Theme 11. Elements of analytical geometry

Lecture plan

1. An equation of a straight line on a plane.
2. Geometric relationship of two straight lines. Distance from the point to a given straight line.

1. An equation of a straight line on a plane.

In the early XVII century, the French mathematician Rene Descartes the idea of a grid or system for location and plotting points on the plane.

Onto this grid, called a Cartesian graph two straights are drawn at right angle; another, called the rectangular axes, we must determine:

1) starting point or origin 0;
2) the positive direction of axes;
3) unit of dimension or scale.

The position of any point (say, A) can then be described by two numbers: 'referring to the horizontal axis X and one to the vertical axis Y. The two are called the Cartesian coordinates, after Descartes.

1) Let's take the point $M_0(x_0, y_0)$ and pass the line l through it. After this we choose any point $M(x, y)$ on this line and plot the vector $\overline{n}=(A;B)$ which is perpendicular to this line l

Let's obtain the vector $\overline{M_0M}$:

$$\overline{M_0M} = (x - x_0; y - y_0).$$

From this picture we have that the vector $\overline{n}=(A;B)$ and the vector $\overline{M_0M}$ are perpendicular, i.e. their scalar product equals 0.

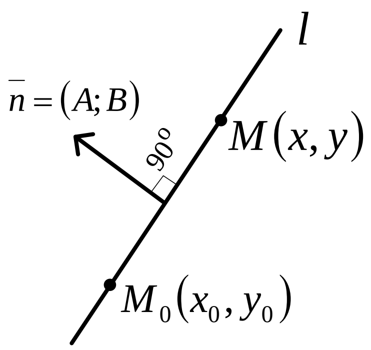
$$\overline{n} \cdot \overline{M_0M} = 0$$

or

$$\overline{n} \cdot \overline{M_0M} = A \cdot (x - x_0) + B \cdot (y - y_0) = 0$$

or

$$A \cdot (x - x_0) + B \cdot (y - y_0) = 0.$$



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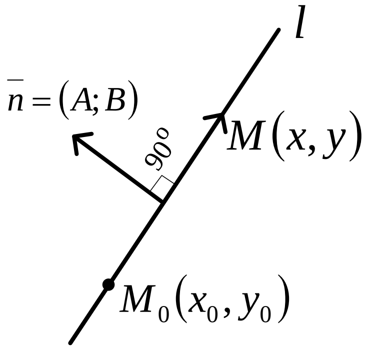


This equation $A\cdot(x-x_0)+B\cdot(y-y_0)=0$ *corresponds to the straight line passing through the point* $M_0(x_0,y_0)$ *with the given normal (perpendicular) vector* $\bar{n}=(A;B)$ *to this straight line* (formula 1).

2) Let's remove the brackets of this equation:

$$A\cdot(x-x_0)+B\cdot(y-y_0)=0$$

or

$$A\cdot x-A\cdot x_0+B\cdot y-B\cdot y_0=0$$

or

$$A\cdot x+B\cdot y\underbrace{-A\cdot x_0-B\cdot y_0}_{C}=0.$$

Let's denote the difference as C and obtain:

