate this method by example.

According to the method by Jordan–Gauss the leading unknown by means of the leading equation on the current step is eliminated not only from equations of the corresponding subsystem but also from the leading equations on previous steps and on any step the leading unknown has the coefficient equal to 1.

Example 3. Let's find a solution of the system from example 1 by Jordan-Gauss method.

Solution. By elementary row operations of the augmented matrix, we obtain

$$A|B=\left(\begin{array}{ccc|c}5&-1&-1&0\\1&2&3&14\\4&3&2&16\end{array}\right)\sim\left(\begin{array}{ccc|c}1&2&3&14\\5&-1&-1&0\\4&3&2&16\end{array}\right)\sim\left[\begin{array}{c}\\ [2]+[1]\cdot(-5)\\ [3]+[1]\cdot(-4)\end{array}\right]\sim$$

$$\sim\left(\begin{array}{ccc|c}1&2&3&14\\0&-5&-10&-40\\0&-11&-16&-70\end{array}\right)\sim\left[\begin{array}{c}\\ [2]:(-5)\\ \\ \end{array}\right]\sim\left(\begin{array}{ccc|c}1&2&3&14\\0&1&2&8\\0&-11&-16&-70\end{array}\right)\sim$$

$$\sim\begin{bmatrix}[1]+[2]\cdot(-2)\\ \\ [3]+[2]\cdot 11\end{bmatrix}\sim\left(\begin{array}{ccc|c}1&0&-1&-2\\0&1&2&8\\0&0&6&18\end{array}\right)\sim\begin{bmatrix}\\ \\ [3]:6\end{bmatrix}\sim\left(\begin{array}{ccc|c}1&0&-1&-2\\0&1&2&8\\0&0&1&3\end{array}\right)\sim$$

$$\sim\begin{bmatrix}[1]+[3]\\ [2]+[3]\cdot(-2)\\ \\ \end{bmatrix}\sim\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&2\\0&0&1&3\end{array}\right).$$

A unit matrix on the left of the vertical line is obtained. The column on the right of the vertical line is values of unknown quantities.

Then write down the received augmented matrix as the system of questions:

$$\begin{cases}x_1=1\\x_2=2\\x_3=3\end{cases}.$$

Thus, $x_1=1$, $x_2=2$, $x_3=3$.

Let's solve the initial system using elementary transformations with equations:

$$\begin{cases}5x_1-x_2-x_3=0\\x_1+2x_2+3x_3=14\\4x_1+3x_2+2x_3=16\end{cases}\Rightarrow\begin{cases}x_1+2x_2+3x_3=14\\5x_1-x_2-x_3=0\\4x_1+3x_2+2x_3=16\end{cases}\Rightarrow\begin{bmatrix}\\ [2]+[1]\cdot(-5)\\ [3]+[1]\cdot(-4)\end{bmatrix}\Rightarrow$$

$$\Rightarrow\begin{cases}x_1+2x_2+3x_3=14\\-5x_2-10x_3=-40\\-11x_2-16x_3=-70\end{cases}\Rightarrow\begin{bmatrix}\\ [2]:(-5)\\ \\ \end{bmatrix}\Rightarrow\begin{cases}x_1+2x_2+3x_3=14\\x_2+2x_3=8\\-11x_2-16x_3=-70\end{cases}\Rightarrow$$

$$\Rightarrow\begin{bmatrix}[1]+[2]\cdot(-2)\\ \\ [3]+[2]\cdot 11\end{bmatrix}\Rightarrow\begin{cases}x_1-x_3=-2\\x_2+2x_3=8\\6x_3=18\end{cases}\Rightarrow\begin{bmatrix}\\ \\ [3]:6\end{bmatrix}\Rightarrow$$

$$\Rightarrow\begin{cases}x_1-x_3=-2\\x_2+2x_3=8\\x_3=3\end{cases}\Rightarrow\begin{bmatrix}[1]+[3]\\ [2]+[3]\cdot(-2)\\ \\ \end{bmatrix}\Rightarrow\begin{cases}x_1=1\\x_2=2\\x_3=3\end{cases}$$

Thus, we obtain the same answer: $x_1=1$, $x_2=2$, $x_3=3$.

5. Investigation of the system compatibility

Kronecker–Capelli theorem. *1.* A linear system (1) is consistent if its basic matrix and its augmented matrix have the same rank, i. e. $rang\ A = rang\ A|B$.

A consistent system is determined if the ranks are equal to the unknowns number, i. e. $rang\ A = rang\ A|B = n$.

A consistent system is undetermined if the ranks are less than the unknowns number, i. e. $rang\ A = rang\ A|B < n$.

3. A linear system is inconsistent if its basic matrix and its augmented matrix have the different rank, i. e. $rang\ A \neq rang\ A|B$.

If $rang\ A = rang\ A|B = n$, then carrying out the backward way we obtain the corresponding values of unknowns.

