Theme 3. Analysis of the function of several variables

Lecture plan

1. Function Definition Domain.
2. Lines and Surfaces of the Level
3. Partial Derivatives of Functions of Several Independent Variables
4. A Total Differential of Functions of Several Independent Variables

1. Function Definition Domain. Lines and Surfaces of the Level

The variable z is called a function of independent variables x, y if for each set of values (x, y) of these variables from the given domain of their variations one or more values of z are established according to some rule or law. The notation: $z = f(x, y)$.

Let's consider two nonempty sets D and U. If each pair $x,\ y$ elements of D set according to a certain rule the only one element z among the set U is put, then one says on the set D there is a mapping or a function is assigned with the set D of U values

$$f : D \to U \text{ or } D \xrightarrow{f} U,$$

where D is the domain of a function; U is the domain of function values.

The domain of the definition of a function can be represented as the whole plane domain XOY or as its part including the boundary or not.

The set of two values x, y is called a "point" in the range of their variation and we deal with the value of the function z at this point. If the function is given by an analytical expression (formula) without any additional conditions, then the range of existence of its analytical expression is considered to be its range of definition (natural domain), i.e. a set of those points, at which the given analytical expression is defined and takes only real and finite values only.

The domain of a function is defined according to the general mathematic requirements: the expression under the root in an even degree must be not negative, under the sign of logarithm it must be positive, the denominator of the fraction is not equal to zero etc.

Example 1. Find domain of the definition of a function

$$z = \ln\left(4 + 4x - y^2\right).$$

Solution. The logarithm is defined for positive values of its argument only and, therefore, $4+4x-y^2>0$ or $4+4x>y^2$. There are no other restrictions on the arguments *x* and *y*.

In order to show the domain *D* geometrically, let us, at first, find its boundary $4+4x=y^2$ or $y^2=4(1+x)$.

The obtained equation defines a parabola (fig. 1). The parabola divides the whole plane into two parts, namely internal and external parts relatively to the parabola. For points belonging to one of its parts the inequality $y^2<4+4x$ is valid, for the other part $y^2>4+4x$ (on the parabola $y^2=4+4x$).

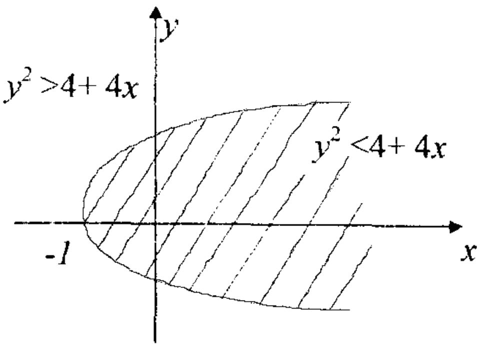

Fig. 1.

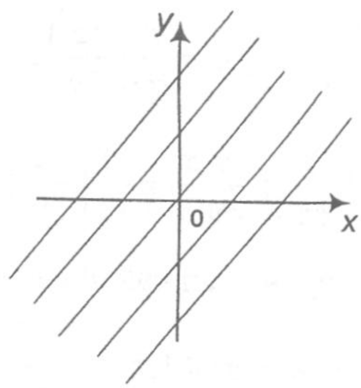

Fig. 2.

In order to understand which of these two parts is the domain of the given function, i.e. satisfies the condition $y^2<4+4x$, we should check this condition for any point not lying on the parabola. For example, the point of origin $O(0,0)$ lies inside the parabola and satisfies the necessary condition $0^2<4+4\cdot 0$. Therefore, the studied domain *D* consists of the points inside the parabola. As the parabola does not belong to the domain *D* we mark the boundary of the domain on the figure (the parabola) with a dotted line.

2. Lines and Surfaces of the Level

The level line of the function $z=f(x,y)$ is a line in the plane domain *XOY* of the function values are constant $f(x,y)=C$. As the example you can consider the lines of the equal heights on the geographical map.

Example 2. Calculate level lines of a function $z=x-y$.

Solution. The level lines of the function are $x-y=C$, $y=x-C$ (fig. 2). At each of these lines a function value is constant.

2. Partial Derivatives of Functions of Several Independent Variables

A partial derivative of the function $z=f(x,y)$ with respect to the variable *x* is defined corresponding to the formula

$$z_x' = \frac{\partial z}{\partial x} = \lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x},$$

calculated at the constant *y*.

