Theme 10. The elements of vector algebra

Lecture plan

1. Definition of a vector, types of vectors.
2. Basic operations with vectors, properties of these operations, their geometric illustration.
3. Formulas of division of a segment in the given ratio.
4. A module of the vector, its properties. The angle between the vectors. A scalar product. A vector product. A mixed product.

1. Definition of a vector, types of vectors

Let's consider a two dimensional space (a plane).

The Cartesian coordinate system is two perpendicular lines that cross at a point called the origin.

The horizontal line is called the x-axis (the abscissa axis).

The vertical line is called the y-axis (the ordinate axis).

The system is denoted by XY or OXY (fig. 1).

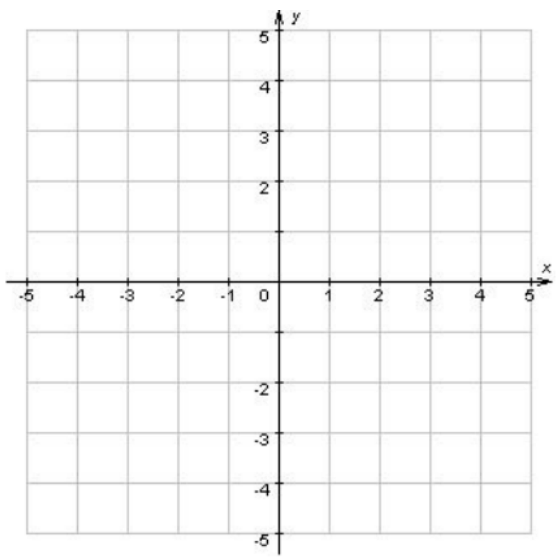


Figure 1. Cartesian coordinate system OXY

The x-axis divides the plane into the upper and the lower half planes, the ordinate axis divides the plane into the right and left half planes.

Two coordinate axes divide the plane into four parts which are called quadrants (fig. 2).

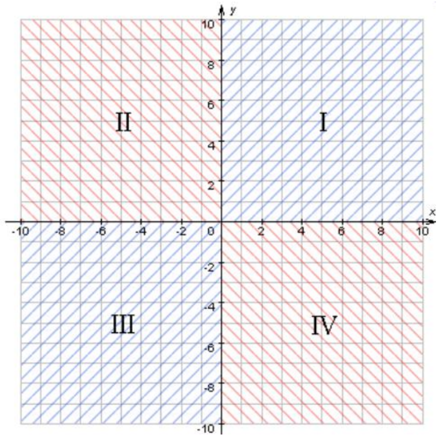


Figure 2. Quadrants of Cartesian coordinate system OXY

Let’s take the point M on a plane and project it onto the coordinate axes (fig. 3).

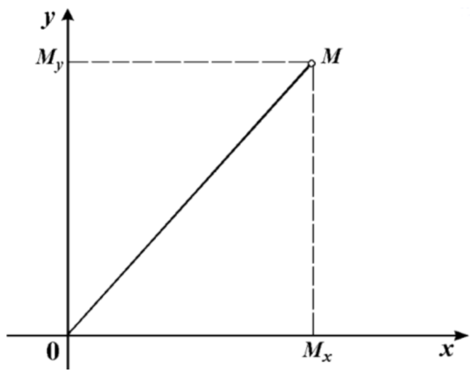


Figure Point M on a plane of Cartesian coordinate system OXY

Each point on a plane has the vertical projection and the horizontal projection.

Let a rectangular Cartesian coordinate system be defined in space.

A rectangular Cartesian coordinate system in a space is three pairwise perpendicular directed lines OX (the abscissa axis), OY (the ordinate axis) and OZ (the applicate axis) with an intersection point O (the origin). These coordinate axes divide the space into eight parts called octants.

Then a position of any spatial point is defined by its coordinates *x, y, z* (fig. 4).

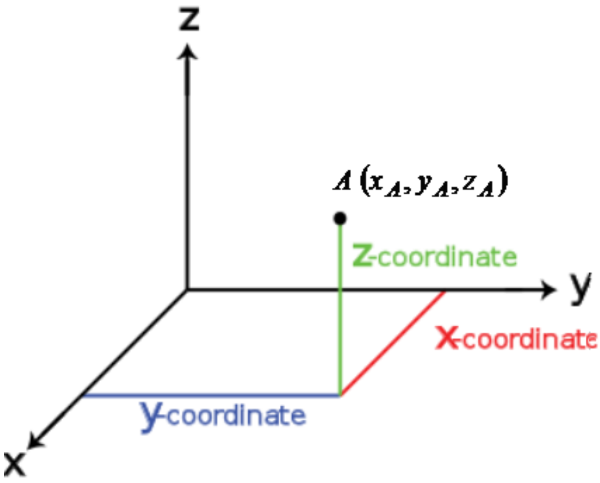


Figure 4. Rectangular Cartesian coordinate system in space

Definition. A segment bounded by points A and B is called *a directed segment* if its initial point (the origin) is A and its end point (the terminus) is B.

Definition. A directed segment (or an ordered couple of points A and B) is called *a vector* (geometrical). A vector is denoted by $\overline{AB}$ (symbol is called "arrow") or $\overline{a}$ (fig. 5).

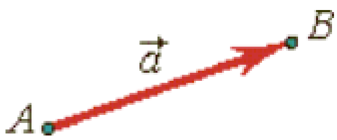


Figure 5. Vector $\overline{AB}$

The vector $\overline{BA}$ has the opposite direction, i.e. $\overline{BA} = -\overline{AB}$.

