Theme 2. Differential calculus of the function of one variable

Lecture plan

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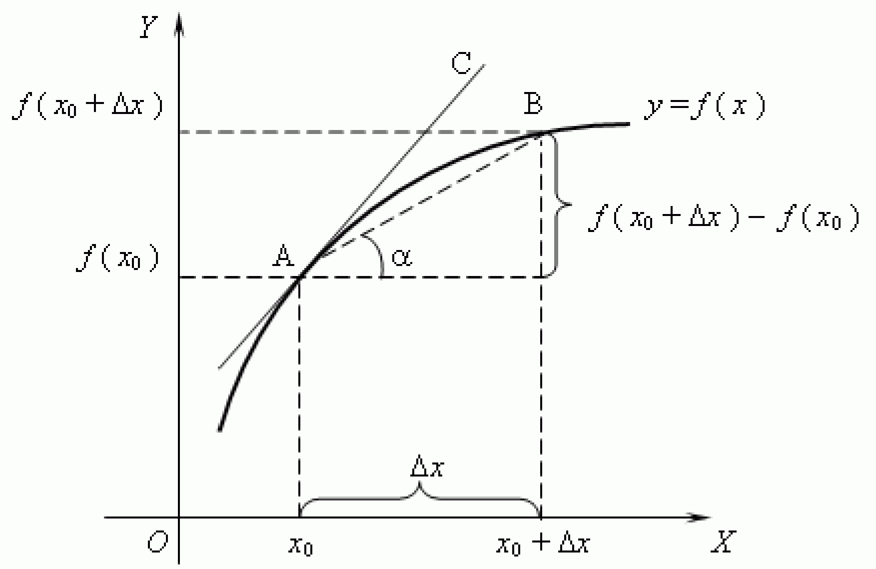
1. A derivative. The elementary rules of derivatives calculation

Definition. The limit of the ratio of the function increment to the argument increment, while the last one approaches zero is called ***a derivative*** of the function $y = f(x)$ at a point x and is denoted as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$



The elementary rules of derivatives calculation

Let the functions $f(x)=u$ and $g(x)=v$ have their derivatives u', v' at some definite point. Then

– the constant multiplier can be taken out from a sign of the derivative, i.e. $(Cu)' = Cu'$;

– the derivative of the algebraic sum (difference) of functions is equal to the algebraic sum (difference) of derivatives of components, i.e. $(u \pm v)' = u' \pm v'$;

– the derivative of the product is equal to the sum of products of the derivative of the first multiplier by the second one without changing and the derivative of the second multiplier by the first one without changing, i.e. $(u \cdot v)' = u' \cdot v + v' \cdot u$;

– derivative of a fraction is equal to a fraction whose denominator is the square of the denominator of the given fraction, and the numerator is the difference between the product of the denominator by the derivative of the numerator and the product of the numerator by the derivative of the denominator, i.e. $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$, if $v \neq 0$.

2. The table of basic derivatives of the simplest elementary functions

The table of derivatives of the simplest elementary functions considering the argument u being some function of *x* is shown below.

The table of basic derivatives

1. $(C)' = 0$	**9.** $(\sin x)' = \cos x$.
2. $\left(x^n\right)' = nx^{n-1}$	**10.** $(\cos x)' = -\sin x$.
3. $(x)' = 1$	**11.** $(tgx)' = \dfrac{1}{\cos^2 x}$.
4. $\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}}$	**12.** $(ctgx)' = -\dfrac{1}{\sin^2 x}$.
5. $\left(a^x\right)' = a^x \cdot \ln a$.	**13.** $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$.
$\left(e^x\right)' = e^x$.	**14.** $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$.
7. $(\ln x)' = \dfrac{1}{x}$.	**15.** $(arctgx)' = \dfrac{1}{1+x^2}$.
8. $(\log_a x)' = \dfrac{1}{x \ln a}$.	**1** $(arcctgx)' = -\dfrac{1}{1+x^2}$.

Example 1. Find the derivative of the function:

$$y = 4x^6 + \frac{1}{2x^3} - 3\sqrt[4]{x^5} + \frac{2}{3}$$

Solution. We transform each summand using the formulas for the power functions: $\dfrac{1}{x^n} = x^{-n}$, $\sqrt[m]{x^n} = x^{\frac{n}{m}}$.

