Theme 8. The elements of the theory of matrices and determinants

Lecture plan

1. A definition and types of matrices, basic matrices.
2. Basic operations with matrices and properties of these operations.
3. A definition of the determinant, rules of calculation and properties of determinants.
4. Notion of an inverse matrix, calculation of an inverse matrix by cofactors and elementary row operations.
5. Notion of a matrix rank and its calculation.

1. A definition and types of matrices, basic matrices.

Definition. A rectangular table of numbers of the form

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ or briefly } A=\left(a_{ij}\right)_{m\times n},\ i=\overline{1,m},\ j=\overline{1,n},$$

which has m rows and n columns is called *the matrix* A *of the size* $m\times n$.

Matrices are denoted by the capital letters A, B, C and so on.

Each element of the matrix A is designated by a_{ij} and represents the entry in the matrix A on the i-th row and j-th column.

Example Consider the 3×4 matrix $A=\begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$.

Here $\begin{pmatrix} 3 & 1 & 5 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$ represent the 2-nd row and the 3-rd column of the matrix A respectively, and the element 5 represents the entry in the matrix on the 2-nd row and 3-rd column ($a_{23}=5$).

Definition. The size of a matrix is called its *order*. *The order* is specified as:

(number of rows) $\times$ (number of columns).

Definition. *A matrix-column (row)* is called a matrix consisting of the only column (row):

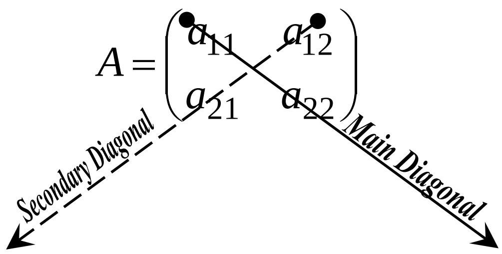
$$A=\begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{pmatrix}_{m\times 1} \quad \text{or} \quad A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix}_{1\times n}.$$

Definition. The matrix is called *zero (or null) matrix* if all its elements are equal to zero and designated by O.

$$O=\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Definition. The matrix is called *square matrix*, if $m=n$. The number of rows is considered to be *the order of this matrix*.

Definition. The set of elements a_{11}, a_{22}, …, a_{nn} makes up *the main diagonal* of the matrix. The set of the elements $a_{1\,n}$, $a_{2\,n-1}$, …, a_{n1} makes up *the secondary diagonal* of the matrix. For example,



Definition. Two matrices of identical sizes are called *equal* if their corresponding elements are equal, i.e. $A=B$ if $a_{ij}=b_{ij}$, $i=\overline{1,m}$, $j=\overline{1,n}$.

Definition. A square matrix is called *diagonal* if all its elements except diagonal ones are equal to zero.

$$D=\begin{pmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}$$

Definition. If all the diagonal elements of the diagonal matrix are equal to 1 then the matrix is called a *unit (identity) matrix* and designated as *E*:

$$E=\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1\end{pmatrix}.$$

Definition. If all the elements of a matrix located below (above) the main diagonal are equal to zero then the matrix is called *an upper (lower) triangular matrix*

For example, $A=\begin{pmatrix}2 & 5 & 1 & 0\\ 0 & -4 & 2 & 3\\ 0 & 0 & -1 & -7\\ 0 & 0 & 0 & 11\end{pmatrix}$ is the upper triangular matrix.

Definition. If $a_{ij}=a_{ji}$, then the matrix *A* is called *a symmetrical matrix*.

For example, $A=\begin{pmatrix}-1 & 8 & -2 & 0\\ 8 & 3 & 4 & 6\\ -2 & 4 & 10 & -5\\ 0 & 6 & -5 & -9\end{pmatrix}$.

2. Basic operations with matrices and properties of these operations.

The addition and subtraction matrices

Definition. Suppose that both matrices $A=\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & & \vdots\\ a_{m1} & \cdots & a_{mn}\end{pmatrix}$ and $B=\begin{pmatrix}b_{11} & \cdots & b_{1n}\\ \vdots & & \vdots\\ b_{m1} & \cdots & b_{mn}\end{pmatrix}$ have *m* rows and *n* columns. Then we write $A\pm B=\begin{pmatrix}a_{11}\pm b_{11} & \cdots & a_{1n}\pm b_{1n}\\ \vdots & & \vdots\\ a_{m1}\pm b_{m1} & \cdots & a_{mn}\pm b_{mn}\end{pmatrix}$ and call this *the sum (or difference) of the two matrices A and B.*

Example 2. If $A=\begin{pmatrix}13 & 30\\ 8 & 15\end{pmatrix}$ and $B=\begin{pmatrix}7 & 35\\ 8 & 4\end{pmatrix}$ what are $A+B$ and $A-B$?

