Theme 4. The indefinite integral

1. Direct integration
2. Integration by substitution (change of variable)
3. Integration by parts
4. Application of the substitution method to calculate indefinite integral of rational functions with quadratic trinomial
5. Integration of rational fractions

1. Direct integration

The function $F(x)$ is called *the antiderivative* of the function $f(x)$ if the derivative of $F(x)$ is the function $f(x)$, i.e. $F'(x) = f(x)$.

The set of all antiderivatives $F(x)+C$ of the function $f(x)$ is called *the indefinite integral* and designated by the symbol:

$$\int f(x)dx = F(x)+C,$$

where C is a constant.

The symbols $\int$ and dx are the symbols of an integration.

The symbol dx is an integration by the variable x.

The integration is the finding of the function $F(x)$ with the help of its derivative $f(x)$.

The basic properties of *an indefinite integral:*

1) $\int k \cdot f(x)dx = k \cdot \int f(x)dx$, i.e.

the indefinite integral of k multiplied by $f(x)$ is k multiplied by the indefinite integral of $f(x)$

2) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

the indefinite integral of the sum (or difference) of two functions is the sum (or difference) of two indefinite integrals of functions

3) $\int f(kx+b)dx = \dfrac{1}{k}F(kx+b)+C$,

$kx+b$ is a linear function

the indefinite integral of $f(kx+b)$ is one over $k \cdot$ multiplied by the capital $F(kx+b)+C$.

4) A derivative of an indefinite integral equals an integrand.

$$\left(\int f(x)dx\right)' = f(x).$$

5) A differential of an indefinite integral equals an integrand expression.

Consider the table of the basic indefinite integrals.

The table of basic indefinite integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$.	**11.** $\int \frac{dx}{\sin^2 x} = -ctg\ x + C.$
2. $\int dx = x + C$.	**12.** $\int \frac{dx}{\cos^2 x} = tg\ x + C.$
3. $\int \frac{dx}{x} = ln\lvert x\rvert + C.$	**13.** $\int \frac{dx}{1+x^2} = arctg\ x + C.$
4. $\int 0 \cdot dx = C$	**14.** $\int \frac{dx}{a^2+x^2} = \frac{1}{a} arctg \frac{x}{a} + C$, $(a \neq 0)$.
5. $\int a^x dx = \frac{a^x}{ln\ a} + C$, $(a > 0, a \neq 1)$.	**15.** $\int \frac{dx}{\sqrt{1-x^2}} = arcsin\ x + C.$
6. $\int e^x dx = e^x + C.$	**16.** $\int \frac{dx}{\sqrt{a^2-x^2}} = arcsin \frac{x}{a} + C.$ $(a \neq 0)$
7. $\int sin\ xdx = -cos\ x + C.$	**17.** $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} ln\left\lvert \frac{x-a}{x+a} \right\rvert + C$, $(a \neq 0)$.
8. $\int cos\ xdx = sin\ x + C.$	**18.** $\int \frac{dx}{\sqrt{x^2 \pm a}} = ln\left\lvert x + \sqrt{x^2 \pm a} \right\rvert + C$, $(a \neq 0)$
9. $\int tgxdx = -ln\lvert cos\ x\rvert + C.$	**19.** $\int \frac{dx}{\sin x} = \ln\left\lvert tg \frac{x}{2} \right\rvert + C$
10. $\int ctgxdx = ln\lvert sin\ x\rvert + C.$	**20.** $\int \frac{dx}{\cos x} = \ln\left\lvert tg\left(\frac{x}{2} + \frac{\pi}{4}\right) \right\rvert + C$

Example 1. Find the integral:

a) $\int \left(2x^2 + \frac{7}{2x^3} - e^x \right) dx = \int \left(2x^2 + \frac{7}{2} \cdot x^{-3} - e^x \right) dx = \lvert \text{apply tabular integrals} \rvert$

$= 2\frac{x^3}{3} + \frac{7}{2} \cdot \frac{x^{-3+1}}{-3+1} - e^x + C = \frac{2}{3}x^3 - \frac{7}{4}x^{-2} - e^x + C$

b) $\int \frac{1}{\sin^2(5x-3)} = \lvert \text{apply the 3} - \text{rd property} \rvert = -\frac{1}{5} ctg(5x-3) + C$

$$\int \sin(2x-12)dx = -\frac{1}{2}\cos(2x-12)+C;$$

$$\int \frac{dx}{5-9x} = -\frac{1}{9}\ln|5-9x|+C;$$

$$\int e^{4x+3}dx = \frac{1}{4}e^{4x+3}+C.$$

2. Integration by substitution (change of variable)

This method is one of the main methods to calculate indefinite integrals. If the integrand has the function $f(g(x))$ and the derivative $g'(x)$, i.e.

$$\int f(g(x))g'(x)dx,$$

then this integral is transformed to the integral $\int f(t)dt$ by the substitution $t = g(x)$. This method reduces such integrals to the tabular ones.

Let us consider the following example.

Example 2. Calculate the following integral: $\int \frac{xdx}{16+5x^2}$

Solution. The given integral is not the tabular integral, because the numerator has the expression x.

Let's consider the derivative of the denominator: $(16+5x^2)' = 10x$.

Thus, the numerator is the derivative of the denominator.

Let's apply the method of integration by substitution and make the substitution of the variable: $t = 16+5x^2$, then the derivative of t is $dt = (16+5x^2)'dx = 10xdx$.