Then we obtain:

$$\frac{1}{2x^3} = \frac{1}{2}x^{-3}; \qquad 3\sqrt[4]{x^5} = 3x^{\frac{5}{4}}$$

and

$$y' = \left(4x^6 + \frac{1}{2x^3} - 3\sqrt[4]{x^5} + \frac{2}{3}\right)' = \left| \begin{array}{l} we\, apply\ the\ property \\ (u \pm v)' = u' \pm v' \end{array} \right| =$$

$$= \left(4x^6\right)' + \left(\frac{1}{2}x^{-3}\right)' - \left(3x^{\frac{5}{4}}\right)' + \left(\frac{2}{3}\right)' =$$

Now, using the table, we calculate the derivative:

$$= \left| \begin{array}{l} C' = 0, (Cu)' = Cu' \\ \left(x^n\right)' = nx^{n-1} \end{array} \right| =$$

$$= 6 \cdot 4x^5 + \frac{1}{2} \cdot (-3) \cdot x^{-3-1} - 3 \cdot \frac{5}{4} x^{\frac{5}{4}-1} + 0 = 24x^5 - \frac{3}{2} x^{-4} - \frac{15}{4} x^{\frac{1}{4}}.$$

Example 2. Find the derivatives of the function: $y = ctgx \cdot x^5$.

Solution.

$$y' = \left[ctgx \cdot x^5\right]' = \left| (u \cdot v)' = u' \cdot v + u \cdot v' \right| =$$

$$= (ctgx)' \cdot x^5 + ctgx \cdot \left(x^5\right)' = \left| \begin{array}{l} (ctgx)' = -\dfrac{1}{\sin^2 x} \\ \left(x^n\right)' = nx^{n-1} \end{array} \right| =$$

$$= \left(-\frac{1}{\sin^2 x}\right) \cdot x^5 + ctgx \cdot 5x^4 = -\frac{x^5}{\sin^2 x} + ctgx \cdot 5x^4.$$

Example 3. Find the derivative of the function: $y = \dfrac{arctgx}{2x + x^3}$.

Solution.

$$y' = \left(\frac{arctgx}{2x + x^3}\right)' = \left| \left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2} \right| =$$

$$= \frac{(arctgx)' \cdot \left(2x + x^3\right) - arctgx \cdot \left(2x + x^3\right)'}{\left(2x + x^3\right)^2} =$$

$$= \frac{\frac{1}{x^2+1} \cdot \left(2x + x^3\right) - arctgx \cdot \left(2 + 3x^2\right)}{\left(2x + x^3\right)^2}.$$

3. The derivative of the composite function

Let the function $u = g(x)$ have the derivative $u' = g'(x)$ at the point x_0, and the function $y = f(u)$ has its derivative $y' = f'(u)$ at the corresponding point $u_0 = g(x_0)$. Then the composite function $y = f(g(x))$ at the mentioned point x_0 will also have a derivative equal to the following

$$y'(x_0) = f'(u_0) \cdot u'(x_0)$$

or

$$y'_x = f'_u \cdot u'_x.$$

In other words the derivative of the composite function is equal to the product of the derivative of the given function with respect to the intermediate argument by the derivative of the intermediate argument with respect to the independent variable.

Example 4. Find the derivative of the function: $y = \left(\sin 5x - e^{3x}\right)^3$.

The given function is composite. Use the rule of deriving of the composite function:

$$y' = \left[\left(\sin 5x - e^{3x}\right)^3\right]' = 3\left(\sin 5x - e^{3x}\right)^{3-1} \cdot \left(\sin 5x - e^{3x}\right)' =$$

$$= 3\left(\sin 5x - e^{3x}\right)^2 \cdot \left(5 \cdot \cos 5x - 3 \cdot e^{3x}\right).$$

4. The derivative of the implicit function

An implicit function $F(x, y) = 0$ is a function that is defined implicitly by a relation between its argument and its value.

If the equation $F(x, y) = 0$ becomes an identity, when *y* is replaced by the function $f(x)$, then one says that $y = f(x)$ is *an implicit function* defined by the equation $F(x, y) = 0$.

In order to find the derivative y' of the function $y = f(x)$ implicitly given by the equation $F(x, y) = 0$ it is necessary to differentiate both the parts of the identity $F(x, y(x)) = 0$ with respect to the variable x using the rule of differentiation of a composite function. Then the obtained equation should be solved for y'.