Solution. Matrices *A* and *B* have the same order 2×2, therefore, we can add them together, or subtract:

$$C = A + B = \begin{pmatrix} 13 & 30 \\ 8 & 15 \end{pmatrix} + \begin{pmatrix} 7 & 35 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 13+7 & 30+35 \\ 8+8 & 15+4 \end{pmatrix} = \begin{pmatrix} 20 & 65 \\ 16 & 19 \end{pmatrix},$$

$$C = A - B = \begin{pmatrix} 13 & 30 \\ 8 & 15 \end{pmatrix} - \begin{pmatrix} 7 & 35 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 13-7 & 30-35 \\ 8-8 & 15-4 \end{pmatrix} = \begin{pmatrix} 6 & -5 \\ 0 & 11 \end{pmatrix}.$$

Example 3. We do not have a definition for "adding" the matrices $A = \begin{pmatrix} 5 & 4 & 12 & 7 \\ 10 & 12 & 9 & 14 \end{pmatrix}$ and $B = \begin{pmatrix} 13 & 30 \\ 8 & 15 \end{pmatrix}$, because matrices have the different order.

2. *The multiplication of a matrix by a scalar value*

Definition. Suppose that the matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ has m rows and n columns and $\alpha \in R$. Then we write $\alpha A = \begin{pmatrix} \alpha a_{11} & \cdots & \alpha a_{1n} \\ \vdots & & \vdots \\ \alpha a_{m1} & \cdots & \alpha a_{mn} \end{pmatrix}$ and call this the product of the matrix A by the scalar α.

The operations of matrix addition (or subtraction) and matrix multiplication by some number satisfy the following laws:

1) $A + B = B + A$
2) $(A + B) + C = A + (B + C)$
3) $A + O = A$
4) $(\alpha \cdot \beta)A = \alpha(\beta \cdot A)$
5) $(\alpha \pm \beta)A = \alpha A \pm \beta A$
6) $\alpha(A \pm B) = \alpha A \pm \alpha B$

Example 4. Calculate the matrix $C = 3B - 2A$, if $A = \begin{pmatrix} 2 & -4 & 1 \\ 3 & 0 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 6 & -1 \end{pmatrix}$.

Solution. Matrices *A* and *B* have the same order 2×3, therefore, we can obtain $C = 3B - 2A$. The entry $2A$ is multiplication the matrix A by 2:

$$2A = \begin{pmatrix} 2\cdot 2 & 2\cdot(-4) & 2\cdot 1 \\ 2\cdot 3 & 2\cdot 0 & 2\cdot 7 \end{pmatrix} = \begin{pmatrix} 4 & -8 & 2 \\ 6 & 0 & 14 \end{pmatrix}.$$

$3B$ like $2A$: $3B=\begin{pmatrix} 3\cdot5 & 3\cdot1 & 3\cdot2 \\ 3\cdot3 & 3\cdot6 & 3\cdot(-1) \end{pmatrix}=\begin{pmatrix} 15 & 3 & 6 \\ 9 & 18 & -3 \end{pmatrix}$. Then

$$C=3B-2A=\begin{pmatrix} 15 & 3 & 6 \\ 9 & 18 & -3 \end{pmatrix}-\begin{pmatrix} 4 & -8 & 2 \\ 6 & 0 & 14 \end{pmatrix}=\begin{pmatrix} 11 & 11 & 4 \\ 3 & 18 & -17 \end{pmatrix}.$$

3. *Operation of multiplying a matrix by a matrix*

Let two matrices $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{p\times q}$ be given and the number of columns at the first matrix be equal to the number of rows of the second one, i.e. $n=p$. In this case we can define the operation of multiplying the matrix A by the matrix B. The matrix $C=A\cdot B$ of the size $m\times q$, which elements are calculated according to the following rule

$$c_{ij}=\sum_{k=1}^{n} a_{ik}\cdot b_{kj}=a_{i1}\cdot b_{1j}+\ldots+a_{in}\cdot b_{nj},\ i=\overline{1,m},\ j=\overline{1,q},$$

is called *a product of the matrix A by the matrix B*.