$$Ax+By+C=0.$$

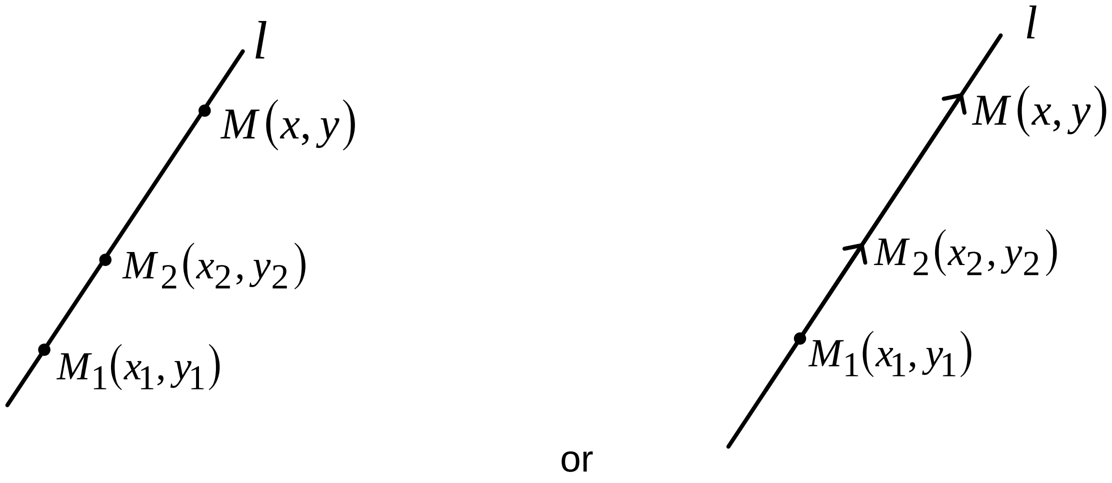
This is a general equation of the straight line (formula 2), if $A^2+B^2\neq 0$ (i.e. $|\bar{n}|\neq 0$ or $\bar{n}=\sqrt{A^2+B^2}\neq 0$).

Definition. *The straight line* is a set of points which coordinates in some system of the Cartesian coordinates satisfy the following equation:

$$Ax+By+C=0\text{, where } A^2+B^2\neq 0.$$

3) Let's pass a straight line through two points.

We take two points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ on the straight line l and choose any point $M(x, y)$.



Let's obtain the vector $\overline{M_1M}$ and the vector $\overline{M_1M_2}$:

$$\overline{M_1M} = (x - x_1; y - y_1)$$

$$\overline{M_1M_2} = (x_2 - x_1; y_2 - y_1).$$

Both vectors lie on the straight line l and it means that $\overline{M_1M} \parallel \overline{M_1M_2}$.

If two vectors are collinear then their vector product is equal to zero, i.e. their coordinates are proportional or equal:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

The straight line equation passing through two points $M_1(x_1, y_1)$ *and* $M_2(x_2, y_2)$ *is presented as follows* (formula 3)*:*

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

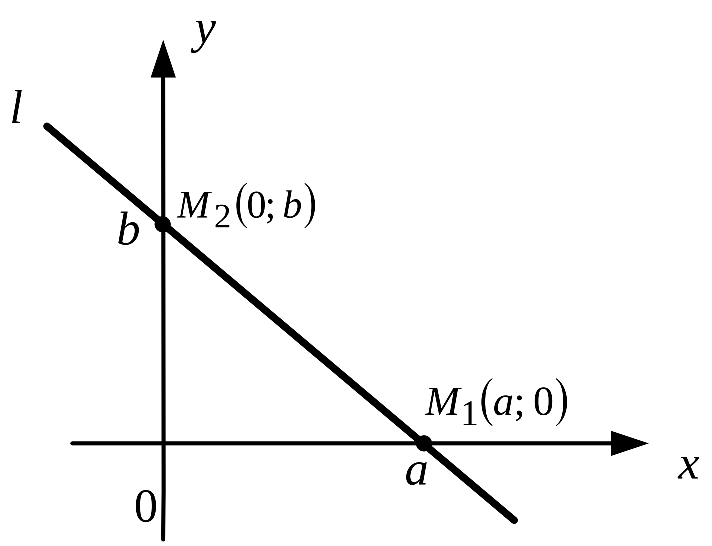
If $x_2 = x_1$, then straight line equation is presented as $x = x_1$.

If $y_2 = y_1$, then the straight line equation is presented as follows $y = y_1$.

An angular coefficient of this straight line is calculated according to the formula $k = \frac{x_2 - x_1}{y_2 - y_1}$.

4) Let's obtain the equation of a straight line with the given intercepts on the axes.

We plot the straight line in Cartesian system of coordinates:



This straight line intersects the x-axis at the point $M_1(a; 0)$ and the y-axis at the point $M_2(0; b)$, i.e. this line passes through two points M_1 and M_2.

