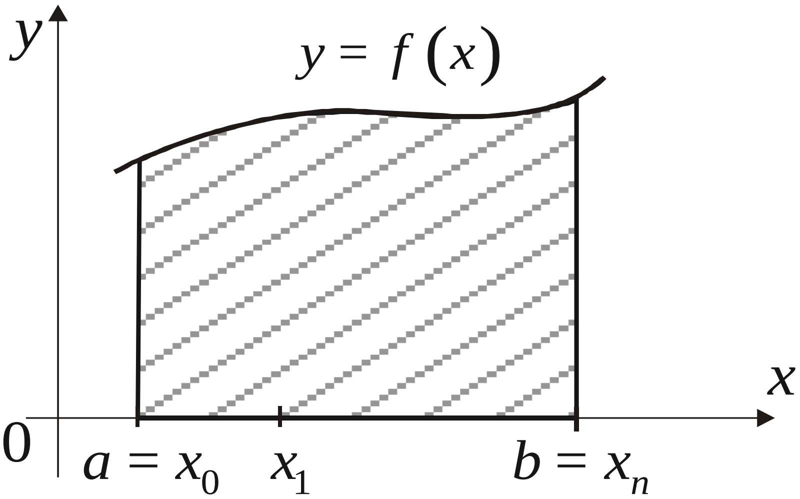
Theme 5. The definite integral and its application

1. Definite integral. Formula of Newton-Leibnitz

Let the function $f(x)$ be determined on the interval $[a,b]$. Let us divide the interval $[a,b]$ into n subintervals by n points $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$.

On each subinterval $[x_{i-1}, x_i]$ we choose some point ξ_i and compound the sum $S_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + ... + f(\xi_n)\Delta x_n$, where $\Delta x_i = x_i - x_{i-1}$ is the length of each interval.



The definite integral of the function $f(x)$ on the interval $[a,b]$ is called the limit

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)\,dx,$$

where a is the upper limit, b is the lower limit of integration.

If $F(x)$ is an antiderivative for a continuous function $f(x)$ on the interval $[a,b]$ then the formula of Newton-Leibnitz is true

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(в) - F(a).$$

Example 1. Calculate the integral:

a) $\int_1^4 5x^4 dx$.   b) $\int_1^2 \left(x^2 - 3x + 1\right)dx.$ c) $\int_{-3}^{-2} \frac{dx}{1-x^2}$.

Solution. Apply the formula of Newton-Leibnitz:

a) $\int_1^4 5x^4dx = 5\cdot\frac{x^5}{5}\Big|_1^4 = x^5\Big|_1^4 = 1024-1=1023.$

b)

$$\int_1^2\left(x^2-3x+1\right)dx=\int_1^2 x^2dx-3\int_1^2 xdx+\int_1^2 dx=\frac{x^3}{3}\Big|_1^2-3\cdot\frac{x^2}{2}\Big|_1^2+$$

$$+x\Big|_1^2=\frac{8}{3}-\frac{1}{3}-\frac{3}{2}(4-1)+2-1=\frac{7}{3}-\frac{9}{2}+1=-\frac{7}{6}.$$

c)

$$\int_{-3}^{-2}\frac{dx}{1-x^2}=\int_{-2}^{-3}\frac{dx}{x^2-1}=\frac{1}{2}ln\left|\frac{x-1}{x+1}\right|\Bigg|_{-2}^{-3}=$$

$$=\frac{1}{2}\left(ln\left|\frac{-3-1}{-3+1}\right|-ln\left|\frac{-2-1}{-2+1}\right|\right)=\frac{1}{2}\left(ln2-ln3\right)=\frac{1}{2}ln\frac{2}{3.}$$

2. The properties of the definite integral:

1) The integral changes its sign on opposite if the limits of integration are replaced:

$$\int_a^b f(x)dx=-\int_b^a f(x)dx.$$

2) If the lower and the upper limits of integration are equal, then the integral is equal to zero: $\int_a^a f(x)dx=0.$

3) The additivity of the integral: $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^в f(x)dx$, where $c\in[a,b]$.