Let the point $A(x_A, y_A, z_A)$ be an origin of the vector $\vec{a}$ and the point $B(x_B, y_B, z_B)$ be its terminus. Coordinates of the vector $\overline{AB}$ are defined as

$$\overline{AB} = (x_B - x_A, y_B - y_A, z_B - z_A).$$

It is known that any vector in space can be presented as

$$\overline{a} = a_x \cdot \overline{i} + a_y \cdot \overline{j} + a_z \cdot \overline{k} \text{ or } \overline{a} = (a_x, a_y, a_z),$$

where a_x, a_y, a_z are projections of the vector $\overline{a}$ on the axes Ox, Oy, Oz respectively; $\overline{i}, \overline{j}, \overline{k}$ are unit vectors (orts) whose directions coincide with the direction of the coordinate axes, $\overline{i} = (1,0,0)$, $\overline{j} = (0,1,0)$, $\overline{k} = (0,0,1)$ (fig. 6).

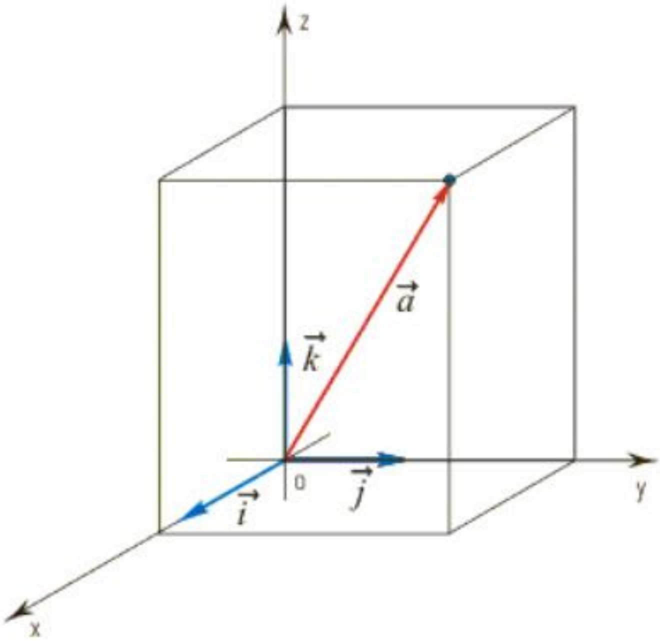


Figure 6. Rectangular Cartesian coordinate system with orts in space

Every of the vectors $\bar{i}, \bar{j}, \bar{k}$ is perpendicular (orthogonal) to the both others. These vectors form a so called orthonormalized basis. The projections a_x, a_y, a_z are coordinates of the vector in the orthonormalized basis.

Definition. So called *the null vector* whose origin coincides with its terminus also relates to vectors: $\bar{0} = (0,0,0)$.

Definition. Vectors located on the same or parallel straight lines are called *parallel* or *collinear.*

Example. Vectors $\bar{a}, \bar{b}, \bar{c}$ are collinear, vectors $\overline{AC}, \overline{BD}, \overline{CB}$ are collinear (fig. 7).

Definition. Vectors are called *complanar* if there exists a plane which they are parallel to (fig. 8).

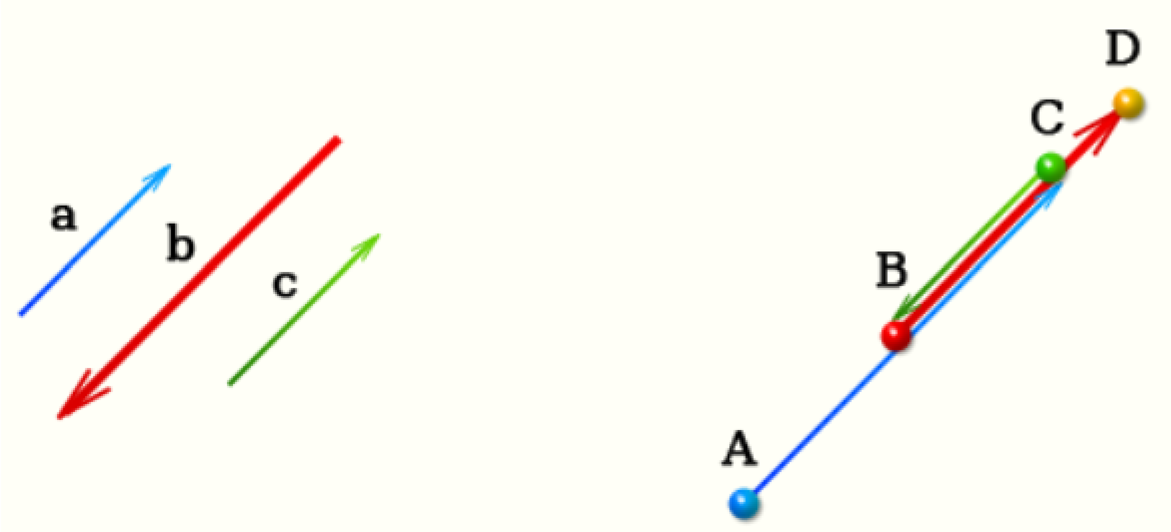


Figure 7. Collinear vectors

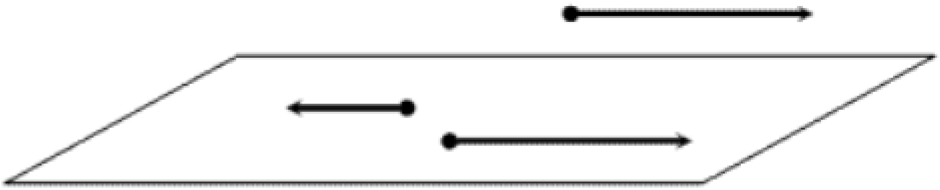


Figure 8. Complanar vectors

Definition. Two vectors should be considered *equal* if they are *collinear,* equally directed and have equal lengths (fig. 9).