A partial derivative of the function $z=f(x,y)$ with respect to the variable *y* calculated by the formula

$$z_y' = \frac{\partial z}{\partial y} = \lim_{\Delta y\to 0}\frac{\Delta_y z}{\Delta y} = \lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

and is calculated at constant *x* .

Calculating partial variables you need to use the common formulae and rules of differentiating of the function with one independent variable.

Example 3. Find the partial derivatives of the function $z=tg\frac{x}{y}$ at the point $M(\pi,1)$.

Solution. Let's find the partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{1}{cos^2\frac{x}{y}}\cdot\frac{1}{y},\quad (y=const);$$

$$\frac{\partial z}{\partial y} = \frac{1}{cos^2\frac{x}{y}}\cdot\frac{-x}{y^2},\quad (x=const).$$

Let's calculate the values $\frac{\partial z}{\partial x}$ and $\frac{\partial x}{\partial y}$ at $x=\pi,\quad y=1$:

$$\frac{\partial z}{\partial x} = \frac{1}{cos^2\frac{\pi}{1}}\cdot 1 = 1, \qquad \frac{\partial z}{\partial y} = \frac{1}{cos^2\pi}\cdot(-\pi) = -\pi .$$

Partial derivatives of the second order are partial derivatives obtained from the derivatives of the first order.

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial z}{\partial x}\right)'_x=\frac{\partial^2 z}{\partial x^2}=z''_{xx};$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial z}{\partial y}\right)'_y=\frac{\partial^2 z}{\partial y^2}=z''_{yy};$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial z}{\partial x}\right)'_y=\frac{\partial^2 z}{\partial x\partial y}=z''_{xy};$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial z}{\partial y}\right)'_x=\frac{\partial^2 z}{\partial y\partial x}=z''_{yx}.$$

The same way we can calculate partial variables at the higher orders.

If mixed partial variables are $\frac{\partial^2 z}{\partial x\partial y}$ and $\frac{\partial^2 z}{\partial y\partial x}$ continuous then they are equal.

Example 4. Find the second order partial derivatives of the function $z=\ln\left(x^2+y\right)$.

Solution. Let's the first order partial derivatives:

$$z_x{}'=\frac{\partial z}{\partial x}=\left(\ln\left(x^2+y\right)\right)_x'=\frac{1}{x^2+y}\cdot\left(x^2+y\right)_x'=$$

$$=\frac{1}{x^2+y}\cdot(2x+0)=\frac{2x}{x^2+y},\qquad (y=const)$$

$$z_y{}'=\frac{\partial z}{\partial y}=\left(\ln\left(x^2+y\right)\right)_y'=\frac{1}{x^2+y}\cdot\left(x^2+y\right)_y'=$$

$$=\frac{1}{x^2+y}\cdot(0+1)=\frac{1}{x^2+y},\qquad (x=const)$$

Let's the second order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2}=\left(\frac{\partial z}{\partial x}\right)'_x=\frac{2\left(x^2+y\right)-2x\cdot 2x}{\left(x^2+y\right)^2}=\frac{2y-2x^2}{\left(x^2+y\right)^2}\ (y = \text{const});$$

$$\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial z}{\partial y}\right)'_y=-\frac{1}{\left(x^2+y\right)^2}$$ (*x* = const);

$$\frac{\partial^2 z}{\partial x\partial y}=\left(\frac{\partial z}{\partial x}\right)'_y=2x\left(-\frac{1}{\left(x^2+y\right)^2}\right)=-\frac{2x}{\left(x^2+y\right)^2}$$ (*x* = const).

$$\frac{\partial^2 z}{\partial y\partial x}=\left(\frac{\partial z}{\partial y}\right)'_x=-\frac{1}{\left(x^2+y\right)^2}\cdot 2x$$ (*y* = const).

Then $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$.

Example 5. Check if the function $z=arctg\,\frac{y}{x}$ satisfies the equation $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$.

Solution. Let's the first order partial derivatives:

$$\frac{\partial z}{\partial x}=\left(arctg\,\frac{y}{x}\right)'_x=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)=\frac{x^2}{x^2+y^2}\cdot\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}$$

(*y* = const);

$$\frac{\partial z}{\partial y}=\left(arctg\,\frac{y}{x}\right)'_y=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}$$ (*x* = const).