The numerator of the integrand does not contain the factor 10, therefore let's divide both parts of the derivative $dt = 10xdx$ by 10: $\frac{dt}{10} = xdx$.

Let's get back to the initial integral. Its numerator is $\frac{dt}{10}$, the denominator is t.

$$= \int \frac{\frac{dt}{10}}{t} =$$

Let's apply one of the basic properties of indefinite integral to the given integral, i.e. the factor $\frac{1}{10}$ can be taken out of the sign of the integral:

$$=\frac{1}{10}\int\frac{dt}{t}=$$

This integral is the tabular integral, and it equals $=\frac{1}{10}\ln|t|+C=$

Let's get back to the previous variable and obtain:

$$=\frac{1}{10}\ln\left|16+5x^2\right|+C.$$

It is the result.

Example 3. Calculate following integrals:

$$\int\frac{dx}{x\sqrt{4-\ln^2 x}}=\left|\begin{array}{l}t=\ln x\\ dt=(\ln x)'\,dx=\frac{1}{x}dx\end{array}\right|=\int\frac{dt}{\sqrt{4-t^2}}=$$

$$=\arcsin\frac{t}{2}+C=\arcsin\frac{\ln x}{2}+C$$

$$\int\frac{\sin x dx}{\cos^5 x}=\left|\begin{array}{l}t=\cos x\\ dt=-\sin x dx\\ -dt=\sin x dx\end{array}\right|=\int\frac{-dt}{t^5}=-\int\frac{dt}{t^5}=$$

$$=-\int t^{-5}dt=-\frac{t^{-5+1}}{-5+1}+C=\frac{1}{4t^4}+C=\frac{1}{4\cos^4 x}+C$$

$$\int\frac{x^5dx}{\sqrt{x^{12}-5}}=\left|\begin{array}{l}t=x^6\\ dt=6x^5dx\end{array}\right|=\frac{1}{6}\int\frac{dt}{\sqrt{t^2-5}}=\frac{1}{6}\ln\left|t+\sqrt{t^2-5}\right|+c=$$

$$=\frac{1}{6}\ln\left|x^6+\sqrt{x^{12}-5}\right|+c;$$

$$\int\frac{x^3dx}{6x^4+5}=\left|\begin{array}{l}t=6x^4+5\\ dt=24x^3dx\end{array}\right|=\frac{1}{24}\int\frac{dt}{t}=\frac{1}{24}\ln|t|+c=\frac{1}{24}\ln\left|6x^4+5\right|+c;$$

$$\int x\cdot 10^{x^2}dx=\left|\begin{array}{l}t=x^2\\ dt=2xdx\end{array}\right|=\frac{1}{2}\int 10^t dt=\frac{1}{2}\cdot\frac{10^t}{\ln 10}+c=\frac{10^{x^2}}{2\ln 10}+c.$$

3. Integration by parts

This method is based on the famous formula of the derivative of the product of two functions:

$$(u\cdot v)'=u'\cdot v+v'\cdot u$$

where $u = u(x)$ and $v = v(x)$ are some functions of x.

In the differential form we have:

$$d(u \cdot v) = u \cdot dv + v \cdot du.$$

We integrate this formula and obtain $\int d(uv) = \int udv + \int vdu$, then we apply the property $\int dF(x) = F(x) + C$ and find:

$$uv = \int udv + \int vdu \qquad \text{or} \qquad \int udv = uv - \int vdu.$$

Let the functions $u = u(x)$ and $v = v(x)$ have continuous derivatives, the following formula of integration by parts is valid:

$$\int u \cdot dv = uv - \int v \cdot du.$$

This method transforms the given integral $\int u \cdot dv$ to calculation of the integral $\int v \cdot du$. The last one may be reduced to the tabular integrals or the similar ones.

Remark. The name "integration by parts" can be explained as follows: the formula does not produce a final result, but only transforms the problem from calculation of the integral $\int u \cdot dv$ to calculation of the integral $\int v \cdot du$, which is simpler at the successful choice of $u = u(x)$ and $v = v(x)$.

As a rule, this a method of integration by parts is used in the case when the integrand includes a product of rational (a power function) and transcendental functions (an exponential function, a logarithmic function, trigonometrical functions and inverse integration trigonometrical functions).

This method is applied to three types of integrals.

The first type			
	Kind of integral	The factor u	The factor dv
1.	$\int P_n(x) \cdot a^x dx$, where $P_n(x)$ is polynomial	$P_n(x)$	$a^x dx$
2.	$\int P_n(x) \cdot e^x dx$	$P_n(x)$	$e^x dx$
3.	$\int P_n(x) \cdot \sin x dx$	$P_n(x)$	$\sin x dx$
4.	$\int P_n(x) \cdot \cos x dx$	$P_n(x)$	$\cos x dx$

The second type			
	Kind of integral	The factor u	The factor dv

5.	$\int P_n(x)\cdot \arccos x dx$	$\arccos x$	$P_n(x)dx$
6.	$\int P_n(x)\cdot \arcsin x dx$	$\arcsin x$	$P_n(x)dx$
7.	$\int P_n(x)\cdot \mathrm{arctg} x dx$	$\mathrm{arctg} x$	$P_n(x)dx$
8.	$\int P_n(x)\cdot \mathrm{arcctg} x dx$	$\mathrm{arcctg} x$	$P_n(x)dx$
9.	$\int P_n(x)\cdot \ln x dx$	$\ln x$	$P_n(x)dx$

Example 4. Find the integral: $\int x\cdot\cos 3x dx$.