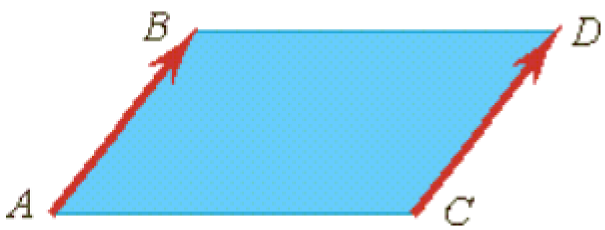


Figure 9. Equal vectors

2. Basic operations with vectors, properties of these operations, their geometric illustration.

Linear operations with the vectors:

1) *a sum or difference of vectors are determined according to the formulas*:

$$\bar{a} \pm \bar{b} = \left(a_x \pm b_x, a_y \pm b_y, a_z \pm b_z\right).$$

Vectors are added and subtracted according to the rule of triangle or parallelogram (fig. 10).

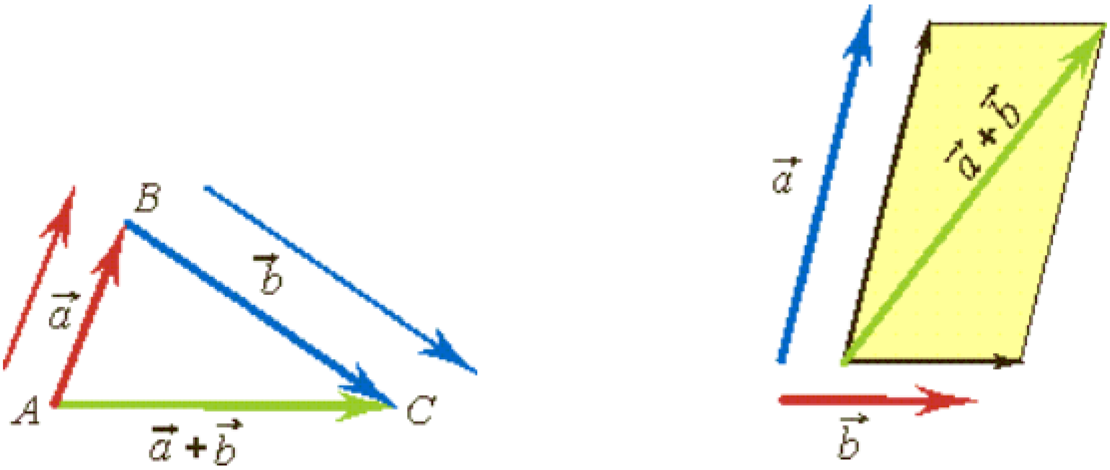


Figure 10. Rule of triangle (on the left) or rule of parallelogram (on the right)

A sum of any number of vectors can be calculated according to the rule of polynomial. For example, a sum of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ is the vector $\vec{R}=\vec{a}+\vec{b}+\vec{c}+\vec{d}$:

The operations of vector addition (subtraction) and vector multiplication by some number satisfy the following laws:

1) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ is a commutative law;

2) $\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$ is an associative law;

3) $\vec{a}+\left(-\vec{a}\right)=\bar{0}$, where $\bar{0}$ is a null vector, $\left(-\vec{a}\right)$ is the opposite vector relative to a vector $\vec{a}$: $-\vec{a}=(-1)\cdot\vec{a}$.

2) *a multiplication of a vector by a number is determined according to the formula*:

$$\alpha\cdot\bar{a}=\left(\alpha\cdot a_x,\alpha\cdot a_y,\alpha\cdot a_z\right)$$

If $\alpha>0$, then $\vec{a}$ and $\alpha\bar{a}$ are parallel (collinear) and directed to the same side (fig. 11),

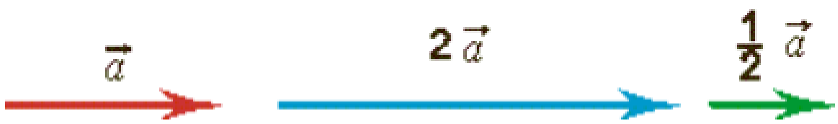


Figure 11. Vectors directed to the same side

if $\alpha<0$, then to the opposite sides (fig. 12).

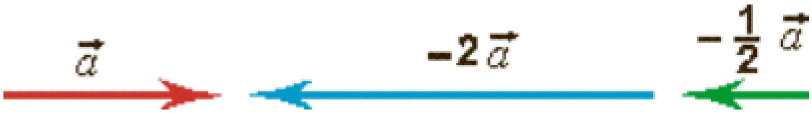


Figure 12. Vectors directed to the opposite sides

From the definition of collinear vectors it follows that two vectors are collinear if and only if one of them can be obtained by multiplying another one by some value α, i.e. $\vec{b} = \alpha\vec{a}$.