If $rang\ A = rang\ A|B = r < n$, then we should choose *the main (basic)* unknowns, i. e. those ones which coefficients generate the unit matrix. The basic variables are remained on the left, and other $n-r$ variables are transposed to the right parts of equations. The variables placed on the right part of the system are called *free variables.* The basic variables are expressed through free ones using the backward way. The obtained equalities are the *general solution of the system.*

Assigning to free variables any numeric values, we can find corresponding values of the basic variables. Thus we can find the *particular solutions* of the initial system of equations.

If free variables are assigned zero value, then the obtained particular solution is called *basic*.

If the values of the basic variables are not negative, then the solution is called *supporting*.

Investigation of the system compatibility is carried out using Gauss method or Jordan–Gauss method.

Example 4. Investigate the compatibility of the given system:

$$\begin{cases} x_1 + 2x_2 - 3x_3 + 4x_4 = 7 \\ 2x_1 + 4x_2 + 5x_3 - x_4 = 2 \\ 5x_1 + 10x_2 + 7x_3 + 2x_4 = 11 \end{cases}.$$

Solution. By elementary row operations of the augmented matrix, we obtain:

$$A|B=\left(\begin{array}{cccc|c}1&2&-3&4&7\\2&4&5&-1&2\\5&10&7&2&11\end{array}\right)\sim\begin{bmatrix}\\ [2]+[1]\cdot(-2)\\ [3]+[1]\cdot(-5)\end{bmatrix}\sim\left(\begin{array}{cccc|c}1&2&-3&4&7\\0&0&11&-9&-12\\0&0&22&-18&-24\end{array}\right)\sim$$

$$\sim\begin{bmatrix}\\ [2]:11\\ \\ \end{bmatrix}\sim\left(\begin{array}{cccc|c}1&2&-3&4&7\\0&0&1&-9/11&-12/11\\0&0&22&-18&-24\end{array}\right)\sim\begin{bmatrix}\\ \\ [3]+[2]\cdot(-22)\end{bmatrix}\sim$$

$$\sim\left(\begin{array}{cccc|c}1&2&-3&4&7\\0&0&1&-9/11&-12/11\\0&0&0&0&0\end{array}\right)\sim\left(\begin{array}{cccc|c}1&2&-3&4&7\\0&0&1&-9/11&-12/11\end{array}\right)\sim\begin{bmatrix}[1]+[2]\cdot 3\\ \\ \end{bmatrix}\sim$$

$$\sim\left(\begin{array}{cccc|c}1&2&0&4&7\\0&0&1&-9/11&-12/11\end{array}\right)\sim\begin{bmatrix}[1]+[2]\cdot 3\\ \\ \end{bmatrix}\sim\left(\begin{array}{cccc|c}1&2&0&17/11&41/11\\0&0&1&-9/11&-12/11\end{array}\right).$$

The initial system is equivalent to the following system of equations:

$$\begin{cases}x_1+2x_2+\dfrac{17}{11}x_4=\dfrac{41}{11}\\ x_3-\dfrac{9}{11}x_4=-\dfrac{12}{11}\end{cases}.$$

Let’s obtain the general solution:

$$\begin{cases}x_1=\dfrac{41}{11}-2x_2-\dfrac{17}{11}x_4\\ x_3=-\dfrac{12}{11}+\dfrac{9}{11}x_4\end{cases},$$

where x_1, x_3 are basic unknowns, x_2, x_4 are free ones.

For example, obtain the particular solution, if $x_2=1, x_4=-1$:

$$x_1=\frac{41}{11}-2+\frac{17}{11}\text{ or }x_1=\frac{36}{11},\qquad x_3=-\frac{12}{11}-\frac{9}{11}\text{ or }x_3=-\frac{21}{11}.$$

Thus $x_1=\dfrac{36}{11}$, $x_2=1$, $x_3=-\dfrac{21}{11}$, $x_4=-1$ are the particular solution.

For example, obtain the basic solution, if $x_2=0, x_4=0$:

$$x_1=\frac{41}{11},\qquad x_3=-\frac{12}{11}.$$

Thus $x_1 = \frac{41}{11}$, $x_2 = 0$, $x_3 = -\frac{12}{11}$, $x_4 = 0$ are the basic solution.

In this example the basic solution is not the supporting one, because $x_3 = -\frac{12}{11} < 0$.

Theoretical questions

1. Give a definition of system of linear algebraic equations.
2. Give a definition of the augmented and matrix forms of its entry.
3. What do you call a solution of system of linear algebraic equations?
4. What system is called consistent?
5. What system is called inconsistent?
6. What system is called determined?
7. What system is called undetermined?
8. What methods are used for finding a solution of system?
9. What system are called equivalent transformations?
10. Write formulas by Cramer. What case are they used?
11. What is Gauss method?
12. What is Jordan-Gauss method?
13. What is matrix method?
14. Formulate Kronecker–Capelli theorem.
15. What solution is called general?
16. What solution is called particular?
17. What solution is called basic?
18. What solution is called supporting?