Solution. It is the integral of the first type:

$$u = x,\ dv = cos 3xdx.$$

Let's find $du = dx$, $v = \int cos 3xdx = \frac{1}{3} sin 3x$ (suppose that $C = 0$).

Let's substitute into the formula $\int u\cdot dv = uv - \int v\cdot du$:

$$\int x cos 3xdx = x\cdot\frac{1}{3} sin 3x - \int\frac{1}{3} sin 3xdx.$$

Let's apply the table of the basic integrals:

$$\int x cos 3xdx = \frac{x}{3} sin 3x - \frac{1}{3}\left(-\frac{1}{3} cos 3x\right) + C =$$

$$= \frac{x}{3} sin 3x + \frac{1}{9} cos 3x + C.$$

Example 5. Find $\int\frac{\ln x}{\sqrt[3]{x}} dx$.

Solution. It is the integral of the second type:

$$u = ln x,\ dv = \frac{dx}{\sqrt[3]{x}}.$$

Then

$$du = (\ln x)' dx = \frac{1}{x} dx, \qquad v = \int dv = \int\frac{dx}{\sqrt[3]{x}} = \int x^{-\frac{1}{3}} dx = \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} = \frac{3}{2} x^{\frac{2}{3}}.$$

Let's substitute into the formula $\int u\cdot dv = uv - \int v\cdot du$:

$$\int \frac{ln\,x}{\sqrt[3]{x}}dx = \frac{3}{2}x^{\frac{2}{3}}\,ln\,x - \int \frac{3}{2}x^{\frac{2}{3}}\frac{dx}{x} = \frac{3}{2}x^{\frac{2}{3}}\,ln\,x - \frac{3}{2}\int x^{-\frac{1}{3}}dx =$$

$$= \frac{3}{2}x^{\frac{2}{3}}\,ln\,x - \frac{3}{2}\cdot\frac{3}{2}x^{\frac{2}{3}} + C = \frac{3}{2}x^{\frac{2}{3}}\,ln\,x - \frac{9}{4}x^{\frac{2}{3}} + C.$$

Example 6.

$$\int x^2 \sin x dx = \begin{cases} u = x^2; & dv = \sin x dx; \\ du = 2xdx; & v = -\cos x \end{cases} = -x^2\cos x + \int \cos x \cdot 2xdx =$$

$$= \begin{cases} u = x; & dv = \cos x dx; \\ du = dx; & v = \sin x \end{cases} = -x^2\cos x + 2\left[x\sin x - \int \sin x dx\right] = -x^2\cos x + 2x\sin x + 2\cos$$

4. Application of the substitution method to calculate indefinite integral of rational functions with quadratic trinomial

The rational functions with quadratic trinomial are the functions of the kinds:

$$\int \frac{A}{ax^2+bx+c}dx,\ \int \frac{Ax+B}{ax^2+bx+c}dx,\ \int \frac{A}{\sqrt{ax^2+bx+c}}dx,\ \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx,$$

where ax^2+bx+c is the quadratic trinomial.

Let's consider the first and third kinds: $\int \frac{A}{ax^2+bx+c}dx$, $\int \frac{A}{\sqrt{ax^2+bx+c}}dx$. They are reduced to tabular integrals if you allocate the perfect square in the denominator with the help of formulas:

$$(y \pm z)^2 = y^2 \pm 2yz + z^2$$

(the square of the sum or the square of the difference).

Example. Find $\int \frac{2}{\sqrt{x^2+8x+25}}dx$

Solution. Let's allocate the perfect square in the denominator with the help of the formula:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$x^2+8x+25 = x^2+2\cdot 4\cdot x+16+9 = \underbrace{x^2+2\cdot 4\cdot x+4^2} + 9 = (x+4)^2+9.$$

Let's substitute

$$\int \frac{2}{\sqrt{x^2+8x+25}}dx = \int \frac{2}{\sqrt{(x+4)^2+9}}dx = 2\cdot\int \frac{dx}{\sqrt{(x+4)^2+9}}.$$

Let’s use the tabular integral: $\int \frac{dx}{\sqrt{x^2 \pm a}} = ln\left|x+\sqrt{x^2 \pm a}\right| + C$

We have

$$2\cdot\int \frac{dx}{\sqrt{(x+4)^2+9}} = 2\cdot\ln\left|\left(x+4+\sqrt{(x+4)^2+9}\right)\right| + C =$$

$$= 2\cdot\ln\left|\left(x+4+\sqrt{(x+4)^2+9}\right)\right| + C = 2\cdot\ln\left|\left(x+4+\sqrt{x^2+8x+25}\right)\right| + C.$$

Let’s consider the second and fourth kinds: $\int \frac{Ax+B}{ax^2+bx+c}dx$, $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$. To integrate these functions we should use the following rules:

1) to allocate the perfect square in the trinomial with the help of formulas $(y \pm z)^2 = y^2 \pm 2yz + z^2$ (the square of the sum or the square of the difference) and obtain a new denominator $(x \pm p)^2 \pm q$;

2) to apply the substitution:

$t = x + p$	$t = x - p$
$dt = dx$	$dt = dx$
$x = t - p$	$x = t + p$

3) to present the initial integral as a sum of two integrals, the first one is the tabular integral and the second one may be integrated by substitution.