Example 5. Find the derivative of the function given by the following equation:

$$x^2 + y^2 = 9.$$

Solution. Differentiating the given identity with respect to x we obtain the following:

$$\left(x^2\right)' + \left(y^2\right)' = (9)',$$
$$2x + 2yy' = 0.$$

Let's express y':

$$2yy' = -2x,$$
$$y' = -x / y.$$

5. The derivative of the power exponential function

Let the function $y = f(x)$ have its derivative $y' = f'(x)$ rather difficult to calculate using previously described methods and formulas, but its Napierian logarithm $\ln f(x)$ is the function that can be easily differentiated. Then, in order to find the derivative, we should use the method of logarithmic differentiating including sequential taking the logarithm of the initial function $\ln y = \ln f(x)$ and then its differentiating as an implicit function.

Thus, if $\ln y = \ln f(x)$, then $\frac{y'}{y} = (\ln f(x))'$, whence we find $y' = y \cdot (\ln f(x))'$.

If the function $y = f(x)^{g(x)}$ is given, then you take the natural logarithm of the initial function $\ln y = \ln f(x)^{g(x)}$ and in differentiation it as an implicit function:

$$\frac{1}{y} y' = (g(x) \cdot \ln f(x))',$$

whence we find

$$y' = y(g(x) \cdot \ln f(x))' \text{ or } y' = f(x)^{g(x)}(g(x) \cdot \ln f(x))'.$$

Example Find the derivative of the following function: $y = x^{\sin x}$.

Solution. There is no formula to differentiate the given function in the table. Therefore we use the method of logarithmic differentiation. Let's take the logarithm of this function:

$$\ln y = \ln x^{\sin x}.$$

Let's use the following property: $\ln a^b = b \cdot \ln a$.

We obtain

$$\ln y = \sin x \cdot \ln x.$$

Differentiating both parts of the equation we obtain the following:

$$(\ln y)' = (\sin x \cdot \ln x)',$$

$$\frac{1}{y} \cdot y' = (\sin x)' \ln x + \sin x \cdot (\ln x)',$$

$$\frac{1}{y} \cdot y' = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}.$$

Let's multiply both parts by y:

$$y' = y \cdot \left(\cos x \cdot \ln x + \sin x \cdot \frac{1}{x} \right),$$

$$y' = x^{\sin x} \cdot \left(\cos x \cdot \ln x + \sin x \cdot \frac{1}{x} \right).$$

Example 7. Find the derivative of the following function: $y = x^{x^3}$.

Solution. The formula for differentiating the given function isn't present in the table. Therefore we use the method of logarithmic differentiating. Let's take the logarithm of this function:

$$\ln y = \ln x^{x^3} = x^3 \cdot \ln x$$

Differentiating both parts of the equation we obtain the following:

$$\frac{y'}{y} = 3x^2 \cdot \ln x + x^3 \cdot \frac{1}{x},$$

whence

$$y' = y \cdot \left(3x^2 \cdot \ln x + x^2\right) = x^{x^3} \cdot \left(3x^2 \cdot \ln x + x^2\right) = x^{x^3+2} \cdot (3\ln x + 1).$$

6. The derivative of the function given in a parametric form

If the function is given in a parametric form, i.e. $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, then its derivative on x can be presented in the following way:

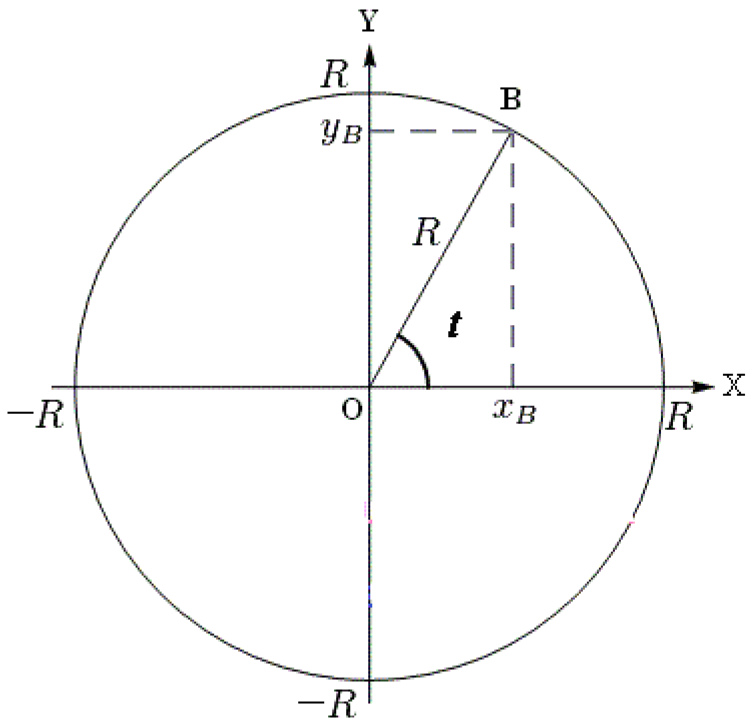
$$y'_x = \frac{dy}{dx} = \frac{y'_t dt}{x'_t dt} = \frac{y'_t}{x'_t}, \text{ i.e. } y'_x = \frac{y'_t}{x'_t}.$$