4) The integral of the algebraic sum of functions equals the algebraic sum of integrals: $\int_a^b\left(f_1(x)\pm f_2(x)\right)dx=\int_a^b f_1(x)dx\pm\int_a^b f_2(x)dx$.

5) It is possible to take the constant C outside the integral:

$$\int_a^b C\cdot f(x)dx=C\cdot\int_a^b f(x)dx.$$

6) If $a<b$ on the segment $[a,b]$ and $f(x)\le\varphi(x)$, then

$$\int_a^b f(x)dx \le \int_a^b \varphi(x)dx.$$

7) The mean-value theorem. If the function $y = f(x)$ is continuous on the interval $[a,b]$, where $a < b$, then there exists a value $\xi \in [a,b]$, such that

$$\int_a^b f(x)dx = f(\xi)(в - a).$$

3. Method of substitution in a definite integral

<u>Theorem.</u> Let the integral $\int_a^b f(x)dx$ is given, where the function $f(x)$ be continuous on the interval $[a,b]$. Let's change a variable t by the formula $x = \varphi(t)$. If

1) $\varphi(\alpha) = a$, $\varphi(\beta) = b$;

2) $\varphi(t)$ and $\varphi'(t)$ are continuous on the interval $[\alpha, \beta]$;

3) $f(\varphi(t))$ is defined and continuous on the interval $[\alpha, \beta]$, then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

Example 2. Calculate: $\int_0^1 \frac{xdx}{x^2 + 3x + 2}$.

Solution. Let's allocate a perfect square in the denominator:

$$x^2 + 3x + 2 = x^2 + 2 \cdot \frac{3}{2}x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}.$$

Let's make the substitution: $x + \frac{3}{2} = t, dx = dt$.

We change the limits of integration: if $x = 0$, then $t = \frac{3}{2}$; if $x = 1$, then $t = \frac{5}{2}$. After this we have

$$\int_0^1 \frac{xdx}{x^2+3x+2} = \int_0^1 \frac{xdx}{\left(x+\frac{3}{2}\right)^2-\frac{1}{4}} = \int_{\frac{3}{2}}^{\frac{5}{2}} \frac{\left(t-\frac{3}{2}\right)dt}{t^2-\frac{1}{4}} = \frac{1}{2}\int_{\frac{3}{2}}^{\frac{5}{2}} \frac{2tdt}{t^2-\frac{1}{4}} -$$

$$-\frac{3}{2}\int_{\frac{3}{2}}^{\frac{5}{2}} \frac{dt}{t^2-\frac{1}{4}} = \frac{1}{2}ln\left|t^2-\frac{1}{4}\right|\Bigg|_{\frac{3}{2}}^{\frac{5}{2}} - \frac{3}{2}\cdot\frac{1\cdot 2}{2\cdot 1}ln\left|\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right|\Bigg|_{\frac{3}{2}}^{\frac{5}{2}} =$$

$$= \frac{1}{2}\left(ln6 - ln2\right) - \frac{3}{2}\left(ln\frac{2}{3} - ln\frac{1}{2}\right) = \frac{1}{2}ln\frac{6}{2} - \frac{3}{2}ln\frac{2\cdot 2}{3\cdot 1} =$$

$$= \frac{1}{2}ln3 - \frac{3}{2}ln\frac{4}{3} = ln\frac{3^{\frac{1}{2}}\cdot 3^{\frac{3}{2}}}{4^{\frac{3}{2}}} = ln\frac{9}{8}.$$

Example 3. Calculate $\int_{-3}^{3} x^2\sqrt{9-x^2}dx$.

Solution. We use the trigonometrical substitution $x = 3cos\,t$, therefore $dx = -3sin\,tdt$. Let's find new limits of integration: $\frac{x}{3} = cos\,t$ or $t = arccos\frac{x}{3}$.

If $x=-3$, then $t = arccos(-1) = \pi$. If $x = 3$, then $t = arccos1 = 0$.