Let’s consider four kinds of such integrals (see the table of basic integrals, 23-26):

23. $\int \frac{xdx}{x^2 \pm a} = \frac{1}{2}\ln\left|x^2 \pm a\right| + C$

24. $\int \frac{xdx}{a - x^2} = -\frac{1}{2}\ln\left|x^2 - a\right| + C$

25. $\int \frac{xdx}{\sqrt{a-x^2}} = -\sqrt{a-x^2} + C$

26. $\int \frac{xdx}{\sqrt{x^2 \pm a}} = \sqrt{x^2 \pm a} + C$

4) to get back to the previous variable by substitution:

$t = x + p \qquad t = x - p$

Let's give the examples.

Example 8. Let's find this integral

$$\int \frac{(7-8x)dx}{x^2-6x+2} = \int \frac{(7-8x)dx}{(x-3)^2-7} = \left| \begin{matrix} t = x-3 \\ dt = dx \\ x = t+3 \end{matrix} \right| = \int \frac{(7-8(t+3))dt}{t^2-7} =$$

$$= \int \frac{(10-8t)dt}{t^2-7} = 10\int \frac{dt}{t^2-7} - 8\int \frac{tdt}{t^2-7} = 10\int \frac{dt}{t^2-\left(\sqrt{7}\right)^2} - 8\int \frac{tdt}{t^2-7} =$$

$$= \frac{10}{2\sqrt{7}} \ln\left|\frac{t-\sqrt{7}}{t+\sqrt{7}}\right| - \frac{8}{2}\ln\left|t^2-7\right| + C =$$

$$= \frac{5}{\sqrt{7}} \ln\left|\frac{x-3-\sqrt{7}}{x-3+\sqrt{7}}\right| - \frac{8}{2}\ln\left|x^2-6x+2\right| + C.$$

Example 9.

Find $\int \frac{2xdx}{3x^2+5x+4}$.

Solution.

We take out the common factor 3 and allocate the perfect square in the denominator:

$$3x^2+5x+4 = 3\left(x^2+\frac{5}{3}x+\frac{4}{3}\right) = 3\left(x^2+2\cdot\frac{5}{6}x+\frac{25}{36}-\frac{25}{36}+\frac{4}{3}\right)$$

$$= 3\left(x^2+2\cdot\frac{5}{6}x+\frac{25}{36}+\frac{23}{36}\right) = 3\left(\left(x+\frac{5}{6}\right)^2+\frac{23}{36}\right).$$

Let's make the substitution: $t = x+\frac{5}{6}$, then $x = t-\frac{5}{6}$, $dx = dt$.

We have

$$\int \frac{2xdx}{3x^2+5x+4} = \int \frac{2xdx}{3\left(\left(x+\frac{5}{6}\right)^2+\frac{23}{36}\right)} = \frac{1}{3}\int \frac{2\left(t-\frac{5}{6}\right)dt}{t^2+\frac{23}{36}} =$$

$$= \frac{1}{3}\int \frac{2t-\frac{5}{3}}{t^2+\frac{23}{36}}dt = \frac{1}{3}\int \frac{2tdt}{t^2+\frac{23}{36}} - \frac{5}{9}\int \frac{dt}{t^2+\frac{23}{36}} =$$

$$= \frac{1}{3} ln\left(t^2+\frac{23}{36}\right) - \frac{5}{9}\cdot\frac{6}{\sqrt{23}} arctg\,\frac{6t}{\sqrt{23}} + C_1.$$

We get back to the previous variable. Thus,

$$\int \frac{2xdx}{3x^2+5x+4} = \frac{1}{3} ln\left(\left(x+\frac{5}{6}\right)^2+\frac{23}{36}\right) -$$

$$-\frac{5}{9}\cdot\frac{6}{\sqrt{23}} arctg\,\frac{6\left(x+\frac{5}{6}\right)}{\sqrt{23}} + C_1 = .$$

$$= \frac{1}{3} ln\left(x^2+\frac{5}{3}x+\frac{4}{3}\right) - \frac{10}{3\sqrt{23}} arctg\,\frac{6x+5}{\sqrt{23}} + C_1 =$$

$$= \frac{1}{3} ln\left(3x^2+5x+4\right) - \frac{10}{3\sqrt{23}} arctg\,\frac{6x+5}{\sqrt{23}} + C,$$

5. Integration of rational fractions

The rational fraction is called the following relation of the polynomials:

$$\frac{P_n(x)}{Q_m(x)} = \frac{a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_{m-1}x + b_m},$$

where $P_n(x), Q_m(x)$ are polynomials, the degree of $P_n(x)$ is n, the degree of $Q_m(x)$ is m.

The rational fraction is called a improper fraction, if $n \geq m$, i.e. if the degree of the numerator is greater than or equal to the degree of the denominator.

The rational fraction is called a proper fraction, if $n < m$, i.e. if the degree of the numerator is less than the degree of the denominator**.**

For example, $\frac{x^3+1}{x^3-x^2}$ is the improper fraction, $\frac{x^2+2x}{x^4+5x^2}$ is the proper fraction.