Example 8. Find the derivative y'_x, if the function is given in a parametric form $\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}$.

Solution. Then

$$y'_t = (R\sin t)'_t = R\cos t,$$

$$x'_t = (R\cos t)'_t = -R\sin t.$$



and

$$y'_x = \frac{y'_t}{x'_t} = \frac{R\cos t}{-R\sin t} = -\frac{x}{y}.$$

7. The geometric meaning of a derivative: the equations of the tangent and the normal line

The derivative of a function at the given point is numerically equal to the slope of the tangent to the curve at this point. Therefore *the equation of a non-vertical tangent line* to the curve $y = f(x)$ at the point x_0 is as follows:

$$y - y_0 = f'(x_0)(x - x_0),$$

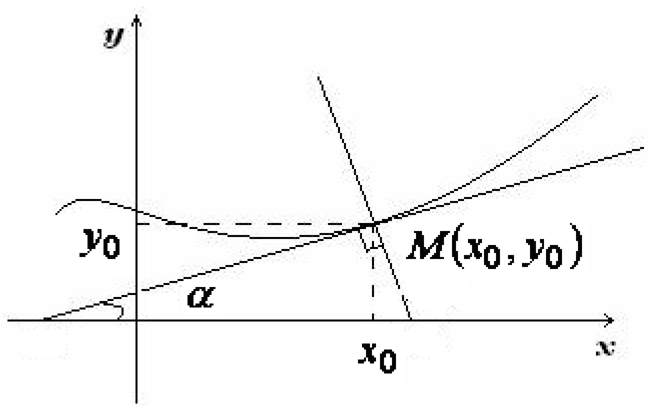
where $f'(x_0) = tg\alpha$, α is the angle of the slope of the tangent to the curve at the point $M(x_0, y_0)$.

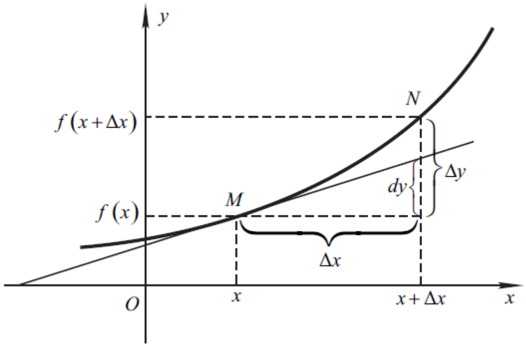
The equation of a vertical tangent line is $x = x_0$.

The normal line to the curve at the point $M(x_0, y_0)$ is a perpendicular to the tangent drawn to this curve at the given point. The equation of a non-horizontal normal line looks like

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0).$$

The equation of a horizontal normal is $y = y_0$.





Example 9. Find the equations of the tangent and the normal line to the curve $y = x^2 + 5$ at the point with the abscissa $x_0 = 1$.

Solution. The ordinate of the tangency point is $y_0 = 1^2 + 5 = 6$. The slope of the tangent is $k = f'(x_0) = f'(1)$. Find it: $f'(x) = y' = \left(x^2 + 5\right)' = 2x$, $f'(1) = 2 \cdot 1 = 2$, then $k_{\tan g} = 2$. The equation of the tangent is as follows:

$$y - y_0 = f'(x_0) \cdot (x - x_0),$$
$$y - 6 = f'(1) \cdot (x - 1),$$
$$y - 6 = 2(x - 1),$$
$$y - 6 = 2x - 2,$$
$$y = 2x - 2 + 6,$$
$$y = 2x + 4.$$

The slope of the normal line is

$k_{norm} = -\dfrac{1}{f'(x_0)} = -\dfrac{1}{f'(1)} = -\dfrac{1}{k_{\tan g}} = -\dfrac{1}{2}$. The equation of the normal line is