We have

$$\int_{-3}^{3} x^2\sqrt{9-x^2}dx = \int_{\pi}^{0} 9cos^2 t\sqrt{9-9cos^2 t}\left(-3sin\,t\right)dt =$$

$$= -81\int_{\pi}^{0} cos^2 t\,sin^2 tdt = 81\int_0^{\pi}\frac{4}{4}cos^2 t\,sin^2 tdt = \frac{81}{4}\int_0^{\pi} sin^2 2tdt =$$

$$= \frac{81}{4}\int_0^{\pi}\frac{1-cos4t}{2}dt = \frac{81}{4}\left(\int_0^{\pi}\frac{1}{2}dt - \frac{1}{2}\int_0^{\pi}cos4tdt\right) =$$

$$= \frac{81}{4}\left(\frac{t}{2}\Big|_0^{\pi} - \frac{1}{2}\frac{sin4t}{4}\Big|_0^{\pi}\right) = \frac{81}{4}\left(\frac{\pi}{2} - 0 - \frac{1}{2}\frac{sin4\pi}{4} + \frac{1}{2}\frac{sin0}{4}\right) =$$

$$= \frac{81}{4}\cdot\frac{\pi}{2} = \frac{81}{8}\pi.$$

4. Method of integration by parts in a definite integral

Let the functions $u=u(x)$ and $v=v(x)$ have continuous derivatives on the interval $[a,b]$. Then $\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$, where $uv\Big|_a^b = u(b)v(b)-u(a)v(a)$.

Example 4. Calculate $\int_0^1 xe^{-x}dx$.

Solution. We use the method of integration by parts:

$$u=x,\ \ dv=e^{-x}dx,$$

then

$$du=dx,\ v=\int e^{-x}dx=-e^{-x}.$$

We have

$$\int_0^1 xe^{-x}dx = x\left(-e^{-x}\right)\Big|_0^1 - \int_0^1\left(-e^{-x}\right)dx = -e^{-1}+\int_0^1 e^{-x}dx =$$

$$=-\frac{1}{e}-e^{-x}\Big|_0^1 = -\frac{1}{e}-\left(e^{-1}-e^{0}\right) = -\frac{1}{e}-\frac{1}{e}+1=\frac{e-2}{e}.$$

Example 5. Calculate $\int_0^{e-1} ln(x+1)dx$.

Solution. We use the method of integration by parts:

$$u=ln(x+1),\ dv=dx,$$

then

$$du=\frac{1}{x+1}dx,\ v=\int dx = x.$$

We have

$$\int_0^{e-1} ln(x+1)dx = x\,ln(x+1)\Big|_0^{e-1} - \int_0^{e-1}\frac{xdx}{x+1}.$$

We allocate a whole part in the last integral and obtain

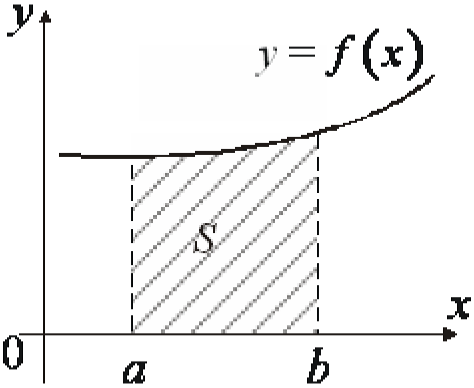
$$\int_0^{e-1} ln(x+1)\,dx = (e-1)\,ln\,e - \int_0^{e-1} \frac{x+1-1}{x+1}\,dx = e-1-$$

$$-\int_0^{e-1}\left(1-\frac{1}{x+1}\right)dx = e-1-\int_0^{e-1}dx+\int_0^{e-1}\frac{dx}{x+1}=$$

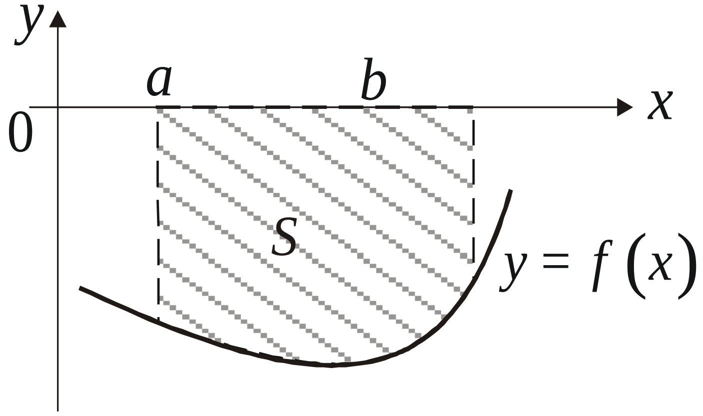
$$= e-1-x\Big|_0^{e-1}+ln\left|x+1\right|\Big|_0^{e-1} = e-1-(e-1)+ln\,e-ln1=1.$$