Remark: in a general case the multiplication of matrices does not possess a commutative property, i. e. $A\cdot B$ and $B\cdot A$ are not equal to each other or one of the products does not exist.

The basic properties of a matrix multiplication:

1) $\alpha(AB)=(\alpha A)B=A(\alpha B)$
2) $(A+B)C=AC+BC$, $C(A+B)=CA+CB$
3) $A(BC)=A(BC)$
4) $AE=EA=A$
5) $AO=OA=O$

Remark: Unlike the operations of the addition and subtraction matrices, the operation of division of two matrices is not defined.

Example 5. Multiply the matrices: $A=\begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}_{3\times4}$ and

$$B=\begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & -2 \\ 3 & 1 \end{pmatrix}_{4\times2}.$$

Solution. Note that A is a 3×4 matrix and B is a 4×2 matrix and the number of columns of the matrix A is equal to the number of rows of the matrix B, so that the product $C = A\cdot B$ is a 3×2 matrix.

Let us calculate the product

$$C = A\cdot B = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}.$$

Consider first of all c_{11}. To calculate this, we need the 1-st row of A and the 1-st column of B, so let us cover up all the unnecessary information, so that

$$C = \begin{pmatrix} 2 & 4 & 3 & -1 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \cdot \begin{pmatrix} 1 & \times \\ 2 & \times \\ 0 & \times \\ 3 & \times \end{pmatrix} = \begin{pmatrix} c_{11} & \times \\ \times & \times \\ \times & \times \end{pmatrix}.$$

From the definition of the product of the matrix A by the matrix B, we have

$$c_{11} = a_{11}\cdot b_{11} + a_{12}\cdot b_{21} + a_{13}\cdot b_{31} + a_{14}\cdot b_{41} = 2\cdot 1 + 4\cdot 2 + 3\cdot 0 + (-1)\cdot 3 =$$
$$= 2 + 8 + 0 - 3 = 7$$

(multiply elements standing in the row 1 of A by the corresponding elements of the column 1 of the B and then summarize the obtained products).

Consider next c_{12}. To calculate this, we need the 1-st row of A and the 2-nd column of B, so let us cover up all the unnecessary information, so that

$$C = \begin{pmatrix} 2 & 4 & 3 & -1 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \cdot \begin{pmatrix} \times & 4 \\ \times & 3 \\ \times & -2 \\ \times & 1 \end{pmatrix} = \begin{pmatrix} \times & c_{12} \\ \times & \times \\ \times & \times \end{pmatrix}.$$

From the definition, we have

$$c_{12} = a_{11}\cdot b_{12} + a_{12}\cdot b_{22} + a_{13}\cdot b_{32} + a_{14}\cdot b_{42} = 2\cdot 4 + 4\cdot 3 + 3\cdot(-2) + (-1)\cdot 1 =$$
$$= -8 + 12 - 6 - 1 = 13.$$

According to the definition, we have the rest of the elements:

$$c_{21} = a_{21}\cdot b_{11} + a_{22}\cdot b_{21} + a_{23}\cdot b_{31} + a_{24}\cdot b_{41} = 3\cdot 1 + 1\cdot 2 + 5\cdot 0 + 2\cdot 3 =$$
$$= 3 + 2 + 0 + 6 = 11,$$
$$c_{22} = a_{21}\cdot b_{12} + a_{22}\cdot b_{22} + a_{23}\cdot b_{32} + a_{24}\cdot b_{42} = 3\cdot 4 + 1\cdot 3 + 5\cdot(-2) + 2\cdot 1 =$$
$$= 12 + 3 - 10 + 2 = 7,$$
$$c_{31} = a_{31}\cdot b_{11} + a_{32}\cdot b_{21} + a_{33}\cdot b_{31} + a_{34}\cdot b_{41} = (-1)\cdot 1 + 0\cdot 2 + 7\cdot 0 + 6\cdot 3 =$$
$$= -1 + 0 + 0 + 18 = 17,$$

$c_{32} = a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} + a_{34} \cdot b_{42} = (-1) \cdot 4 + 0 \cdot 3 + 7 \cdot (-2) + 6 \cdot 1 =$
$= -4 + 0 - 14 + 6 = -12.$

Therefore we conclude that $C = A \cdot B = \begin{pmatrix} 7 & 13 \\ 11 & 7 \\ 17 & -12 \end{pmatrix}$.