Let's the second order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2}=\left(\frac{\partial z}{\partial x}\right)'_x=\left(-\frac{y}{x^2+y^2}\right)'_x=\frac{2xy}{\left(x^2+y^2\right)^2}$$ (*y* = const);

$$\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial z}{\partial y}\right)'_y=\left(\frac{x}{x^2+y^2}\right)'_y=-\frac{x\cdot 2y}{\left(x^2+y^2\right)^2}$$ (*x* = const).

Let's substitute $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ into $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$:

$$\frac{2xy}{\left(x^2+y^2\right)^2}-\frac{2xy}{\left(x^2+y^2\right)^2}=0.$$

Which was to be proved.

4. A Total Differential of Functions of Several Independent Variables

The function $z=f\left(x,y\right)$ is called differentiable at the point M_0 if its total increment

$$\Delta z=f\left(x_0+\Delta x,y_0+\Delta y\right)-f\left(x_0,y_0\right)$$

can be presented as follows

$$\Delta z=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y+\gamma_1\Delta x+\gamma_2\Delta y,$$

where γ_1 and γ_2 tend to zero, if $\Delta x\to 0$ and $\Delta y\to 0$.

The expression $\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y$ is called the main part.

The main part of the total increment in the function z at the point M_0 linear for the increments $\Delta x,\Delta y$ is called a *total differential* of the function z at the point M_0.

Therefore, $dz=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$.

A total differential of the second order is

$$d^2z=\frac{\partial^2 f}{\partial x^2}dx^2+2\frac{\partial^2 f}{\partial x\partial y}dxdy+\frac{\partial^2 f}{\partial y^2}dy^2.$$

At rather small increments in independent variables, i.e. $\rho=\sqrt{\left(\Delta x\right)^2+\left(\Delta y\right)^2}$ the total increment in a function can be approximately changed by its total differential:

$$dz\approx\Delta z$$

or

$$f\left(x_0+\Delta x,y_0+\Delta y\right)\approx f\left(x_0,y_0\right)+df\left(x,y\right).$$

Example 6. Calculate approximately $1,02^{4,03}$.

Solution. Let’s consider the function $z = x^{y}$.

We can consider the value $1,02^{4,03}$ as a partial value of this function at the point $M\left(1,02;4,03\right)$.

Then

$x=1$, $y=4$, $\Delta x=0,02$, $\Delta y=0,03$.

$f\left(1,4\right)=1^{4}=1$, $\frac{\partial f}{\partial x}=yx^{y-1}$, $\left.\frac{\partial f}{\partial x}\right|_{\substack{x=1\\y=4}}=4\cdot 1^{3}=4$,

$\frac{\partial f}{\partial y}=x^{y}\ln x$, $\left.\frac{\partial f}{\partial y}\right|_{\substack{x=1\\y=4}}=1^{4}\cdot \ln 1=0$.

Let’s apply the formula $\Delta z\approx \frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y$ and obtain

$1,02^{4,03}\approx 1+4\cdot 0,02+0\cdot 0,03$

or

$1,02^{4,03}\approx 1,08.$

Example 7. Find dz and $d^{2}z$, if $z=\sin x\cdot \sin y$.

Solution. Let’s find the first order partial derivatives:

$\frac{\partial z}{\partial x}=\left(\sin x\cdot \sin y\right)'_{x}=\sin y\cdot \cos x$;

$\frac{\partial z}{\partial y}=\left(\sin x\cdot \sin y\right)'_{y}=\sin x\cdot \cos y$.

Let’s apply the formula $dz=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$ and obtain

$$dz=\sin y\cdot \cos x dx+\sin x\cdot \cos y dy.$$

We find the second order partial derivatives:

$\frac{\partial^{2}z}{\partial x^{2}}=\left(\sin y\cdot \cos x\right)'_{x}=\sin y\left(-\sin x\right)=-\sin x\sin y$;

$\frac{\partial^{2}z}{\partial x\partial y}=\left(\sin y\cdot \cos x\right)'_{y}=\cos x\cos y$;

$$\frac{\partial^2 z}{\partial y^2} = \left(\sin x \cos y\right)'_y = \sin x\left(-\sin y\right) = -\sin x \sin y .$$

Let’s apply the formula $d^2 z = \frac{\partial^2 f}{\partial x^2} dx^2 + 2\frac{\partial^2 f}{\partial x \partial y} dxdy + \frac{\partial^2 f}{\partial y^2} dy^2$.

Then

$$d^2 z = -\sin x \sin y dx^2 + 2\cos x \cos y dxdy - \sin x \sin y dy^2 .$$