Any rational fraction can be presented as a sum of a polynomial and partial fractions. The partial fractions are fractions of four following types:

$$1)\ \frac{A}{x-a};\quad 2)\ \frac{A}{(x-a)^k};\quad 3)\ \int\frac{Ax+B}{x^2+px+q}dx;\quad 4)\ \int\frac{Ax+B}{\left(x^2+px+q\right)^r}dx,$$

where $D=p^2-4q<0$.

Finding integrals of rational fractions should be carried out according to the following scheme:

1. If $n\geq m$ (an improper fraction), we should secure (выделять) an integer part, representing the integrand as a sum of an integer part (a polynomial) and a proper rational fraction.
2. Decompose the denominator of the proper rational fraction $Q_m(x)$ into factors like $(x-a)^k$ and $\left(x^2+px+q\right)^r$, where $D=p^2-4q<0$.
3. Decompose the proper rational fraction into partial ones according to the following theorem:

Theorem. If $Q_m(x)=b_0(x-a)^\alpha\cdot(x-b)^\beta\cdot\ldots\cdot\left(x^2+px+q\right)^\mu\cdot\ \cdot\ldots\cdot\left(x^2+lx+s\right)^\nu$, then the proper irreducible (несократимая) rational fraction $\frac{P_n(x)}{Q_m(x)}$ represented by the following way:

$$\frac{P_n(x)}{Q_m(x)}=\frac{A_0}{(x-a)^\alpha}+\frac{A_1}{(x-a)^{\alpha-1}}+\ldots+\frac{A_{\alpha-1}}{(x-a)}+\frac{B_0}{(x-b)^\beta}+\frac{B_1}{(x-b)^{\beta-1}}+\ldots+\frac{B_{\beta-1}}{(x-b)}+$$

$$+\frac{M_0x+N_0}{\left(x^2+px+q\right)^\mu}+\frac{M_1x+N_1}{\left(x^2+px+q\right)^{\mu-1}}+\ldots+\frac{M_{\mu-1}x+N_{\mu-1}}{\left(x^2+px+q\right)}+\ldots+$$

$$+\frac{P_0x+Q_0}{\left(x^2+lx+s\right)^\nu}+\frac{P_1x+Q_1}{\left(x^2+lx+s\right)^{\nu-1}}+\ldots+\frac{P_{\nu-1}x+Q_{\nu-1}}{\left(x^2+lx+s\right)}.$$

The coefficients $A_0,A_1,\ldots,B_0,B_1,\ldots$ can be defined according to the following. The obtained equation is an identity, therefore, reducing the fractions to a common denominator, we obtain identical polynomials in the numerators on the right and the left. Equating the coefficients of the same power of *x*, we obtain a system of equations for defining undetermined coefficients $A_0,A_1,\ldots,B_0,B_1,\ldots$.

We also can use the following remark to define coefficients: since the polynomials obtained in numerators on the left and right sides of the equation, after reducing to the common denominator are identically equal, their values are equal at any x. Assigning some values to x we will get equations to define unknown coefficients. It is convenient to choose real roots of the denominator. In practice we can use the both methods to find undetermined coefficients simultaneously.

1. Integrals of partial rational fractions are calculated according to the following formulas:

a) $\int \frac{A}{x-a}dx = A\ln|x-a| + C$;

b) $\int \frac{A}{(x-a)^k}dx = \int A(x-a)^{-k}dx = A\frac{(x-a)^{-k+1}}{-k+1} + C$;

c) in this case $\int \frac{Ax+B}{x^2+px+q}dx$ we apply the rule of integration of a quadrate trinomial.

There are some **examples** of integrating the rational fractions.

Example 10. Let’s find $\int \frac{x^3+1}{x^3-x^2}dx$.

This fraction $\frac{x^3+1}{x^3-x^2}$ is improper, because the degree of the numerator is equal to the degree of the denominator. Let us separate the integer part by the division of the polynomials:

$$\begin{array}{r|l} x^3+1 & x^3-x^2 \\ \cline{2-2} -\;\underline{x^3-x^2} & 1 \\ x^2+1 & \end{array}$$

and obtain

$$\frac{x^3+1}{x^3-x^2} = 1 + \frac{x^2+1}{x^3-x^2}.$$

Then the initial integral will be transformed to the following two integrals:

$$\int \frac{x^3+1}{x^3-x^2}dx = \int\left(1+\frac{x^2+1}{x^3-x^2}\right)dx = \int dx + \int \frac{x^2+1}{x^3-x^2}dx.$$

2) The denominator of the second integral has real multiple roots and can be represented as a product $x^2(x-1)$.

Let us decompose the integrand into partial fractions.

$$\frac{x^2+1}{x^2(x-1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1},$$

where A, B, C are undetermined coefficients.

Since the denominators on the left and right are equal and the fractions are identically equal, then the numerators are equal too.

Let us reduce to the common denominator and equate the numerators:

$$x^2+1=A(x-1)+Bx(x-1)+Cx^2.$$

Let us collect (приводить подобные):

$$x^2+1=x^2(B+C)+x(A-B)+(-A).$$

Consider the first method of finding of undetermined coefficients.

To find the coefficients, we equate the coefficients at the equal powers of *x* on the left and on the right:

at x^2: $1=B+C$,

at x^1: $0=A-B$,

at x^0: $1=-A$.

We find the **undetermined coefficients:**

from the 3-rd equation: $A=-1$;

from the 2-nd equation: $A=B=-1$;

from the 1-st equation: $C=1-B=1-(-1)=2$.