Example 6. Consider the same matrices in example 5.

Note that B is a 4×2 matrix and A is a 3×4 matrix, so that we do not have a definition for the product $B \cdot A$, because the number of columns of the matrix A is not equal to the number of rows of the matrix B.

4. *Transposition matrix*

Definition. Consider the $m \times n$ matrix $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$. By the transpose A^T of *A*, we mean *the transposed matrix* $A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{pmatrix}$ obtained from *A* by transposing rows and columns.

Example 7. Consider the matrix $A = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}$. Then $A^T = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 1 & 0 \\ 3 & 5 & 7 \\ -1 & 2 & 6 \end{pmatrix}$. Note that A is a 3×4 matrix and A^T is a 4×3 matrix.

4. *Raising to power*

For the $n \times n$ matrix and a positive integer m, *the* m*-th power of* A is $A^m = \underbrace{A \cdot A \cdot \ldots \cdot A}_{m \text{ copies of } A}$. It is also convenient to define $A^0 = E$.

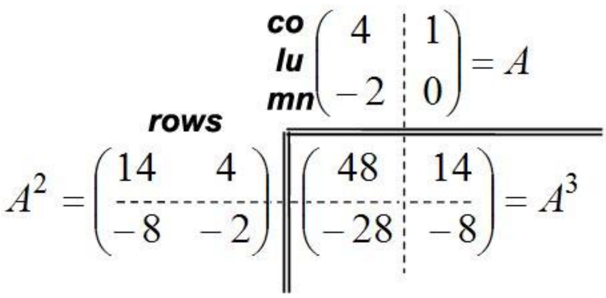
Example 5. A matrix is given: $A = \begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix}$. Find A^2 and A^3.

Solution. This operation can be applied to the square matrix, i.e. for the $n \times n$ matrix. Find A^2 and A^3. By the definition of the m-th power of A we have $A^2 = \underbrace{A \cdot A}_{2 \text{ copies of } A}$ and $A^3 = \underbrace{A \cdot A \cdot A}_{3 \text{ copies of } A}$ or $A^3 = A^2 \cdot A = A \cdot A^2$.

We obtain:

$$A^2 = A \cdot A = \begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 4 \cdot 4 + 1 \cdot (-2) & 4 \cdot 1 + 1 \cdot 0 \\ -2 \cdot 4 + 0 \cdot (-2) & -2 \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 14 & 4 \\ -8 & -2 \end{pmatrix}.$$

The matrix A^3 can be obtained by another way:



Each element of A^3 is a result of the matrix product of a row of A^2 and a column of A, which is found on intersection of the corresponding row and column.

For instance, the element -28 represents the result of the matrix product of the 2-nd row of A^2 and the 1-st column of A, i.e. it is $(-8) \cdot 4 + (-2) \cdot (-2)$.

3. A definition of the determinant, rules of calculation and properties of determinants.

Any square matrix A can be associated with some value (number) called its *determinant* and designated as $\det A$ or $|A|$.

For example, a determinant of a matrix of the 2-nd order is calculated according to the following formula:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}.$$

Example 8. Calculate the determinant of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

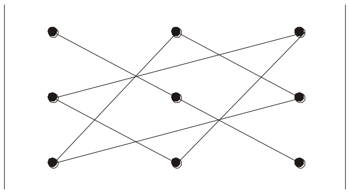
Solution. $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$.

A determinant of the 3-rd order is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} + a_{21} \cdot a_{32} \cdot a_{13} + a_{31} \cdot a_{12} \cdot a_{23} -$$

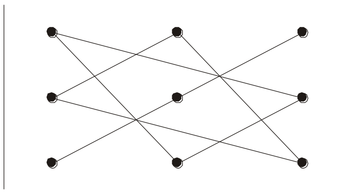
$$- a_{13} \cdot a_{22} \cdot a_{31} - a_{21} \cdot a_{12} \cdot a_{33} - a_{32} \cdot a_{23} \cdot a_{11}.$$

To memorize the last formula ***the rule of triangle (or Sarrus formula)*** is often used. It says: a product of elements from the main diagonal and 2 products of elements forming in a matrix isosceles triangles with their bases parallel to the main diagonal are taken with the sign *plus:*



+

a product of elements from the secondary diagonal and 2 products of elements forming triangles with their bases parallel to the secondary diagonal are taken with the sign *minus:*



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Example 9. Calculate the determinant of the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$.