Let's substitute coordinates of these points into the previous formula:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Here $x_1 = a$, $y_1 = 0$, $x_2 = 0$, $y_2 = b$. We obtain:

$$\frac{y - 0}{b - 0} = \frac{x - a}{0 - a}$$

or

$$\frac{y}{b} = \frac{x-a}{-a}$$

or

$$\frac{y}{b} = \frac{x}{-a} + \frac{-a}{-a}$$

or

$$\frac{y}{b} = -\frac{x}{a} + 1$$

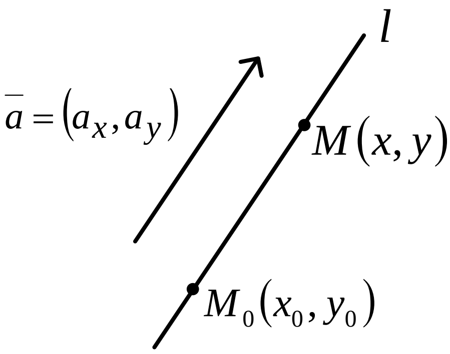
or

$$\frac{x}{a} + \frac{y}{b} = 1.$$

This equation is called the equation of a straight line with the given intercepts on the axes, where a, b are lengths of intercepts cut by a straight line from the coordinate axes OX and OY accordingly.

5) Let the straight line l passing through the point $M_0(x_0, y_0)$ and the vector $\bar{a} = (a_x, a_y)$ parallel to this line be given.

We take any point $M(x, y)$ on this line l and plot the vector $\overline{M_0M}$.



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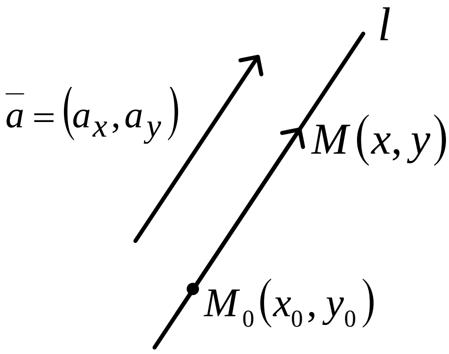


Let's obtain the vector $\overline{M_0M}$:

$$\overline{M_0M} = (x - x_0; y - y_0).$$

From this picture we have that the vector $\overline{M_0M}$ and the vector $\bar{a} = (a_x, a_y)$ are parallel $(\overline{M_0M} \parallel \bar{a})$, i.e. their vector product is equal to zero, i.e. their coordinates are proportional or equal:

$$\overline{M_0M} \times \bar{a} = \bar{0}$$

If two vectors are collinear then

$$\frac{y - y_0}{a_y} = \frac{x - x_0}{a_x}.$$

The straight line is defined by the point $M_0(x_0, y_0)$ and by the vector $\bar{a} = (a_x, a_y)$ collinear to the line which is called the directing vector of the straight line.

6) Let’s obtain a parametric equation of a straight line.

We introduce an arbitrary parameter t into an equation of a straight line l passing through the point $M_0(x_0, y_0)$ and the vector $\bar{a} = (a_x, a_y)$ parallel to this line as a coefficient of proportionality:

$$\frac{y - y_0}{a_y} = \frac{x - x_0}{a_x} = t.$$

Let’s transform this expression as

$$\frac{y - y_0}{a_y} = t \quad \text{and} \quad \frac{x - x_0}{a_x} = t.$$

We express *x* and *y*:

$$y = a_y t + y_0 \quad \text{and} \quad x = a_x t + x_0.$$

The obtained expressions are written as

$$\begin{cases} y = a_y t + y_0 \\ x = a_x t + x_0 \end{cases}.$$

Such a form is called a parametric equation of a straight line with a parameter t.

7) Let's obtain the straight line equation passing through the point $M_0(x_0, y_0)$ with the given angular coefficient k.

We plot this straight line and the point $M_0(x_0, y_0)$ belonging to it.