5. Application of definite integral for geometry problems calculation of the area of a plane figure

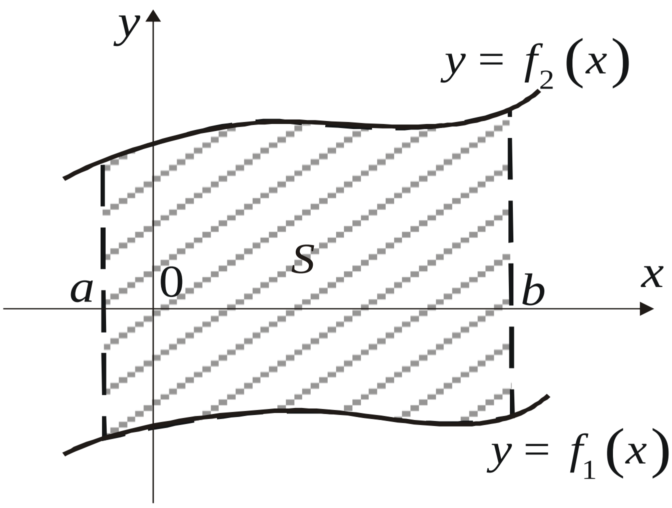
If $f(x)\geq 0\ \ \forall x\in[a,b]$ then $\int_a^b f(x)dx$ is numerically equal to the area of the curvilinear trapezoid with the base $[a,b]$, bounded by the straight lines $x=a$, $x=b$, $y=0$ and the curve $y=f(x)$: $S=\int_a^b f(x)dx$.



If the curvilinear trapezoid is bounded by the straight lines $x=a$, $x=b$, $y=0$ and the curve $y=f(x)$, $f(x)\leq 0$, then its area is $S=-\int_a^b f(x)dx$.



If $f_2(x) \geq f_1(x)$ then the area, bounded by these curves and the straight lines $x=a$ and $x=b$ equals $S=\int_a^b \left(f_2(x) - f_1(x) \right) dx$.



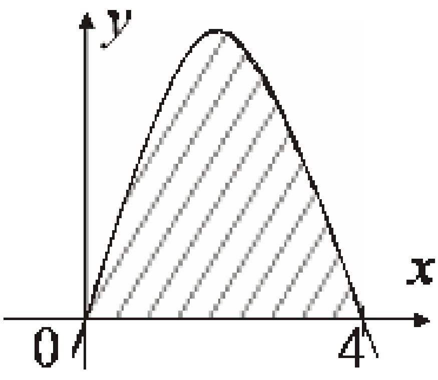
Example 6. Calculate the area of the figure bounded by the parabola $y=4x-x^2$ and the x-axis.

Solution. Find the limits of integration, i.e. the points of intersections of the given curves:

$$4x-x^2=0,\ x(4-x)=0,$$

thus

$$x=0,\ \ x=4.$$



Calculate the area of the figure by the formula:

$$S=\int_0^4 \left(4x-x^2\right)dx = 4\int_0^4 xdx - \int_0^4 x^2 dx = 2x^2\Big|_0^4 - \frac{x^3}{3}\Big|_0^4 =$$

$$=32-\frac{64}{3}=\frac{32}{3}.$$

Example 7. Calculate the area of the figure bounded by the curves $y=x^2-2$, $y=x$.