Let the vectors $\bar{a}$ and $\bar{b}$ be defined by their coordinates, i.e. $\bar{a} = (a_x, a_y, a_z)$ and $\bar{b} = (b_x, b_y, b_z)$, then the vector equality $\bar{b} = \alpha\bar{a}$ is equivalent to three numerical ones:

$$b_x = \alpha a_x, \qquad b_y = \alpha a_y, \qquad b_z = \alpha a_z,$$

from which it follows that

$$\frac{b_x}{a_x} = \alpha, \qquad \frac{b_y}{a_y} = \alpha, \qquad \frac{b_z}{a_z} = \alpha \quad \text{or} \quad \frac{b_x}{a_x} = \frac{b_y}{a_y} = \frac{b_z}{a_z} = \alpha.$$

Rule. Thus vectors are *collinear* if their *coordinates are proportional.*

Example 1. Two vectors $\vec{a} = (3,-2,-6)$ and $\vec{b} = (-2,1,0)$ are given. Determine the projections on the coordinate axes of the following vectors:

1) $\vec{a} + \vec{b}$; 2) $\vec{a} - \vec{b}$; 3) $2\vec{a}$; 4) $2\vec{a} - 3\vec{b}$.

Solution. By the rule of vector addition and vector multiplication by a number we have:

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z) = (3 + (-2), -2 + 1, -6 + 0) = (1,-1,-6);$$

$$\vec{a} - \vec{b} = (a_x - b_x, a_y - b_y, a_z - b_z) = (3 - (-2), -2 - 1, -6 - 0) = (5,-3,-6);$$

$$2 \cdot \vec{a} = (2 \cdot a_x, 2 \cdot a_y, 2 \cdot a_z) = (2 \cdot 3, 2 \cdot (-2), 2 \cdot (-6)) = (6,-4,-12);$$

$$2\vec{a} - 3\vec{b} = (2a_x + 3b_x, 2a_y + 3b_y, 2a_z + 3b_z) =$$

$$= (2 \cdot 3 - 3 \cdot (-2), 2 \cdot (-2) - 3 \cdot 1, 2 \cdot (-6) - 3 \cdot 0) = (12,-7,-12).$$

Homework 1. The first centre sells products of three types A, B and C for the year 2011:

$\vec{a} = (200,100,150)$ and the second centre sells products of three types A, B and C for the year 2011: $\vec{b} = (130,70,110)$.

Find:

1) the total sales position of two centers for the year 2011;

2) the sales position of the third centre if it has the sales position in two times greater then the first centre.

Example 2. Determine values of α and β , when the vectors $\vec{a} = -2\vec{i} + 3\vec{j} + \beta\vec{k}$ and $\vec{b} = \alpha\vec{i} - 6\vec{j} + 2\vec{k}$ are collinear.

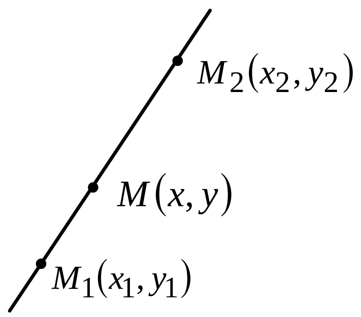
Solution. From the condition of collinear vectors we get

$$\frac{-2}{\alpha} = \frac{3}{-6} = \frac{\beta}{2}.$$

From this it follows that $\frac{-2}{\alpha} = \frac{3}{-6}$ and $\frac{3}{-6} = \frac{\beta}{2}$. Thus $\alpha = \frac{-6}{3} \cdot (-2) = 4$ and $\beta = \frac{3}{-6} \cdot 2 = -1$.

Formulas of division of a segment in the given ratio

Let us assume that the point $M(x_M, y_M, z_M)$ divides a segment between the points $M_1(x_{M_1}, y_{M_1}, z_{M_1})$ and $M_2(x_{M_2}, y_{M_2}, z_{M_2})$ in the ratio λ, that is $\lambda = \frac{|M_1M|}{|MM_2|}$.



In this case the following formulas should be used to find the coordinates of the point M :

$$x_M = \frac{x_{M_1} + \lambda \cdot x_{M_2}}{1+\lambda},\ y_M = \frac{y_{M_1} + \lambda \cdot y_{M_2}}{1+\lambda},\ z_M = \frac{z_{M_1} + \lambda \cdot z_{M_2}}{1+\lambda}.$$

In the particular case, if the point M bisects the segment M_1M_2 $(\lambda = 1)$ then $x_M = \frac{x_{M_1} + x_{M_2}}{2}$, $y_M = \frac{y_{M_1} + y_{M_2}}{2}$, $z_M = \frac{z_{M_1} + z_{M_2}}{2}$.

4. A module of the vector, its properties. The angle between the vectors. The distance between the vectors.

The distance between an origin and a terminus of a vector is called its length or module and designated by $|\vec{a}|$ or $|\overrightarrow{AB}|$.

The module of a vector $\vec{a}$ is calculated according to the following formula:

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}\ .$$

The module of a vector $\overrightarrow{AB}$ is calculated according to the following formula:

$$|\overrightarrow{AB}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}\ .$$

The module of a null vector is equal to zero.

Direction cosines are called cosines of the angles between the vector $\vec{a}$ and positive directions of the corresponding coordinate axes and defined as follows:

$$\cos\alpha = \frac{a_x}{|\vec{a}|} = \frac{a_x}{\sqrt{a_x^2 + a_y^2 + a_z^2}}; \qquad \cos\beta = \frac{a_y}{|\vec{a}|} = \frac{a_y}{\sqrt{a_x^2 + a_y^2 + a_z^2}};$$

$$\cos\gamma = \frac{a_z}{|\vec{a}|} = \frac{a_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}}\ .$$

They are related to the equality

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Example Find the direction cosines of the vector $\overrightarrow{AB}$ if the points $A(1,2,0)$ and $B(3,1,-2)$ are given.