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0),$$
$$y - y_0 = -\frac{1}{f'(1)}(x - x_0),$$

$$y - y_0 = k_{norm} \cdot (x - x_0)$$

$$y-6=-\frac{1}{2}(x-1),$$

$$y-6=-\frac{1}{2}x+\frac{1}{2},$$

$$y=-\frac{1}{2}x+\frac{1}{2}+6,$$

$$y=-0.5x+6.5.$$

8. The physical meaning of a derivative

If the function $y=f(x)$ is the equation of the way, then $y'=f(x)'$ is the velocity and $y''=f(x)''$ is the acceleration.

9. The economic meaning of a derivative

An elasticity of function is defined by the formula:

$$E_x(y)=\frac{x}{y}\cdot y'_x.$$

Elasticity of a function $y=f(x)$ approximately shows the change of one variable (y) if the other one (x) is changed within 1 %.

If $|E_x(y)|>1$, then the function is elastic; if $|E_x(y)|=1$, then the function is neutral; if $|E_x(y)|<1$, then the function is inelastic.

Price elasticity of demand (**PED**) is a measure used in economics to show the responsiveness, or elasticity, of the quantity demanded of a good or service to a change in its price. More precisely, it gives the percentage change in quantity demanded in response to a one percent change in price. In general, the demand for a good is said to be *inelastic* (or *relatively inelastic*) when the PED is less than one (in absolute value): that is, changes in price have a relatively small effect on the quantity of the good demanded. The demand for a good is said to be *elastic* (or *relatively elastic*) when its PED is greater than one (in absolute value): that is, changes in price have a relatively large effect on the quantity of a good demanded.

10. The main theorems (the mean-value theorem)

Rolle's theorem. If the function $f(x)$ is continuous on some interval $[a,b]$ and differentiable at all interior points of the interval (a,b), and $f(a)=f(b)$, then there is at least one point $x=c$ belonging to the interval (a,b), where the derivative of the given function vanishes, i.e. it is equal to zero, $f'(c)=0$. The geometrical illustration of Rolle's theorem is presented on figure 1.

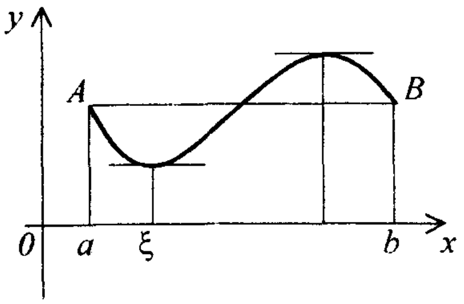


fig. 1.

Lagrange's theorem. If the function $f(x)$ is continuous on some interval $[a,b]$ and has the derivative at all interior points of the interval (a,b), then there is at least one point in the interval (a,b), that $\frac{f(b)-f(a)}{b-a}=f'(\xi)$.

Geometrically Lagrange's theorem is interpreted in the following way. On the arc *AB* (fig. 2) of the continuous curve, to which is possible to draw a tangent at any point, there is at least one point *C* at which the tangent is parallel to the chord. Rolle's theorem is a special case of Lagrange's theorem at $f(a)=f(b)$, i.e. when the chord *AB* is parallel to the axis *OX.*

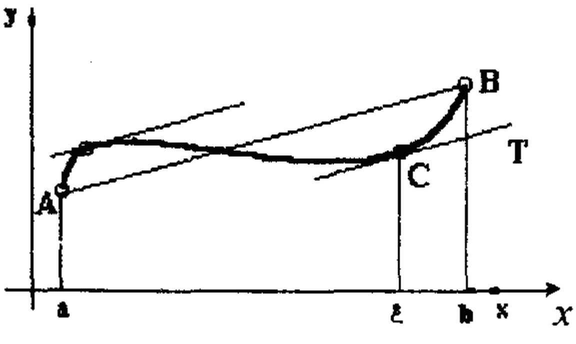


Fig. 2.

Cauchy's Theorem. If the functions $f(x)$ and are continuous on the interval $[a,b]$, have their derivatives $f'(x)$ and $g'(x)$ at all interior points of the interval (a,b), and $g'(x) \neq 0$, then there is at least one point ξ on the interval (a,b) that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}, \text{. where } a < \xi < b.$$

Cauchy's theorem is the generalization of LaGrange's theorem.