So, $\int\frac{x^3+1}{x^3-x^2}dx=\int dx+\int\frac{x^2+1}{x^3-x^2}dx=\int dx+\int\left(\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1}\right)dx=$

$=\int dx+\int\left(\frac{-1}{x^2}+\frac{-1}{x}+\frac{2}{x-1}\right)dx=\int dx-\int\frac{1}{x^2}dx-\int\frac{1}{x}dx+2\int\frac{1}{x-1}dx=$

$=x+\frac{1}{x}-\ln|x|+2\ln|x-1|+C.$

Example 11. $\int\frac{dx}{x^3-1}$.

Let us decompose the fraction into partial fractions:

$$\frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{(x^2+x+1)}.$$

Since the denominators on the left and right are equal and the fractions are identically equal, then the numerators are equal too.

Then $1=A(x^2+x+1)+(Bx+C)(x-1)$.

Let us collect (приводить подобные):

$$1=x^2(A+B)+x(A-B+C)+(A-C).$$

Equating the factors at the identical degrees of *x* we obtain the coefficients:

at x^2: $0 = A + B$,

at x^1: $0 = A - B + C$,

at x^0: $1 = A - C$,

we obtain the coefficients:

from the 1-st equation: $B = -A$,

from the 2-nd equation: $C = -A + B = -A - A = -2A$,

from the 3-rd equation: $1 = A - (-2A)$ or $1 = 3A$ or $A = \frac{1}{3}$,

then $B = -A = -\frac{1}{3}$, $C = -2A = -\frac{2}{3}$.

So, $\int \frac{dx}{x^3 - 1} = \int \left(\frac{A}{x-1} + \frac{Bx + C}{(x^2 + x + 1)} \right) dx = \int \left(\frac{1/3}{x-1} + \frac{-1/3\, x - 2/3}{(x^2 + x + 1)} \right) dx =$

$$= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}dx = \left| \begin{matrix} t = x + \frac{1}{2} \\ dt = dx \\ x = t - \frac{1}{2} \end{matrix} \right| =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{t - \frac{1}{2}}{t^2 + \frac{3}{4}}dt = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{t}{t^2 + \frac{3}{4}}dt + \frac{1}{6}\int \frac{dt}{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\frac{1}{2}\ln\left|t^2 + \frac{3}{4}\right| + \frac{1}{6}\frac{1}{\sqrt{3}/2}\, arctg\, \frac{t}{\sqrt{3}/2} + C =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right| + \frac{1}{6}\frac{1}{\sqrt{3}/2}\, arctg\, \frac{x + \frac{1}{2}}{\sqrt{3}/2} + C =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2 + x + 1\right| + \frac{1}{3\sqrt{3}}\, arctg\, \frac{2x+1}{\sqrt{3}} + C.$$

Example 12. $\int \frac{15x^2 - 4x - 81}{(x-3)(x+4)(x-1)}dx$.

The integrand is a proper fraction, let us decompose it into partial fractions.

$$\frac{15x^2-4x-81}{(x-3)(x+4)(x-1)}=\frac{A}{x-3}+\frac{B}{x+4}+\frac{C}{x-1}.$$

Since the denominators on the left and right are equal and the fractions are identically equal, then the numerators are equal too.

Let us reduce to the common denominator and equate the numerators:

$$15x^2-4x-81=A(x+4)(x-1)+B(x-3)(x-1)+C(x-3)(x+4).$$

Consider the second method of finding of undetermined coefficients.

Let us substitute the first root of the denominator $x=3$ into this expression:

$$15\cdot 9-4\cdot 3-81=A(3+4)(3-1)+B\cdot 0+C\cdot 0,$$

$$42=14A,$$

$$A=3.$$

Let us substitute the second root of the denominator $x=-4$ into this expression:

$$15\cdot 16+4\cdot 4-81=A\cdot 0+B(-4-3)(-4-1)+C\cdot 0,$$

$$175=35B,$$

$$B=5.$$

Let us substitute the third root of the denominator $x=1$ into this expression:

$$15\cdot 1-4\cdot 1-81=A\cdot 0+B\cdot 0+C(1-3)(1+4),$$

$$-70=-10B,$$

$$B=7.$$

So, $\int\frac{15x^2-4x-81}{(x-3)(x+4)(x-1)}dx=\int\left(\frac{A}{x-3}+\frac{B}{x+4}+\frac{C}{x-1}\right)dx=$

$$=\int\left(\frac{3}{x-3}+\frac{5}{x+4}+\frac{7}{x-1}\right)dx=3\int\frac{1}{x-3}dx+5\int\frac{1}{x+4}dx+7\int\frac{1}{x-1}dx=$$

$$=3\ln|x-3|+5\ln|x+4|+7\ln|x-1|+C.$$

6. Integration of expressions containing trigonometric functions

6.1. Let's consider the integrals of kind: $\int sin\, mx\cdot cos\, nxdx$, $\int cos\, mx\cdot cos\, nxdx$, $\int sin\, mx\cdot sin\, nxdx$. These products of trigonometric functions are transformed in sums by formulas:

$$sin\,\alpha\cdot cos\,\beta=\frac{1}{2}\left(sin(\alpha+\beta)+sin(\alpha-\beta)\right),$$

$$cos\,\alpha\cdot cos\,\beta=\frac{1}{2}\left(cos(\alpha+\beta)+cos(\alpha-\beta)\right),$$

$sin\,\alpha \cdot sin\,\beta = \frac{1}{2}\left(cos\left(\alpha-\beta\right)-cos\left(\alpha+\beta\right)\right).$

Example 6. Find the integral: $\int \sin 5x \cos 2x dx$.