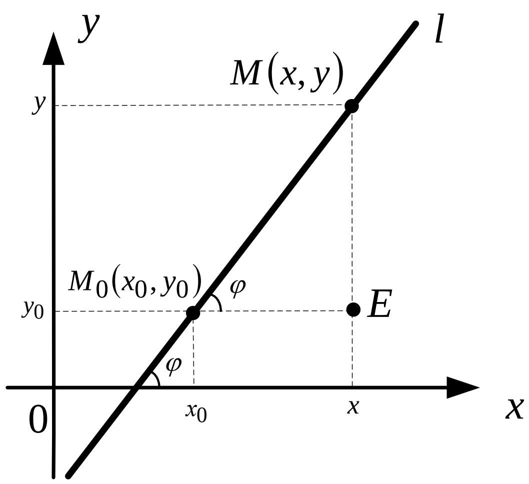
Let's look at this picture. We have the right triangle M_0EM with the right angle $\angle E = 90^{\circ}$.

Let's find $tg\varphi$:

$$tg\varphi = \frac{ME}{M_0E} = \frac{opposite \quad leg}{adjcent \quad leg}$$

or

$$tg\varphi = \frac{ME}{M_0E} = \frac{y - y_0}{x - x_0}.$$



Let's denote $tg\varphi$ as the coefficient k:

$$\frac{y-y_0}{x-x_0}=tg\varphi=k$$

or

$$\frac{y-y_0}{x-x_0}=k$$

or

$$y-y_0=k\cdot(x-x_0).$$

The straight line equation passing through the point $M_0(x_0,y_0)$ with the given angular coefficient k is presented as follows:

$$y-y_0=k(x-x_0),$$

where k is an angular coefficient $k=tg\varphi$, φ is an angle of slope between the given line and the positive direction of the axis *OX*, $0\le\varphi\le\pi$.

8) Let us transform the previous equation and remove the brackets:

$$y-y_0=k\cdot(x-x_0)$$

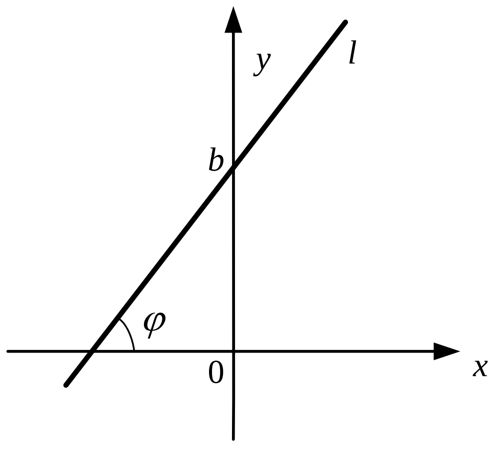
or

$$y=kx\underbrace{-kx_0+y_0}_{b}$$

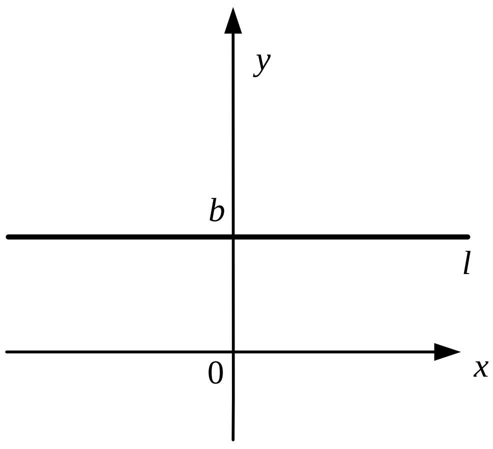
or

$$y=kx+b, \qquad \text{where } b=-kx_0+y_0,$$

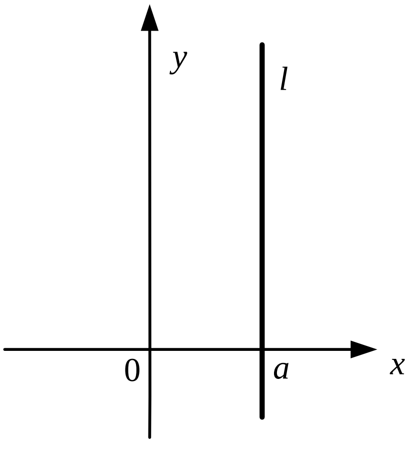
then we obtain the equation of a straight line with the slope k .