11. Differential of a function

The function $y = f(x)$ is considered to be differential at the given point x if the increment Δy of this function at the point x corresponding to be increment of the argument Δx can be shown as:

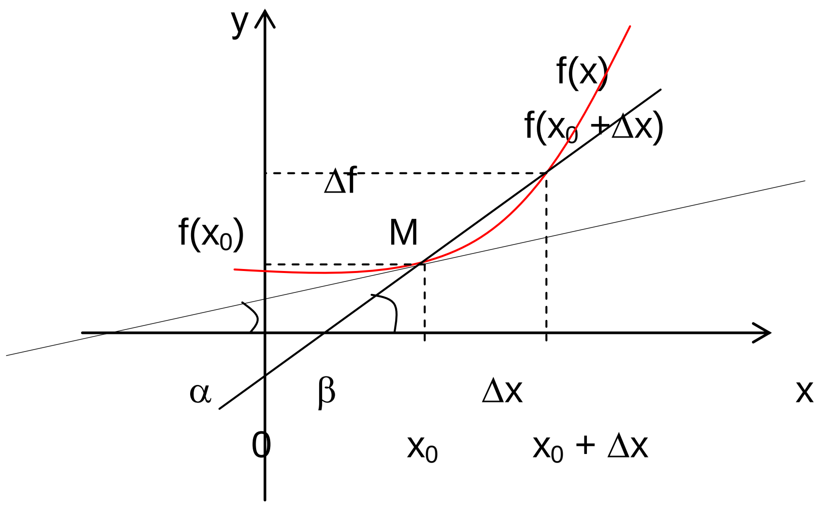
$$\Delta y = A \cdot \Delta x + \alpha \cdot \Delta x,$$

where A is some value, not depending on Δx, and α is a function of the argument Δx being infinitesimal as Δx approaches to zero.

Definition. The main part of the function increment $A \cdot \Delta x$, linear relatively Δx, is called the *differential of the function* and designated as $dy = A \cdot \Delta x$.

The differential of the variable x is equal to its increment $dx = \Delta x$, therefore

$$dy = y' \cdot dx.$$



As we see $y' = \dfrac{dy}{dx}$, i.e. the derivative of the function may be regarded as the ratio of the function differential y to the differential (increment) of the independent variable dx.

12. The application of the differential to approximate calculations

If Δx is small enough it is possible to use the differential of the function in the state of its increment , i.e. $f(x_0+\Delta x)-f(x_0)\approx f'(x_0)\Delta x$ and then get an approximate value of the required according to the following formula:

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x.$$

Example 10. Calculate an approximate value $\sqrt[3]{8,03}$.

Solution. Let's consider the function $y=\sqrt[3]{x}$. Its derivative is $y'=\dfrac{1}{3\sqrt[3]{x^2}}$. Then we obtain from the previous formula:

$$\sqrt[3]{x+\Delta x}\approx\sqrt[3]{x}+\frac{1}{3\sqrt[3]{x^2}}\cdot\Delta x.$$

We have $x=8$, $\Delta x=0,03$. Then $\sqrt[3]{8,03}\approx\sqrt[3]{8}+\dfrac{1}{3\sqrt[3]{8^2}}\cdot 0,03=2,0025$.

13. Derivatives and differentials of the higher orders

Let the function $y=f(x)$ be differentiable in some interval (a,b). Generally, the value of the derivative $f'(x)$ depends on *x*, i.e. the derivative $f'(x)$ is also a function of *x.* If this function is differentiable at some point *x* of the interval (a,b), i.e. has the derivative at this point, then this derivative is called the *second derivative* (or second order derivative) and designated as:

$$y''=(y')'=f''(x).$$

The same way we can introduce the concept of the third order derivative then the concept of the fourth order derivative, etc.

The second order differential is called the differential of the differential function, i.e. $d(dy)=d(y'dx)=y''dx^2=d^2y$ or $d^2y=y''dx^2$.

Generally, the n^{th} order differential is called the first differential of a differential of $(n-1)^{th}$ order

$$d^n y=d\left(d^{n-1}y\right)=y^{(n)}dx^n.$$

Example 11. Find the second order derivative of the function $y=\sqrt{1+x^2}$.