Solution. Let's find the limits of integration, i.e. the points of intersections of the given curves:

$$\begin{cases} y = x^2 - 2, \\ y = x \end{cases},$$

thus

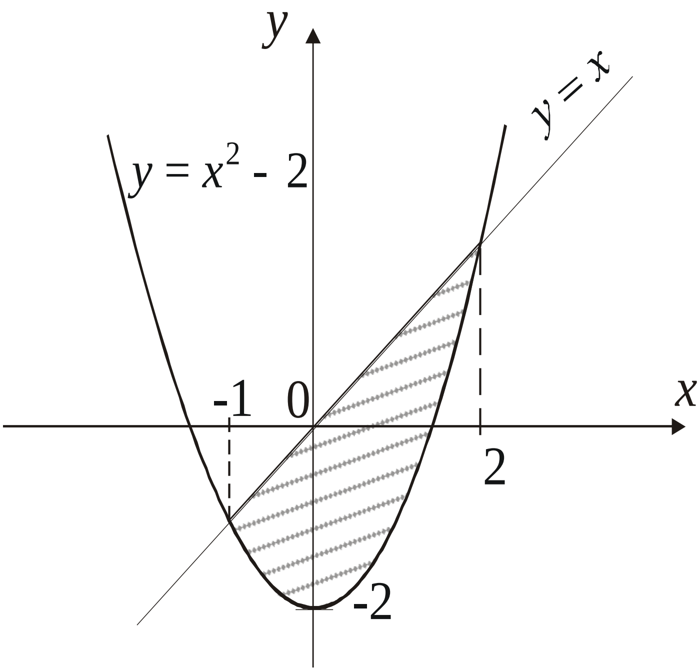
$$x^2 - 2 = x \text{ or } x^2 - x - 2 = 0.$$

Let's find the roots:

$$x = \frac{1 \pm \sqrt{1+8}}{2},$$

$$x_1 = -1, \quad x_2 = 2;$$

$y_1 = -1, \quad y_2 = 2.$



Since $f_2(x) \geq f_1(x)$ then the area, bounded by these curves is defined by the formula:

$$S = \int_{-1}^{2} \left(x - \left(x^2 - 2\right)\right)dx = \int_{-1}^{2} \left(x - x^2 + 2\right)dx = \left.\frac{x^2}{2}\right|_{-1}^{2} - \left.\frac{x^3}{3}\right|_{-1}^{2} +$$

$$+ \left.2x\right|_{-1}^{2} = \frac{1}{2}\left(4 - (-1)^2\right) - \frac{1}{3}\left(2^3 - (-1)^3\right) + 2\left(2 - (-1)\right) =$$

$$= \frac{3}{2} - 3 + 6 = \frac{9}{2}.$$

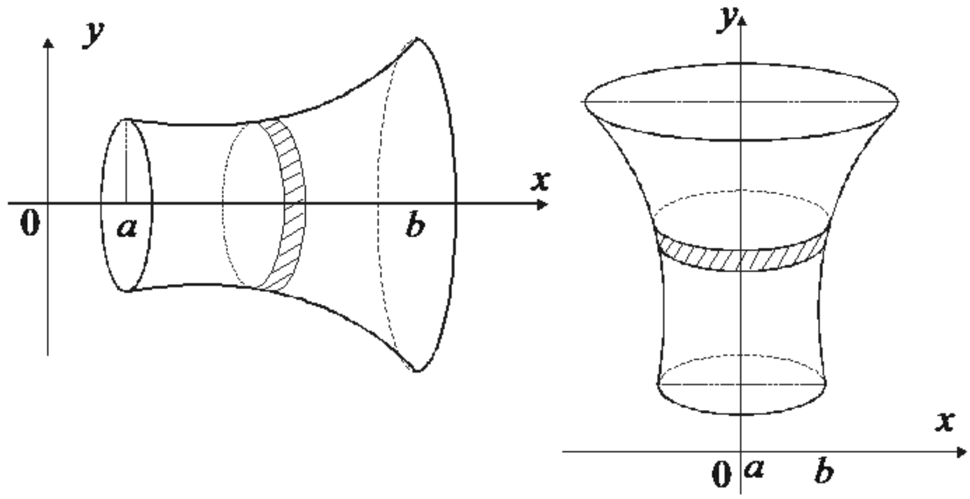
6. Application of definite integral for geometry problems calculation of the volumes of revolution solids