Solution. Let's use the formula $sin\,\alpha \cdot cos\,\beta = \frac{1}{2}\left(sin\left(\alpha+\beta\right)+sin\left(\alpha-\beta\right)\right)$:

$$\int sin\,5x\,cos\,2xdx = \int \frac{1}{2}\left(sin\,7x + sin\,3x\right)dx =$$

$$= \frac{1}{2}\int sin\,7xdx + \frac{1}{2}\int sin\,3xdx = \frac{1}{2}\left(-\frac{cos\,7x}{7}\right) +$$

$$+\frac{1}{2}\left(-\frac{cos\,3x}{3}\right) + C = -\frac{cos\,7x}{14} - \frac{cos\,3x}{6} + C.$$

6.2. The integral of kind $\int R\left(sin\,x, cos\,x\right)dx$, where $R\left(sin\,x, cos\,x\right)$ is a rational function of $sin\,x$ and $cos\,x$, is reduced to a rational function using the general trigonometric substitution $tg\frac{x}{2} = t\left(-\pi < x < \pi\right)$, then $sin\,x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2t}{1+t^2}$, $cos\,x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$, i.e. $tg\frac{x}{2} = t$ or $\frac{x}{2} = arctgt$ or $x = 2arctgt$. Then $dx = \frac{2}{1+t^2}dt$.

We have $\int R\left(sin\,x, cos\,x\right)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2tdt}{1+t^2}$.

Example 7. Find the integral: $\int \frac{dx}{3+5\,sin\,x+3\,cos\,x}$

Solution. Let's use the substitution $tg\frac{x}{2} = t$ or $\frac{x}{2} = arctgt$ or $x = 2arctgt$, $dx = 2\frac{dt}{1+t^2}$.

Then

$$\int \frac{dx}{3+5\sin x+3\cos x}=\int \frac{2}{3+5\cdot\frac{2t}{1+t^2}+3\cdot\frac{1-t^2}{1+t^2}}\cdot\frac{dt}{1+t^2}=$$

$$=\int \frac{2\left(1+t^2\right)}{3\left(1+t^2\right)+5\cdot 2t+3\left(1-t^2\right)}\cdot\frac{dt}{1+t^2}=$$

$$=\int \frac{2dt}{3+3t^2+10t+3-3t^2}=\int \frac{2dt}{10t+6}=\int \frac{dt}{5t+3}=$$

$$=\frac{1}{5}ln\left|5t+3\right|+C=\frac{1}{5}ln\left|5tg\frac{x}{2}+3\right|+C.$$

6.3. Let's consider other three cases this integral $\int R\left(\sin x,\cos x\right)dx$. We have the substitutions:

a) $t=\sin x$, if $R\left(\sin x,-\cos x\right)=-R\left(\sin x,\cos x\right)$;

b) $t=\cos x$, if $R\left(-\sin x,\cos x\right)=-R\left(\sin x,\cos x\right)$;

c) $t=tgx$, if $R\left(-\sin x,-\cos x\right)=R\left(\sin x,\cos x\right)$;

or

a) if we have $\int R\left(\sin x\right)\cos xdx$, then the substitution $\sin x=t$ reduces to the integral $\int R\left(t\right)dt$;

b) if we have $\int R\left(\cos x\right)\sin xdx$, then the substitution $\cos x=t$ reduces to the integral $\int R\left(t\right)dt$.

Example 8. Find the integral: $\int \sin^5 x\cos xdx$.

Solution. We have the substitution $t=\sin x$ and $dt=\cos xdx$, because $\left(\sin x\right)'=\cos x$. Then $\int \sin^5 x\cos xdx=\int t^5dt=\frac{t^6}{6}+C=\frac{\sin^6 x}{6}+C$.

6.4. Let's consider the integrals of kind: $\int tg^m xdx$ (or $\int ctg^m xdx$), where m is a whole positive number. We use the substitution $tgx=t$ (or $ctgx=t$) and the formulas $tg^2x=\frac{1}{\cos^2 x}-1$ (or $ctg^2x=\frac{1}{\sin^2 x}-1$).

Example 9. Find the integral: $\int tg^4 xdx$.

Solution. Let's transform the integral:

$$\int tg^4 x dx = \int tg^2 x tg^2 x dx = \int tg^2 x\left(sec^2 x - 1\right)dx =$$

$$= \int tg^2 x\left(\frac{1}{cos^2 x} - 1\right)dx = \int \frac{tg^2 x}{cos^2 x}dx - \int tg^2 x dx.$$

We find the first integral by the substitution $tgx = t$, then $\frac{1}{cos^2 x}dx = dt$.