Solution. Calculate the determinant, using the rule of triangle:

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = 1 \cdot (-2) \cdot 1 + 2 \cdot 3 \cdot 3 + 0 \cdot 1 \cdot 1 - 1 \cdot (-2) \cdot 3 - 0 \cdot 1 \cdot 2 - 3 \cdot 1 \cdot 1 =$$

$$= -2 + 18 + 6 - 3 = 19.$$

Basic properties of determinants:

If we add to all elements of a row (column) of the determinant the corresponding elements of other row (column) multiplied by some number then the value of the determinant will not change, i. e.

$$\begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ where } k \in R.$$

Example 10. Check the property:

$$\begin{vmatrix} 3 & 2 & 5 \\ 4 & 1 & 8 \\ -1 & 2 & 4 \end{vmatrix} = 12+40-16+5-32-48=-39.$$

Let's transpose rows and columns and obtain:

$$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 1 & 2 \\ 5 & 8 & 4 \end{vmatrix} = 12-16+40+5-32-48=-39.$$

2. *Transposing of two any rows (columns) the determinant changes its sign. For example,* $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

Example 1 Check the property:

$$\begin{vmatrix} 5 & 1 & 0 \\ 2 & 5 & 6 \\ 3 & 2 & -1 \end{vmatrix} = -25+18+0-0+2-60=-65.$$

Transpose the first row and the second one and obtain:

$$\begin{vmatrix} 2 & 5 & 6 \\ 5 & 1 & 0 \\ 3 & 2 & -1 \end{vmatrix} = -2+60+0-18+25-0=65.$$

3. *If any row (column) of the determinant completely consists of zeros then the determinant is equal to zero.*

Example 12. Check the property:

$$\begin{vmatrix} 3 & 5 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 0 \end{vmatrix} = 0+0+0-0-0-0=0.$$

4. *A common factor of all elements of a row (column) can be taken out of the determinant. For example,* $\begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, *where* $k \in R$.

Example 13. Check the property.

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ 6 & 5 & 4 \end{vmatrix} = 16 + 60 - 6 - 36 + 16 - 10 = 40.$$

The first column has a common factor 2. We take it out of the determinant and obtain

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ 6 & 5 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & -1 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 2(8 + 30 - 3 - 18 + 8 - 5) = 2 \cdot 20 = 40.$$

5. *A determinant does not change its value at the transposition of the matrix, i. e.*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \text{ or } \det A = \det A^T.$$

Example 14. Check the property.

$$\begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 1 & 2 & 3 \end{vmatrix} = 9 + 20 + 4 - 5 + 6 + 24 = 58.$$

For example, calculate this determinant by adding to all elements of row 3 of the determinant the corresponding elements of row 1 multiplied by 2. Thus

$$\begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 1 + 3 \cdot 2 & 2 + (-1) \cdot 2 & 3 + 5 \cdot 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ 7 & 0 & 13 \end{vmatrix} =$$

$$= 39 + 28 - 35 + 26 = 58.$$

6. *The determinant possessing two identical or proportional rows (columns) is equal to zero.*

Example 15. Check the property.

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 5 & 1 & 3 \end{vmatrix} = 18 + 8 + 120 - 120 - 18 - 8 = 0.$$

The determinant is equal to zero, because the first row and the second one are proportional rows: $\frac{a_{21}}{a_{11}}=\frac{a_{22}}{a_{12}}=\frac{a_{23}}{a_{13}}=2$.

Minors and algebraic cofactors.

Definition. *The minor* M_{ij} of the element a_{ij} of the determinant of n-th order is called the determinant of the $(n-1)$-th order obtained from the given one by crossing the row and the column on which intersection the element a_{ij} is located.

Definition. *The algebraic cofactor* A_{ij} (or *the cofactor*) of the element a_{ij} of the determinant is called the following value $A_{ij}=(-1)^{i+j}\cdot M_{ij}$.

Theorem *(concerning decomposition of a determinant in its rows or columns)*. The sum of products of elements of any row (column) by their cofactors is equal to this determinant, i. e.

$$|A|=\sum_{k=1}^{n}a_{kj}A_{kj}\,,\ \ j=\overline{1,n}.$$

Example 10. Calculate the determinant on the base of the rule of triangle and check the result using the theorem concerning decomposition of the determinant in its 1-st row.