The volume of revolution solid by rotation of the curvilinear trapezoid, bounded by the curve $y = f(x)$ and the lines $x = a$, $x = b$ and $y = 0$ round the Ox-axis, can be obtain by the formula:

$$V_x = \pi \int_a^{в} y^2 dx$$

by rotation round the Oy-axis :

$$V_y = 2\pi \int_a^b xydx.$$



The volume of a solid formed by rotation of a figure, bounded by the lines $y_1 = f_1(x)$, $y_2 = f_2(x)$ $\left(0 \le f_1(x) \le f_2(x)\right)$, $x = a$, $x = b$, round the Ox-axis :

$$V_x = \pi \int_a^b \left(y_2^2 - y_1^2\right) dx = \pi \int_a^b \left(f_2^2(x) - f_1^2(x)\right) dx.$$

by rotation round the Oy-axis :

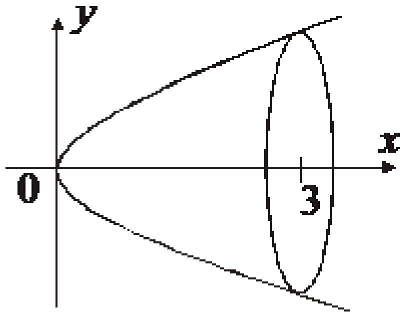
$$V_y = 2\pi \int_a^b x\left(f_2(x) - f_1(x)\right)dx.$$

Example 8. Calculate the volume of a solid formed by rotation of a figure, bounded by the curves $y^2 = 2x$ and $x = 3$ round the Ox-axis.

Solution. Let's find the limits of integration: $a = 0$, $b = 3$.

After this we use the formula for V_x:

$$V_x = \pi\int_0^3 y^2 dx = \pi\int_0^3 2x dx = \pi \cdot 2\frac{x^2}{2}\Big|_0^3 = 9\pi .$$



Example 9. Calculate the volume of a solid formed by rotation of a figure, bounded by the curves $y = x^3$ and $y = 4x$ round the Ox-axis and the Oy-axis.

Solution. Let's find the limits of integration, i.e. solve the system of equations:

$$\begin{cases} y = x^3, \\ y = 4x. \end{cases}$$

We have $x^3 = 4x$ or $x\left(x^2 - 4\right) = 0$.

The roots are $x_1 = 0$, $x_2 = -2$, $x_3 = 2$.

Let's find a half of the volume $\frac{1}{2}V_x$, and we multiply by 2.

We have

$$\frac{1}{2}V_x = \pi\int_0^2 \left(y_2^2 - y_1^2\right)dx = \pi\int_0^2 \left(\left(4x\right)^2 - \left(x^3\right)^2\right)dx =$$

$$= \pi\int_0^2 \left(16x^2 - x^6\right)dx = \pi\left(16\frac{x^3}{3}\Big|_0^2 - \frac{x^7}{7}\Big|_0^2\right) =$$

$$= \pi\left(\frac{128}{3} - \frac{128}{7}\right) = \frac{512}{21}\pi.$$

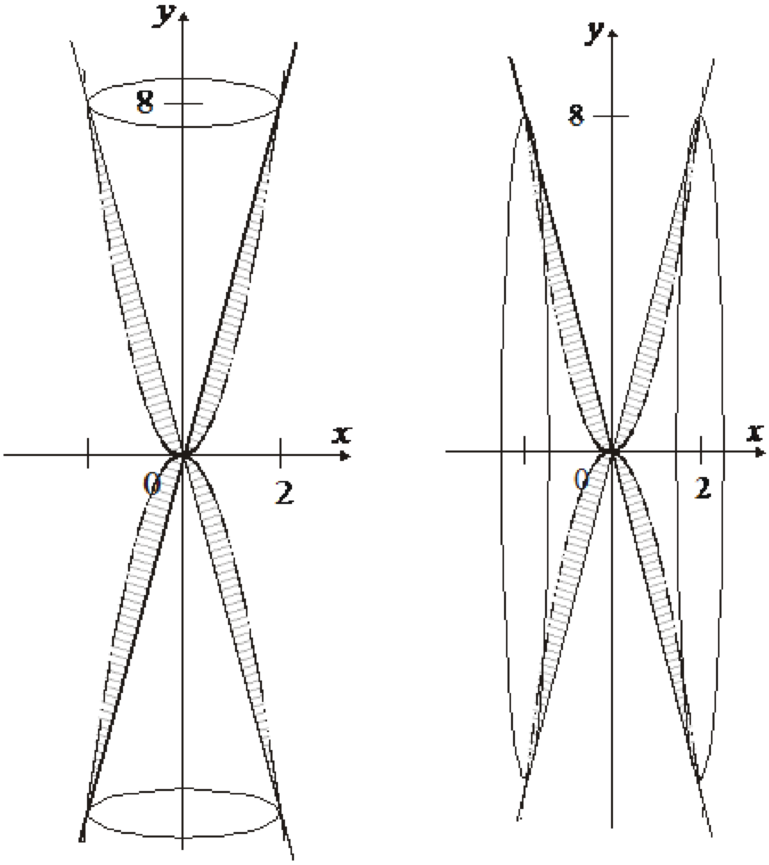
Thus, $V_x = \dfrac{1024}{21}\pi$.