At $x=0$ then $y=b$, i.e. b is an ordinate of the point of intersection of a straight line and the ordinate axis *OY*.



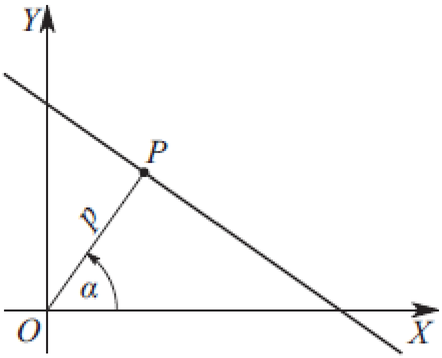
At $y=0$ then $x=a$, i.e. a is an abscissa of the point of intersection of a straight line and the ordinate axis *OX*.



9) Let's multiply a general equation of a straight line $Ax + By + C = 0$ by a normalized multiplier $\mu = \pm\frac{1}{\sqrt{A^2 + B^2}}$, which sign is opposite to the absolute term C sign, and obtain a normal equation of a straight line:

$$x \cdot \cos\alpha + y \cdot \sin\alpha - p = 0,$$

where p is the length of perpendicular dropped from the beginning of coordinates to the straight line, α is an angle obtained from this perpendicular with the positive direction of *OX* axis.

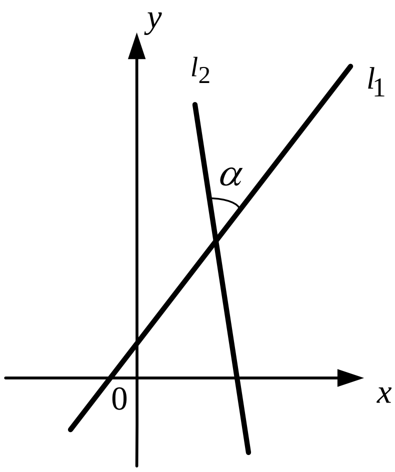


2. Geometric relationship of two straight lines. Distance from the point to a given straight line.

A tangent of the angle between two straight lines $y = k_1x + b_1$ (l_1) and $y = k_2x + b_2$ (l_2) calculated with the help of the formula:

$$tg\alpha = \left|\frac{k_2 - k_1}{1 + k_1k_2}\right| \quad \text{or} \quad ctg\alpha = \left|\frac{1 + k_1k_2}{k_2 - k_1}\right|.$$

This formula defines the acute angle between the straight lines.



If equations of the straight lines are given as

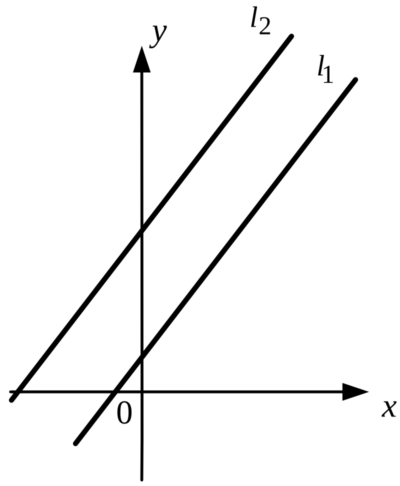
$$A_1x + B_1y + C_1 = 0$$
$$A_2x + B_2y + C_2 = 0$$

then the collinearity condition of two straight lines can be written as $\frac{A_1}{A_2} = \frac{B_1}{B_2}$ (the coordinates of two collinear vectors are proportional) and the condition of perpendicularity as $A_1 \cdot B_1 + A_2 \cdot B_2 = 0$ (or the scalar product of two perpendicular vectors equals 0).

If equations of the straight lines are given as

$$y = k_1x + b_1 \quad (l_1)$$
$$y = k_2x + b_2 \quad (l_2)$$

then we obtain the collinearity condition of these straight lines.