Solution. The coordinates of the vector $\overrightarrow{AB}$ are calculated in this way:

$$\overrightarrow{AB} = (x_B - x_A; y_B - y_A; z_B - z_A) = (3-1, 1-2, -2-0) = (2,-1,-2).$$

Its length is

$$|\overrightarrow{AB}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} = \sqrt{2^2 + (-1)^2 + (-2)^2} =$$
$$= \sqrt{9} = 3$$

The direction cosines are

$$\cos\alpha = \frac{a_x}{|\vec{a}|} = \frac{2}{3}, \qquad \cos\beta = \frac{a_y}{|\vec{a}|} = -\frac{1}{3}, \qquad \cos\gamma = \frac{a_z}{|\vec{a}|} = -\frac{2}{3}.$$

Scalar product. The value $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi\cdot$ is called *a scalar product* of the vectors $\vec{a}$ and $\vec{b}$ (fig. 13).

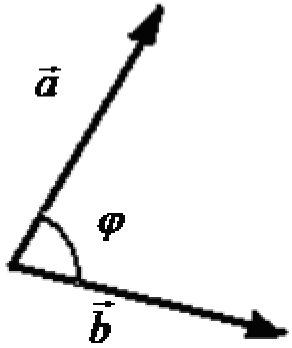


Figure 1 Vectors $\vec{a}$, $\vec{b}$ and the angle φ between them

Let's consider types of angles between vectors (fig. 14).

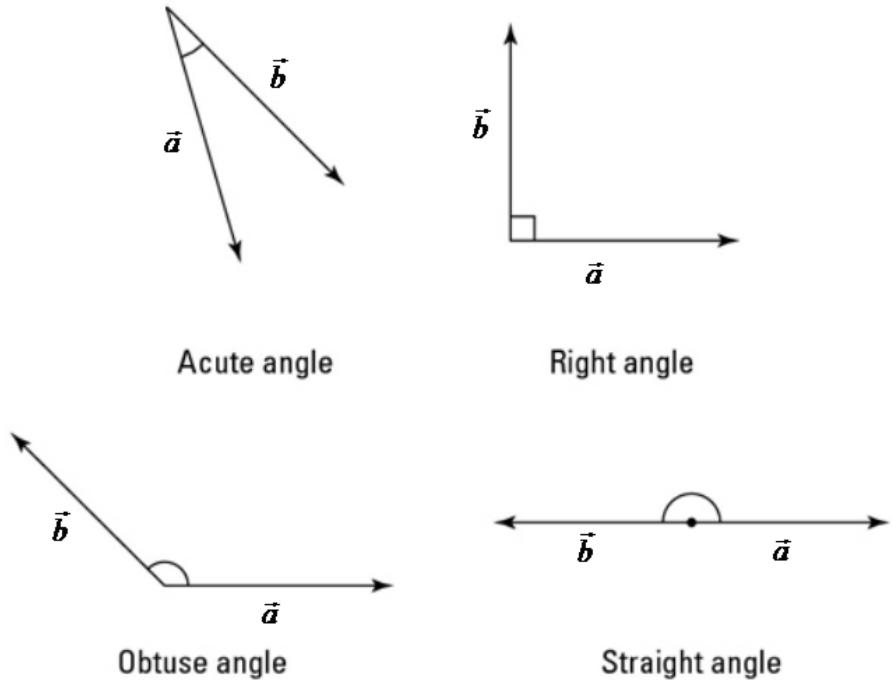


Figure 14. Types of angles between vectors $\vec{a}$ and $\vec{b}$

If the vectors $\vec{a}$ and $\vec{b}$ are given by their coordinates then their scalar product is as follows:

$$\vec{a}\cdot\vec{b}=a_x\cdot b_x+a_y\cdot b_y+a_z\cdot b_z.$$

It is obvious that if vectors are perpendicular then their *scalar product is equal to zero*, i.e.

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot\left|\vec{b}\right|\cdot\cos\varphi=|\vec{a}|\cdot\left|\vec{b}\right|\cdot\cos 90^{o}=0 \text{ or}$$

$$\vec{a}\cdot\vec{b}=0 \text{ or } a_x\cdot b_x+a_y\cdot b_y+a_z\cdot b_z=0.$$

The basic properties of a scalar product:

1) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$;
2) $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$;
3) $(\alpha\vec{a})\cdot\vec{b}=\vec{a}\cdot(\alpha\vec{b})=\alpha(\vec{a}\cdot\vec{b})$.

The length of the vector $\vec{a}$ is defined by means of a scalar product as

$$|\vec{a}|=\sqrt{\vec{a}\cdot\vec{a}} \text{ or } \vec{a}\cdot\vec{a}=|\vec{a}|^2$$

and the cosine of an angle between the vectors $\vec{a}$ and $\vec{b}$ is calculated as

$$\cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot\left|\vec{b}\right|}=\frac{a_x\cdot b_x+a_y\cdot b_y+a_z\cdot b_z}{\sqrt{a_x{}^2+a_y{}^2+a_z{}^2}\sqrt{b_x{}^2+b_y{}^2+b_z{}^2}}.$$

Scalar product of the coordinate axes orts:

$$\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1 \text{ and } \vec{i}\cdot\vec{k}=\vec{j}\cdot\vec{k}=\vec{i}\cdot\vec{j}=0.$$

Example 4. Find the angle between the vectors $\vec{a}=(2,-1,-2)$ and $\vec{b}=(0,3,4)$.