Solution. Calculate the determinant, using the rule of triangle:

$$\begin{vmatrix}1 & 2 & 1\\ 0 & -2 & 3\\ 3 & 1 & 1\end{vmatrix}=1\cdot(-2)\cdot1+2\cdot3\cdot3+0\cdot1\cdot1-1\cdot(-2)\cdot3-0\cdot1\cdot2-3\cdot1\cdot1=$$
$$=-2+18+6-3=19.$$

Checking the determinant value by decomposing in the 1-st row

$$\begin{vmatrix}1 & 2 & 1\\ 0 & -2 & 3\\ 3 & 1 & 1\end{vmatrix}=a_{11}\cdot A_{11}+a_{12}\cdot A_{12}+a_{13}\cdot A_{13}=1\cdot(-1)^{1+1}\cdot M_{11}+2\cdot(-1)^{1+2}\cdot M_{12}+$$
$$+1\cdot(-1)^{1+3}\cdot M_{13}=1\cdot1\cdot\begin{vmatrix}-2 & 3\\ 1 & 1\end{vmatrix}+2\cdot(-1)\cdot\begin{vmatrix}0 & 3\\ 3 & 1\end{vmatrix}+1\cdot1\cdot\begin{vmatrix}0 & -2\\ 3 & 1\end{vmatrix}=(-2-3)-$$
$$-2\cdot(0-9)+(0-(-6))=-5+18+6=19.$$

4. Notion of an inverse matrix, calculation of an inverse matrix by cofactors and elementary row operations.

Definition. The square matrix A is called *invertible or nonsingular*, if $\det A\neq0$, otherwise it is called *not invertible or singular*.

Definition. The matrix A^{-1} is called *inverse* relatively to the square nonsingular matrix A if $A\cdot A^{-1}=A^{-1}\cdot A=E$, where E is the unit matrix.

The square matrix can have an inverse matrix if its determinant is non-zero, i. e. A is a nonsingular matrix.

4. Finding inverses by cofactors

An inverse matrix can be obtained according to the following formula:

$$A^{-1}=\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix},$$

where A_{ij} are cofactors of the elements a_{ij} of the matrix A, $i, j=\overline{1,n}$.

Example 1 Find the inverse matrix for the following matrix $A=\begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$.

$$A=\begin{pmatrix} 1_{11} & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$$

Solution. Let us find the determinant of the given matrix:

$$|A|=\begin{vmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{vmatrix}\begin{matrix} 1 & 1 \\ 3 & 0 \\ -2 & 3 \end{matrix}=1\cdot 0\cdot 0+1\cdot 3\cdot(-2)+3\cdot 3\cdot 2-$$

$$-0\cdot(-2)\cdot 2-3\cdot 3\cdot 1-3\cdot 1\cdot 0=0-6+18-0-9-0=3\neq 0.$$

Its determinant is non-zero. Find the cofactors for the elements of the matrix A:

$A_{11}=(-1)^{1+1}\cdot\begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix}=1\cdot(0-9)=-9$, $A_{12}=(-1)^{1+2}\cdot\begin{vmatrix} 3 & 3 \\ -2 & 0 \end{vmatrix}=(-1)\cdot(0-(-6))=-6$,

$A_{13}=(-1)^{1+3}\cdot\begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix}=1\cdot(9-0)=9$, $A_{21}=(-1)^{2+1}\cdot\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}=(-1)\cdot(0-6)=6$,

$A_{22}=(-1)^{2+2}\cdot\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix}=1\cdot(0-(-4))=4$,

$A_{23}=(-1)^{2+3}\cdot\begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix}=(-1)\cdot(3-(-2))=-5$,

$$A_{31}=(-1)^{3+1}\cdot\begin{vmatrix}1&2\\0&3\end{vmatrix}=1\cdot(3-0)=3,$$

$$A_{32}=(-1)^{3+2}\cdot\begin{vmatrix}1&2\\3&3\end{vmatrix}=(-1)\cdot(3-6)=3,$$

$$A_{33}=(-1)^{3+3}\cdot\begin{vmatrix}1&1\\3&0\end{vmatrix}=1\cdot(0-3)=-3.$$

We obtain the inverse matrix:

$$A^{-1}=\frac{1}{\Delta}\cdot\begin{pmatrix}A_{11}&A_{21}&A_{31}\\A_{12}&A_{22}&A_{32}\\A_{13}&A_{23}&A_{33}\end{pmatrix}=\frac{1}{3}\cdot\begin{pmatrix}-9&6&3\\-6&4&3\\9&-5&-3\end{pmatrix}=\begin{pmatrix}-3&2&1\\-2&4/3&1\\3&-5/3&-1\end{pmatrix}.$$