If $l_1 \parallel l_2$ then the angle between these lines equals $\alpha = 0$. Thus, $tg\,0 = 0 = \left|\frac{k_2 - k_1}{1 + k_1 k_2}\right|$. From this equation we have

$$\frac{k_2 - k_1}{1 + k_1 k_2} = 0$$

or

$$k_2 - k_1 = 0.$$

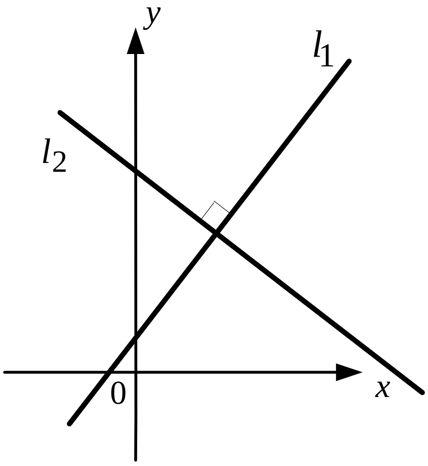
The collinearity condition of two straight lines can be written as

$$k_2 = k_1.$$

If equations of the straight lines are given as

$$y = k_1 x + b_1 \quad (l_1)$$
$$y = k_2 x + b_2 \quad (l_2)$$

then we obtain the perpendicularity condition of these straight lines.



If $l_1 \perp l_2$ then the angle between these lines equals $\alpha = 90^{o}$. Thus, $ctg 90^{o} = 0$ or $\left| \dfrac{1 + k_1 k_2}{k_2 - k_1} \right| = 0$.

From this equation we have

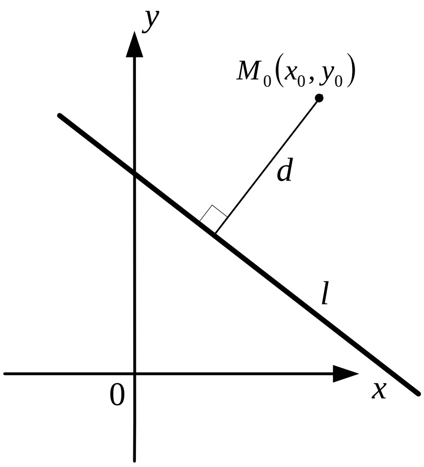
$$\frac{1 + k_1 k_2}{k_2 - k_1} = 0$$

or

$$1 + k_1 k_2 = 0.$$

The perpendicularity condition of two straight lines can be written as

$$k_1 k_2 = -1.$$

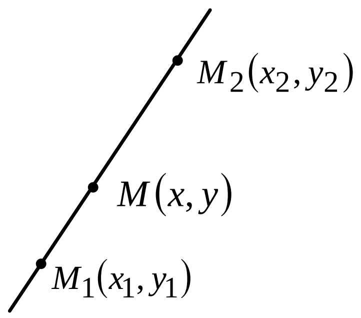


Distance d from point $M_0(x_0, y_0)$ to the straight line $Ax + By + C = 0$ is calculated according to the formula:

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

Formulas of division of a segment in the given ratio

Let us assume that the point $M(x_M, y_M, z_M)$ divides a segment between the points $M_1(x_{M_1}, y_{M_1}, z_{M_1})$ and $M_2(x_{M_2}, y_{M_2}, z_{M_2})$ in the ratio λ, that is $\lambda = \frac{|M_1M|}{|MM_2|}$.



Let's obtain the vector $\overline{M_1M}$ and the vector $\overline{MM_2}$:

$$\overline{M_1M} = (x - x_1; y - y_1)$$

$$\overline{MM_2} = (x_2 - x; y_2 - y).$$

It is known that $\overline{M_1M} \parallel \overline{MM_2}$. Since $\lambda = \frac{|M_1M|}{|MM_2|}$, we have $\overline{M_1M} = \lambda \cdot \overline{MM_2}$.