<u>Solution.</u> We have

$\vec{a}\cdot\vec{b}=2\cdot 0+(-1)\cdot 3+(-2)\cdot 4=-11$;

$|\vec{a}|=\sqrt{2^2+(-1)^2+(-2)^2}=\sqrt{4+1+4}=\sqrt{9}=3$;

$\left|\vec{b}\right|=\sqrt{0^2+3^2+4^2}=\sqrt{25}=5$;

$$\cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot\left|\vec{b}\right|}=\frac{-11}{3\cdot 5}=-\frac{11}{15},$$

$$\varphi=\pi-\arccos\frac{11}{15}.$$

Example 5. The vectors $\vec{a}$ and $\vec{b}$ form an angle $\varphi=60^{\text{o}}$, $|\vec{a}|=3$, $\left|\vec{b}\right|=2$. Find a scalar product of the vectors $\vec{a}$ and $\vec{b}$.

<u>Solution.</u> $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot\left|\vec{b}\right|\cdot\cos\varphi=3\cdot 2\cdot\cos 60^{\text{o}}=6\cdot\frac{1}{2}=3.$

Vector product. *The vector product* of the vectors $\vec{a}$ and $\vec{b}$ is the vector $\vec{c}$ which satisfies the following conditions:

1) it is perpendicular to both the vector-multiplicands $\vec{a}$ and $\vec{b}$,
2) it is directed in such a way that, looking from its terminus, the shortest turn on the angle φ from $\vec{a}$ to $\vec{b}$ occurs anticlockwise, i.e. the triple of vectors $\vec{a}$, $\vec{b}$, $\vec{a}\times\vec{b}$ is right-hand triple,
3) the length of this vector is equal to $|\vec{c}|=|\vec{a}|\cdot|\vec{b}|\cdot\sin\varphi$.

The vector product of the vectors $\vec{a}$ and $\vec{b}$ is designated by $\vec{c}=\vec{a}\times\vec{b}$ or $\vec{c}=[\vec{a},\vec{b}]$ (fig. 15).

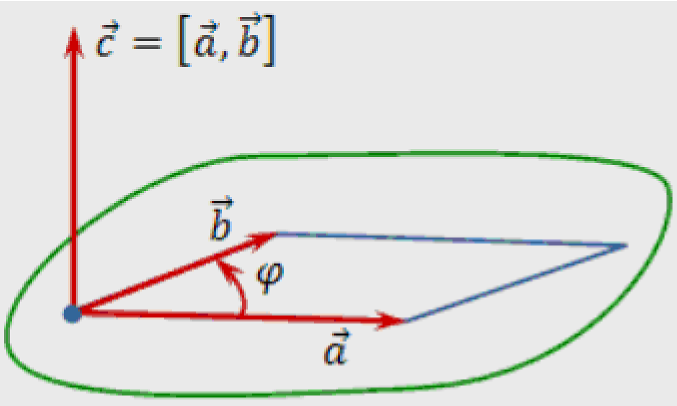


Figure 15. Vector product of the vectors $\vec{a}$ and $\vec{b}$

A module of a vector product is numerically equal to the area of a parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$ (fig. 16):

$$S=|\vec{c}|=|\vec{a}\times\vec{b}|=|\vec{a}|\cdot|\vec{b}|\cdot\sin\varphi.$$

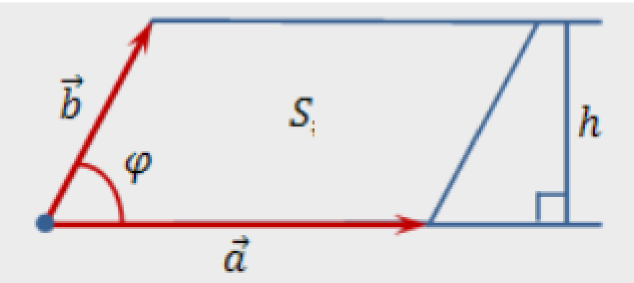


Figure 16. Area of a parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$

The basic properties of a vector product:

1. $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$, i.e. the sign of the vector product changes on opposite if the factors are transposed (fig. 17);

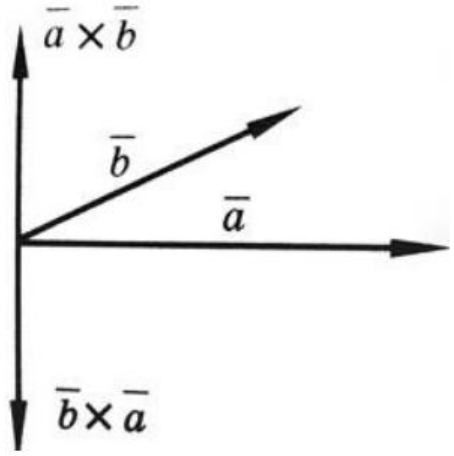


Figure 17. Sign of the vector product changes on opposite
if the factors are transposed

2. $\alpha\left(\vec{a}\times\vec{b}\right)=\left(\alpha\vec{a}\right)\times\vec{b}=\vec{a}\times\left(\alpha\vec{b}\right)$, i.e. the scalar factor α can be taken outside of the vector product;

$\vec{a}\times\left(\vec{b}\pm\vec{c}\right)=\vec{a}\times\vec{b}\pm\vec{a}\times\vec{c}$;

4. $\vec{a}\times\vec{a}=0$.

The vector product through the coordinates of the vector-multiplicands is expressed as it follows:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

If two vectors are collinear then their vector product is equal to zero, i.e. their coordinates are proportional or equal:

$$\vec{a}\times\vec{b}=0 \text{ or } \frac{b_x}{a_x}=\frac{b_y}{a_y}=\frac{b_z}{a_z}.$$

Example 6. Vectors $\bar{a}=(4,-5,0)$ and $\bar{b}=(0,4,-3)$ are given. Calculate the area of the parallelogram constructed on these vectors.