Solution. Let’s find the first order derivative:

$$y'=\left(\sqrt{1+x^2}\right)'=\frac{1}{2\sqrt{1+x^2}}\cdot 2x=\frac{x}{\sqrt{1+x^2}}.$$

Let’s find the derivative of the function $\dfrac{x}{\sqrt{1+x^2}}$, i.e.

$$y''=(y')'=\left(\frac{x}{\sqrt{1+x^2}}\right)'=\frac{\sqrt{1+x^2}-x\cdot\dfrac{1}{2\sqrt{1+x^2}}\cdot 2x}{1+x^2}=$$

$$=\frac{1+x^2-x^2}{\left(1+x^2\right)\sqrt{1+x^2}}=\frac{1}{\sqrt{\left(1+x^2\right)^3}}.$$

Example 12. Find the third order derivative of the function $y=x\,ln\,x$.

Solution. Let’s find the first order derivative:

$$y'=(x\,ln\,x)'=ln\,x+x\cdot\frac{1}{x}=ln\,x+1.$$

Then the second order derivative is

$$y''=(ln\,x+1)'=\frac{1}{x}.$$

Let’s find the third order derivative:

$$y'''=(y'')'=\left(\frac{1}{x}\right)'=-\frac{1}{x^2}.$$

Example 13. Find the n^{th} order derivative of the function $y=a^x$.

Solution. $y'=a^x\,ln\,a$, $y''=a^x\left(ln\,a\right)^2$, $y'''=a^x\left(ln\,a\right)^3$,...,
$y^{(n)}=a^x\left(ln\,a\right)^n$.

Example 14. Find the second order derivative y'' of the implicit function $x+y-xy=1$.

Solution. Let's find y'. We find the derivative of both parts

$$1 + y' - y - xy' = 0.$$

Then $y' = \dfrac{y-1}{1-x}$.

We find the derivative of both parts of the previous equation:

$$y'' = \frac{y'(1-x) + (y-1)}{(1-x)^2}.$$

Let's substitute $y' = \dfrac{y-1}{1-x}$. Then $y'' = \dfrac{\frac{y-1}{1-x}\cdot(1-x) + y - 1}{(1-x)^2} = \dfrac{2(y-1)}{(1-x)^2}$.

Example 15. Find the second order derivative $\dfrac{d^2y}{dx^2}$ of the function given in parametric form $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$

Solution. Let's find $\dfrac{dy}{dx}$: $\dfrac{dy}{dx} = \dfrac{y'_t}{x'_t} = \dfrac{a \sin t}{a(1-\cos t)} = \dfrac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = ctg\,\dfrac{t}{2}$.

Then let's find:

$$\frac{d^2y}{dx^2} = \frac{\left(y'_x\right)'_t}{x'_t} = \frac{\left(ctg\,\frac{t}{2}\right)'_t}{a(1-\cos t)} = \frac{-\frac{1}{2\sin^2\frac{t}{2}}}{a(1-\cos t)} =$$

$$= -\frac{1}{2a\sin^2\frac{t}{2}\cdot 2\sin^2\frac{t}{2}} = -\frac{1}{4a\sin^4\frac{t}{2}}.$$

Example 1 Find the second order differential d^2y of the function $y = \cos 5x$.

Solution.

$$d^2y = y''dx^2. \qquad y' = -5\sin 5x,$$

$$y'' = -25\cos 5x, \qquad d^2y = -25\cos 5x dx^2.$$

14. Evaluation of indeterminate forms by L'Hospital's rule

L'Hospital's rule for evaluation of the indeterminate forms $\left\|\frac{0}{0}\right\|$ and $\left\|\frac{\infty}{\infty}\right\|$ is formulated as the following theorem:

Theorem. Let the single-valued functions $f(x)$ and $g(x)$ be differentiable everywhere in some neighborhood of the point *a*, i.e. at $|x-a|<e$ and $g(x)\neq 0$, then if there exists a limit (finite or infinite) of the ratio of derivatives, then the ratio of the functions has the same limit.

1) $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\left(\left\|\frac{0}{0}\right\| \quad or \quad \left\|\frac{\infty}{\infty}\right\|\right)=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$.

2) $\lim\limits_{x\to \infty}\frac{f(x)}{g(x)}=\left(\left\|\frac{0}{0}\right\| \quad or \quad \left\|\frac{\infty}{\infty}\right\|\right)=\lim\limits_{x\to \infty}\frac{f'(x)}{g'(x)}$.