From this expression $\overline{M_1M} = \lambda \cdot \overline{MM_2}$ we obtain:

$$\begin{cases} x - x_1 = \lambda \cdot (x_2 - x) \\ y - y_1 = \lambda \cdot (y_2 - y) \end{cases}.$$

Let’s remove the brackets and find like terms:

$$\begin{cases} x - x_1 = \lambda x_2 - \lambda x \\ y - y_1 = \lambda y_2 - \lambda y \end{cases}$$

or

$$\begin{cases} x + \lambda x = \lambda x_2 + x_1 \\ y + \lambda y = \lambda y_2 + y_1 \end{cases}$$

or

$$\begin{cases} x \cdot (1 + \lambda) = \lambda x_2 + x_1 \\ y \cdot (1 + \lambda) = \lambda y_2 + y_1 \end{cases}.$$

Let’s express *x* and *y*:

$$\begin{cases} x = \dfrac{\lambda x_2 + x_1}{1 + \lambda} \\ y = \dfrac{\lambda y_2 + y_1}{1 + \lambda} \end{cases}.$$

In this case the following formulas should be used to find the coordinates of the point M :

$$x_M = \frac{x_{M_1} + \lambda \cdot x_{M_2}}{1 + \lambda}, \qquad y_M = \frac{y_{M_1} + \lambda \cdot y_{M_2}}{1 + \lambda}.$$

In the particular case, if the point M bisects the segment $M_1 M_2$ $(\lambda = 1)$ then

$$x_M = \frac{x_{M_1} + x_{M_2}}{2}, \qquad y_M = \frac{y_{M_1} + y_{M_2}}{2}.$$

Example. The coordinates apexes of a triangle ABC $A(-2,-2)$, $B(4,1)$, $C(0,4)$ are given. Using methods of the analytical geometry do the following:

1) find the distance between point A and point B;
2) form equation of the sides $AB,\ AC$;
3) form equation of the altitude dropped from the apex C;
4) find the inner angle of the triangle at the apex A;
5) calculate length of the altitude dropped from the apex C;
6) find the area of the ΔABC;
7) form equation of the median dropped from the apex C;
8) draw ΔABC.

Solution. 1) Let's find the distance between point A and point B:

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} = \sqrt{(4-(-2))^2 + (1-(-2))^2} =$$
$$= \sqrt{(4+2)^2 + (1+2)^2} = \sqrt{6^2 + 3^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5};$$

2) Let's form equation of the sides $AB,\ AC$:

$$AB:\ \frac{y - y_A}{y_B - y_A} = \frac{x - x_A}{x_B - x_A};$$
$$\frac{y-(-2)}{1-(-2)} = \frac{x-(-2)}{4-(-2)};$$
$$\frac{y+2}{1+2} = \frac{x+2}{4+2};$$
$$\frac{y+2}{3} = \frac{x+2}{6};$$
$$\frac{y+2}{1} = \frac{x+2}{2};$$
$$2\cdot(y+2) = 1\cdot(x+2);$$
$$2y + 4 = x + 2;$$
$$x - 2y - 4 + 2 = 0;$$
$$x - 2y - 2 = 0;$$
$$2y = x - 2;$$

$$y=\frac{1}{2}x-1,\ k_{AB}=\frac{1}{2};$$

$$AC:\ \frac{y-y_A}{y_C-y_A}=\frac{x-x_A}{x_C-x_A};$$

$$\frac{y-(-2)}{4-(-2)}=\frac{x-(-2)}{0-(-2)};$$

$$\frac{y+2}{6}=\frac{x+2}{2};$$

$$\frac{y+2}{3}=\frac{x+2}{1};$$

$$1\cdot(y+2)=3\cdot(x+2);$$

$$y+2=3x+6;$$

$$3x+6-y-2=0;$$

$$3x-y+4=0;$$

$$y=3x+4,\ k_{AC}=3.$$

3) Let's form equation of the altitude dropped from the apex C:

$$y-y_C=k_{CN}(x-x_C),$$

$$k_{CN}\cdot k_{AB}=-1,\ k_{CN}=-\frac{1}{k_{AB}},$$

$$y-y_C=-\frac{1}{k_{AB}}(x-x_C);$$

$$y-4=-\frac{1}{1/2}(x-0);$$

$$y-4=-2\cdot(x-0);$$

$$y-4=-2x;$$

$$y=-2x+4.$$

4) Let's find the inner angle of the triangle at the apex:

$$tg\alpha = \left|\frac{k_{AC} - k_{AB}}{1 + k_{AC}k_{AB}}\right| = \left|\frac{3 - 1/2}{1 + 3 \cdot 1/2}\right| = \left|\frac{5/2}{5/2}\right| = 1, \ \alpha = arctg1 = \frac{\pi}{4}.$$