Checking the condition $A\cdot A^{-1}=E$:

$$A\cdot A^{-1}=\begin{pmatrix}1&1&2\\3&0&3\\-2&3&0\end{pmatrix}\cdot\begin{pmatrix}-3&2&1\\-2&4/3&1\\3&-5/3&-1\end{pmatrix}=$$

$$=\begin{pmatrix}-3-2+6&2+4/3-10/3&1+1-2\\-9+0+9&6+0-5&3+0-3\\6-6+0&-4+4+0&-2+3+0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=E.$$

5. Notion of a matrix rank and its calculation.

The rank of the $m\times n$-matrix A is the highest order of its non-zero minor and denoted by $r(A)$, $rg(A)$ or $rang\,A$.

For a non-zero matrix $0\le rang\,A\le\min\{m,n\}$. If the $rang\,A=k$, then any non-zero minor of the k-th order is called *basic.*

To find a rank of a matrix we can use elementary row operations to reduce the given matrix to the triangular form:

$$\begin{pmatrix}a_{11}&a_{12}&\dots&a_{1n}\\0&a_{22}&\dots&a_{2n}\\\dots&\dots&\dots&\dots\\0&0&\dots&a_{nn}\end{pmatrix}\qquad(1)$$

or to the truncated-triangular form:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} & a_{1\,k+1} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2k} & a_{2\,k+1} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{kk} & a_{k\,k+1} & \dots & a_{kn} \end{pmatrix} \qquad (2)$$

then the number of non-zero rows of the transformed matrix defines the rank of the initial matrix.

Elementary row operations (or elementary transformations) are:

1) interchanging (exchanging) two different rows;
2) adding a multiple of one row to another row;
3) multiplying one row by a non-zero constant;
4) crossing out one of the same row;
5) crossing out of zero row.

Example 12. Calculate the rank of the matrix: $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & -3 & 2 \\ 3 & 3 & 9 \end{pmatrix}$.

Solution. Let us exchange the 1-st and the 2-nd rows:

$$A \sim \begin{pmatrix} 1 & -3 & 2 \\ 2 & -2 & 3 \\ 3 & 3 & 9 \end{pmatrix} \sim \begin{bmatrix} \\ [1]\cdot(-2)+[2] \\ [1]\cdot(-3)+[3] \end{bmatrix} \sim \begin{pmatrix} 1 & -3 & 2 \\ 0 & -4 & -1 \\ 0 & 12 & 3 \end{pmatrix} \sim \begin{bmatrix} \\ [2]:(-1) \\ [3]:3 \end{bmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & -3 & 2 \\ 0 & 4 & 1 \\ 0 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 2 \\ 0 & 4 & 1 \end{pmatrix}.$$

The number of non-zero rows of the transformed matrix equivalent to the initial one is 2. Therefore $rang\, A = 2$.

Theoretical questions

1. What do you call the matrix?
2. What is the size of a matrix or its order?
3. What matrices do you know? Call all the types of matrices.
4. What matrix is called
   a) transposed? b) singular? c) nonsingular?
5. What matrices can be:
a) added? b) subtracted?
6. How can a matrix be multiplied by a scalar value?

7. What matrices can be multiplied? What is the rule of multiplying a matrix by a matrix?
8. How many operations on matrices do you know?
9. What matrix is called inverse to a given matrix? Does an inverse matrix exist for any matrix?
10. What methods are used for finding an inverse matrix?
11. What transformations are called elementary?
12. What do you call 2-nd and 3-rd order determinant?
13. What is the rule of triangle (or Sarrus formula)?
14. What are the basic properties of determinants?
15. What do you call minor and algebraic cofactor of any element?
16. How can the determinant order be defined?
17. What ways of calculating determinants do you know?
18. What is a rank of a matrix?
19. What method of finding a rank of a matrix do you know?