Solution. The area is calculated by the formula: $S=\left|\bar{a}\times\bar{b}\right|$.

The vector product of the vectors $\bar{a}$ and $\bar{b}$ is equal to:

$$\bar{a}\times\bar{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -5 & 0\\ 0 & 4 & -3\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ 4 & -5\\ 0 & 4\end{matrix}=\vec{i}\cdot 15+\vec{j}\cdot 0+\vec{k}\cdot 16-\vec{k}\cdot 0-\vec{i}\cdot 0+\vec{j}\cdot 12=$$

$=15\vec{i}+12\vec{j}+16\vec{k}$.

Calculate the area of the parallelogram (fig. 18):

$S=\left|\bar{a}\times\bar{b}\right|=\sqrt{15^2+12^2+16^2}=\sqrt{625}=25$ (sq. units)

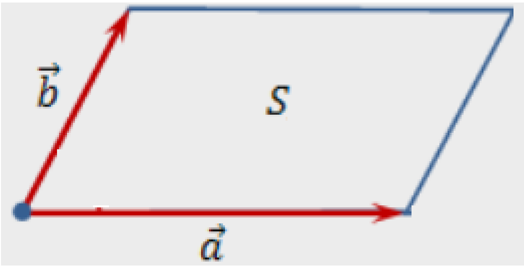


Figure 18. Area of a parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$

Example 7. Let three apexes of the parallelogram $A(1,-1,2)$, $B(5,-6,2)$ and $C(1,3,-1)$ be given. Calculate the area of the parallelogram.

Solution. The area is $S=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$.

Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$\overrightarrow{AB}=(5-1,-6-(-1),2-2)=(4,-5,0)$,

$\overrightarrow{AC}=(1-1,3-(-1),-1-2)=(0,4,-3)$.

The vector product of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is equal to:

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -5 & 0\\ 0 & 4 & -3\end{vmatrix}=\vec{i}\cdot\begin{vmatrix}-5 & 0\\ 4 & -3\end{vmatrix}-\vec{j}\cdot\begin{vmatrix}4 & 0\\ 0 & -3\end{vmatrix}+\vec{k}\cdot\begin{vmatrix}4 & -5\\ 0 & 4\end{vmatrix}=$$

$=15\vec{i}+12\vec{j}+16\vec{k}$.

Calculate the area of the parallelogram:

$S=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{15^2+12^2+16^2}=\sqrt{625}=25$ (sq. units)

Mixed product of vectors. *The mixed product* of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is equal to the value obtained after scalar-multiplying one vector by a vector product of two others:

$$(\vec{a},\vec{b},\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c}).$$

The module of the mixed product is equal to the volume of the parallelepiped constructed on the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (fig. 19), i.e.

$$V=\left|(\vec{a},\vec{b},\vec{c})\right|.$$

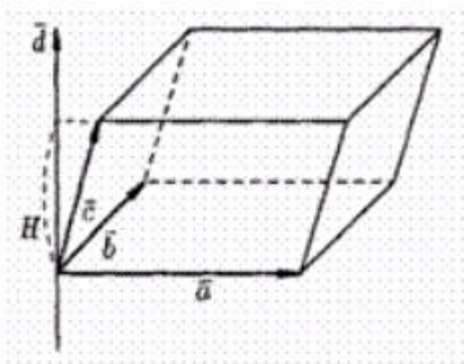


Figure 19. Volume of the parallelepiped constructed on the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

The volume of a tetrahedron is equal to $V=\frac{1}{6}\left|(\vec{a},\vec{b},\vec{c})\right|=\frac{1}{3}\cdot S\cdot h$.

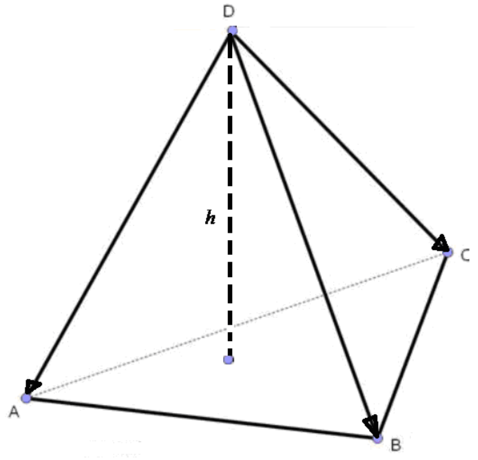
If the vectors are given by their coordinates: $\vec{a}=(a_x,a_y,a_z)$, $\vec{b}=(b_x,b_y,b_z)$ and $\vec{c}=(c_x,c_y,c_z)$, then their mixed product can be found according to the formula:

$$(\vec{a},\vec{b},\vec{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

From the definition of a mixed product it follows that *the condition of complanarity* of vectors is equality to zero of their mixed product:

$$(\vec{a},\vec{b},\vec{c})=0 \text{ or } \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=0.$$

Example 8. Let the following apexes of the pyramid $A(2,3,1)$, $B(4,1,-2)$, $C(6,3,7)$, $D(-5,-4,2)$ be given. Calculate pyramid volume and the length of the altitude put down from the apex D.