We have

$$\int \frac{tg^2 x}{cos^2 x}dx = \int t^2 dt = \frac{t^3}{3} + C_1 = \frac{tg^3 x}{3} + C_1.$$

To find the second integral we use the formula $tg^2 x = \frac{1}{\cos^2 x} - 1$:

$$\int tg^2 x dx = \int\left(sec^2 x - 1\right)dx = \int \frac{1}{cos^2 x}dx - \int dx = tgx - x + C_2.$$

The initial integral is

$$\int tg^4 x dx = \frac{tg^3 x}{3} - tgx + x + C, \text{ де } C_1 - C_2 = C.$$

6.5. Let's consider the integrals of kind: $\int sin^m x\, cos^n x dx$.

<u>Case 1.</u> The exponent of the power of sine or cosine is an odd number. If n is an odd number, then we apply the substitution $sin\, x = t$. If m is an odd number, then we apply the substitution $cos\, x = t$.

<u>Case 2.</u> The exponents of the power of sine m and cosine n are even numbers. We apply to transform the integrand using formulas $sin\, x\, cos\, x = \frac{1}{2} sin\, 2x$, $sin^2 x = \frac{1}{2}\left(1 - cos\, 2x\right)$, $cos^2 x = \frac{1}{2}\left(1 + cos\, 2x\right)$.

Example 10. Find the integral: $\int sin^4 x\, cos^5 x dx$.

Solution. n is odd. Let's transform the integral:

$$\int sin^4 x\, cos^5 x dx = \int sin^4 x\, cos^4 x\, cos\, x dx =$$

$$= \int sin^4 x\left(1 - sin^2 x\right)^2 cos\, x dx.$$

We use the substitution $t = sin\, x$ and $dt = cos\, x dx$. Then

$$\int sin^4 x\, cos^5 x dx = \int t^4 \left(1-t^2\right)^2 dt = \int t^4\left(1-2t^2+t^4\right)dt =$$

$$= \int\left(t^4-2t^6+t^8\right)dt = \frac{t^5}{5}-\frac{2t^7}{7}+\frac{t^9}{9}+C =$$

$$= \frac{sin^5 x}{5}-\frac{2\,sin^7 x}{7}+\frac{sin^9 x}{9}+C.$$

Example 11. Find the integral: $\int sin^2 x\, cos^2 x dx$.

Solution. m and n are even. Let's transform the integral:

$$\int sin^2 x\, cos^2 x dx = \int\left(sin\,x\,cos\,x\right)^2 dx = \int\left(\frac{1}{2}sin\,2x\right)^2 dx =$$

$$= \int\frac{1}{4}sin^2 2xdx.$$

Now let's decrease the power by the formula:

$$\int\frac{1}{4}sin^2 2xdx = \frac{1}{4}\int sin^2 2xdx = \frac{1}{4}\int\frac{1-cos\,4x}{2}dx =$$

$$= \frac{1}{8}\int\left(1-cos\,4x\right)dx = \frac{1}{8}x-\frac{1}{8}\int cos\,4xdx.$$

The last integral is found by the substitution $4x = t$, then $4dx = dt.$
We have

$$\int cos\,4xdx = \int cos\,t\frac{dt}{4} = \frac{1}{4}sin\,t+C_1 = \frac{1}{4}sin\,4x+C_1.$$

The initial integral is

$$\int sin^2 x\, cos^2 x dx = \frac{1}{8}x-\frac{1}{8}\left(\frac{1}{4}sin\,4x+C_1\right) =$$

$$= \frac{1}{8}x-\frac{1}{32}sin\,4x+C, \qquad \text{де } -\frac{1}{8}C_1 = C.$$

7. Integration of irrational functions

7.1. Integrals of the form: $\int R\left(x, x^{\frac{m}{n}},\ldots,x^{\frac{r}{s}}\right)dx$, where *R* is a rational function of its arguments are calculated by the substitution $x = t^k$ (*k* is a common denominator of the fractions $\frac{m}{n},\ldots,\frac{r}{s}$) allowing to get rid of irrationality.

Example 12. Find the integral: $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx.$

Solution. Let's find the common denominator of fractions $\frac{2}{3},\frac{1}{6},\frac{1}{3}$. It's 6, then the substitution is $x=t^6$ or $dx=6t^5dt$.

We have

$$\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx=\int \frac{t^6+t^4+t}{t^6\left(1+t^2\right)}\cdot 6t^5dt=$$

$$=6\int \frac{t\left(t^5+t^3+1\right)t^5}{t^6\left(1+t^2\right)}dt=6\int \frac{t^5+t^3+1}{t^2+1}dt.$$

The integrand is an improper fraction. Let's pick out a whole part:

$$\begin{array}{r|l} -\,t^5+t^3+1 & \underline{t^2+1} \\ \underline{t^5+t^3\quad\;\;} & t^3 \\ 1 & \end{array}$$

We have

$$\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx=6\int\left(t^3+\frac{1}{t^2+1}\right)dt=6\left(\frac{t^4}{4}+arctgt\right)+C=$$

$$=6\left(\frac{\sqrt[6]{x^4}}{4}+arctg\sqrt[6]{x}\right)+C=\frac{3}{2}\sqrt[3]{x^2}+6arctg\sqrt[6]{x}+C.$$

7.2. Integrals of the form:

a) $\int R\left(x,\sqrt{a^2-x^2}\right)dx$, b) $\int R\left(x,\sqrt{a^2+x^2}\right)dx$, c) $\int R\left(x,\sqrt{x^2-a^2}\right)dx.$

Such integrals are found by the substitution:

a) $x=a\sin t$ or $x=a\cos t$, b) $x=atgt$ or $x=actgt$, c) $x=\frac{a}{\cos t}$ or $x=\frac{a}{\sin t}$.