5) Let's calculate length of the altitude dropped from the apex C:

$$CN = \frac{|1 \cdot x_C - 2 \cdot y_C - 2|}{\sqrt{1^2 + (-2)^2}} = \frac{|1 \cdot 0 - 2 \cdot 4 - 2|}{\sqrt{1 + 4}} = \frac{|0 - 8 - 2|}{\sqrt{5}} = \frac{10}{\sqrt{5}}.$$

6) Let's find area of the ΔABC:

$$S_{\Delta ABC} = \frac{1}{2} AB \cdot CN = \frac{1}{2} \cdot 3\sqrt{5} \cdot \frac{10}{\sqrt{5}} = 15.$$

7) Let's find the coordinates of the middle M of the segment AB:

$$x_M = \frac{x_A + x_B}{2} = \frac{-2 + 4}{2} = \frac{2}{2} = 1, \ y_M = \frac{y_A + y_B}{2} = \frac{-2 + 1}{2} = -\frac{1}{2},$$

$$CM: \ \frac{y - y_M}{y_C - y_M} = \frac{x - x_M}{x_C - x_M};$$

$$\frac{y - \left(-\frac{1}{2}\right)}{4 - \left(-\frac{1}{2}\right)} = \frac{x - 1}{0 - 1};$$

$$\frac{y + \frac{1}{2}}{4 + \frac{1}{2}} = \frac{x - 1}{-1};$$

$$\frac{y + \frac{1}{2}}{\frac{9}{2}} = \frac{x - 1}{-1};$$

$$(-1) \cdot \left(y + \frac{1}{2}\right) = \frac{9}{2} \cdot (x - 1);$$

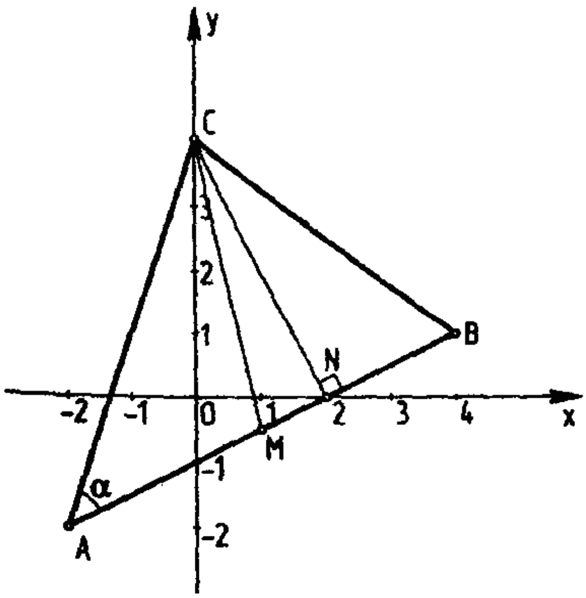
$$-y-\frac{1}{2}=\frac{9}{2}x-\frac{9}{2};$$

$$\frac{9}{2}x-\frac{9}{2}+y+\frac{1}{2}=0;$$

$$\frac{9}{2}x+y-4=0;$$

$$y=-\frac{9}{2}x+4.$$

8) Let's draw ΔABC:



Theoretical questions

1. What form has a general equation of a straight line?
2. Write down an equation of a straight line with a slope.
3. Write down an equation of a straight line passing through the point with a slope.
4. Write down an equation of a straight line passing through two points.
5. Write down an equation of a straight line with given intercepts on the axes.
6. Write down a canonical equation of a straight line.
7. Write down a parametric equation of a straight line.
8. Write down a normal equation of a straight line.

9. Write down the formula of the distance between the point and the straight line.
10. Write down the condition of collinearity of two straight lines.
11. Write down the condition of perpendicularity of two straight lines.
12. Write down the formula of the angle between two straight lines.
13. Describe finding of the slope from a general equation.