After this we find $\frac{1}{2}V_y$:

$$\frac{1}{2}V_y = \frac{1}{2}2\pi\int_a^b x\left(f_2(x) - f_1(x)\right)dx = \pi\int_0^2 x\left(4x - x^3\right)dx = \pi\int_0^2 \left(4x^2 - x^4\right)dx =$$

$$= \pi\left(4\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^2 = \pi\left(4\frac{2^3}{3} - \frac{5^5}{5}\right) = \pi\frac{128}{15}.$$

Thus, $V_y = \frac{256}{15}\pi$.



7. Application of definite integral for economic problems

Problem 1. Let the function $z = f(t)$ describe the changing of the productivity of an enterprise under the time t. Then the volume of production V produced by the time $[t_1, t_2]$ is defined by the formula

$$V = \int_{t_1}^{t_2} f(t)dt.$$

Example 10. Determine the volume of production V produced by an employee over the second working hour if the productivity is defined by the function:

$$f(t)=\frac{2}{3t+4}+3.$$

Solution. The volume of production V is defined by the formula

$$V=\int_{t_1}^{t_2} f(t)dt .$$

In this case

$$V=\int_1^2\left(\frac{2}{3t+4}+3\right)dt=\left(\frac{2}{3}\ln|3t+4|+3t\right)\Big|_1^2=\frac{2}{3}\ln 10+6-\frac{2}{3}\ln 7-3=\frac{2}{3}\ln\frac{10}{7}+3.$$

The second working hour is the time from the first to the second hour.

Problem 2. The mean value of the function $f(x)$ under x from x_1 to x_2 is defined by the formula (the mean value theorem):

$$f(c)=\frac{1}{x_2-x_1}\int_{x_1}^{x_2} f(x)dx .$$

Example 11. Find the mean value of the inputs $K(x)=3x^2+4x+1$, expressed in monetary units (units of money), if the volume of production changes from 0 to 3 units. Indicate the volume of production, under which the inputs take on the mean value.

Solution. Apply the mean value theorem: $K(c)=\dfrac{1}{x_2-x_1}\displaystyle\int_{x_1}^{x_2} K(x)dx$.

In this case

$$K(c)=\frac{1}{3-0}\int_0^3\left(3x^2+4x+1\right)dx=\frac{1}{3}\left(3\frac{x^3}{3}+4\frac{x^2}{2}+x\right)\Bigg|_0^3=\frac{1}{3}\left(x^3+2x^2+x\right)\Big|_0^3=$$

$$=\frac{1}{3}(27+18+3)=16 \text{ (units of money).}$$

The mean value of the inputs equals 16, i.e. $K(c)=16$.

Define the volume of production, under which the inputs take on the mean value, i.e. solve the equation:

$K(c)=16$, $3c^2+4c+1=16$, $3c^2+4c-15=0$.

Find the discriminant and roots of quadratic equation:

$$D=196,\ c_1=-3,\ c_2=\frac{5}{3}.$$

This root $c_1=-3$ is not the value of the volume of production, because it is less than 0. Therefore c equals $\frac{5}{3}$, i.e. $c=\frac{5}{3}$ units of production.