Solution. The volume of the pyramid is equal to:

$$V=\frac{1}{6}\left|\left(\overrightarrow{DA},\overrightarrow{DB},\overrightarrow{DC}\right)\right| \text{ or } V=\frac{1}{3}\cdot S\cdot h, \text{ thus } h=\frac{3V}{S}.$$

Let's find $\overrightarrow{DA}$, $\overrightarrow{DB}$ and $\overrightarrow{DC}$:

$$\overrightarrow{DA}=\left(x_A-x_D, y_A-y_D, z_A-z_D\right)=\left(2-(-5), 3-(-4), 1-2\right)=(7,7,-1),$$

$$\overrightarrow{DB}=\left(x_B-x_D, y_B-y_D, z_B-z_D\right)=\left(4-(-5), 1-(-4), -2-2\right)=(9,5,-4),$$

$$\overrightarrow{DC}=\left(x_C-x_D, y_C-y_D, z_C-z_D\right)=\left(6-(-5), 3-(-4), 7-2\right)=(11,7,5).$$

Then $\left(\overrightarrow{DA},\overrightarrow{DB},\overrightarrow{DC}\right)=\begin{vmatrix}7 & 7 & -1\\ 9 & 5 & -4\\ 11 & 7 & 5\end{vmatrix}\begin{matrix}7 & 7\\ 9 & 5\\ 11 & 7\end{matrix}=$

$$=7\cdot5\cdot5+7\cdot(-4)\cdot11+(-1)\cdot9\cdot7-11\cdot5\cdot(-1)-7\cdot(-4)\cdot7-7\cdot9\cdot5=$$
$$=175-308-63+55+196-315=-260.$$

Thus $V=\frac{1}{6}\left|\left(\overrightarrow{DA},\overrightarrow{DB},\overrightarrow{DC}\right)\right|=\frac{1}{6}\cdot|-260|=\frac{260}{6}=\frac{130}{3}$ (cubed units).

Let's find the area of triangle *ABC* constructed on vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

For this let's find vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\overrightarrow{AB}=\left(x_B-x_A, y_B-y_A, z_B-z_A\right)=\left(4-2, 1-3, -2-1\right)=(2,-2,-3),$$

$$\overrightarrow{AC}=\left(x_C-x_A, y_C-y_A, z_C-z_A\right)=\left(6-2, 3-3, 7-1\right)=(4,0,6).$$

Let's calculate $\overrightarrow{AB}\times\overrightarrow{AC}$:

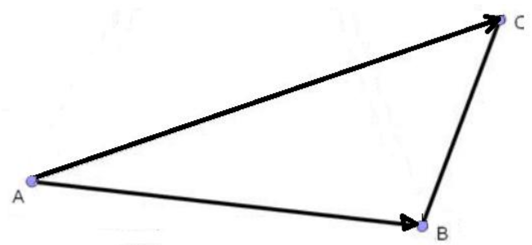
$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -2 & -3\\ 4 & 0 & 6\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ 2 & -2\\ 4 & 0\end{matrix}=-12\cdot\vec{i}-12\cdot\vec{j}+0\cdot\vec{k}+8\cdot\vec{k}-0\cdot\vec{i}-12\cdot\vec{j}=$$

$$=-12\vec{i}-24\vec{j}+8\vec{k}.$$

Let's find the area of triangle *ABC*:

$$S_{ABC}=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{(-12)^2+(-24)^2+8^2}=$$

$$=\frac{1}{2}\sqrt{784}=\frac{28}{2}=14 \text{ (sq. units).}$$



Thus, let's calculate the length of the altitude put down from the apex D:

$$h=\frac{3\cdot V}{S}=\frac{3\cdot V}{S_{ABC}}=\frac{3\cdot\frac{130}{3}}{14}=\frac{65}{7} \text{ (units of length).}$$

Let's consider the projection of the vector $\vec{c}$ on the x-axis (fig. 20):

$$pr_X\vec{c}=c_X=\left|\vec{c}\right|\cdot\cos\alpha.$$

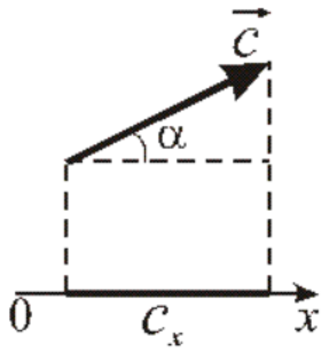


Figure 20. Projection of the vector $\vec{c}$ on the x-axis

Let's consider the projection of the vector $\vec{a}$ on the vector $\vec{b}$:

$$pr_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}.$$

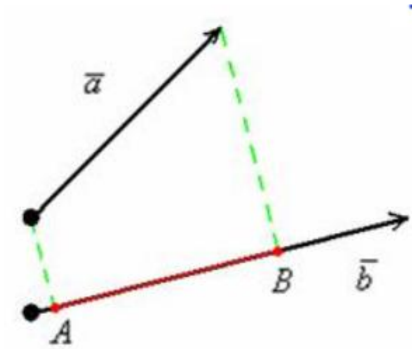


Figure 21. Projection of the vector $\vec{a}$ on the vector $\vec{b}$

Theoretical questions

1. What do you call a scalar product?
2. Write down the property of a scalar product.
3. Write down the formula of a scalar product if two vectors are given by their coordinates.
4. Write down the formula of the angle between vectors.
5. Write down the condition of perpendicularity of vectors.
6. Write down the condition of parallelity of vectors.
7. What do you call a vector product?
8. Write down the property of a vector product.
9. Write down the formula of a vector product if two vectors are given by their coordinates.
10. Write down the formulas of the area of a parallelogram constructed on two vectors.
11. Write down the formulas of division of a segment in the given ratio.
12. What do you call a mixed product?
13. Write down the formulas of the volume of a parallelepiped constructed on three vectors.
14. Write down the condition of complanarity of three vectors.