Note. Let's note once again, that the existence of the ratio limit of the derivatives guarantees the existence of the ratio limit of the functions. The converse is false, i.e. *the ratio limit of the functions can exist without the ratio limit of derivatives.*

Example 17. If $f(x)=x+\sin x$, $f'(x)=1+\cos x$,
$g(x)=x$, $g'(x)=1$,

then $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\frac{x+\sin x}{x}=\left\|\frac{\infty}{\infty}\right\|=\lim\limits_{x\to\infty}\left(1+\frac{\sin x}{x}\right)=1$,

$\lim\limits_{x\to\infty}\frac{f'(x)}{g'(x)}=\lim\limits_{x\to\infty}\frac{1+\cos x}{1}=\lim\limits_{x\to\infty}(1+\cos x)$ - doesn't exist.

Calculate the limits of the following functions using L'Hospital's rule.

Example 18. Calculate the limit: $\lim\limits_{x\to 1}\frac{x^3-1-\ln x}{e^x-e}$.

Solution. $\lim\limits_{x\to 1}\frac{x^3-1-\ln x}{e^x-e}=\left\|\frac{0}{0}\right\|=\lim\limits_{x\to\infty}\frac{3x^2-\frac{1}{x}}{e^x}=\frac{2}{e}$.

Example 19. Calculate the limit: $\lim\limits_{x\to +0}\left(\sqrt{x}\ln x\right)$.

Solution. $\lim\limits_{x\to +0}\left(\sqrt{x}\ln x\right)=\|0\cdot\infty\|$.

Let's transform the indeterminate form $\left\|0 \cdot \infty\right\|$ to the indeterminate form $\left\|\frac{\infty}{\infty}\right\|$.

$$\lim_{x\to+0}\left(\sqrt{x}\,ln\,x\right)=\left\|0\cdot\infty\right\|=\lim_{x\to+0}\frac{lnx}{\frac{1}{\sqrt{x}}}=\left\|\frac{\infty}{\infty}\right\|=\lim_{x\to+0}\frac{\frac{1}{x}}{-\frac{1}{2\sqrt{x^3}}}=-2\lim_{x\to+0}\sqrt{x}=0.$$

Example 20. Calculate the limit: $\lim\limits_{x\to+0}\left(\sqrt{x}\,ln\,x\right)$.

Solution. If $x\to+0$ we have the indeterminate form $0\cdot(-\infty)$. According to L'Hospital's rule we must have the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. To obtain this indeterminate form we transform this limit as: $\lim\limits_{x\to+0}\frac{lnx}{\frac{1}{\sqrt{x}}}$.

The numerator tends to $(-\infty)$ and the denominator tends to ∞, thus, we obtain the indeterminate form $\frac{\infty}{\infty}$.

We can use L'Hospital's rule:

$$\lim_{x\to+0}\frac{lnx}{\frac{1}{\sqrt{x}}}=\lim_{x\to+0}\frac{\frac{1}{x}}{-\frac{1}{2\sqrt{x^3}}}=-2\lim_{x\to+0}\sqrt{x}=0.$$

Example 21. Calculate the limit: $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$.

Solution. In this example we have the indeterminate form $(\infty-\infty)$.

To find the limit of this function we reduce the fractions to the common denominator: $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim\limits_{x\to 0}\frac{e^x-1-x}{x\left(e^x-1\right)}$.

The obtained indeterminate form is $\frac{0}{0}$.

Let's use L'Hospital's rule:

$$\lim_{x\to 0}\frac{e^x-1-x}{x\left(e^x-1\right)}=\lim_{x\to 0}\frac{e^x-1}{e^x-1+xe^x}=\lim_{x\to 0}\frac{e^x}{e^x+e^x+xe^x}=\frac{1}{2}.$$

Example 22. Calculate the limit: $\lim\limits_{x\to 0}\left(sin\right)^x$.

Solution. If x tends to zero we obtain the indeterminate form 0^0. Let's designate this function as y, i.e. $y=\left(sin\,x\right)^x$, and take the natural logarithm: $ln\,y=x\,ln\,sin\,x$.

We find the limit of the natural logarithm of this function:

$$\lim_{x\to 0} ln\,y=\lim_{x\to 0} x\,ln\,sin\,x.$$

We have the indeterminate form $0\cdot\infty$. Let's reduce it to the indeterminate form $\frac{\infty}{\infty}$ and use L'Hospital's rule:

$$\lim_{x\to 0} x\,ln\,sin\,x=\lim_{x\to 0}\frac{ln\,sin\,x}{\frac{1}{x}}=\lim_{x\to 0}\frac{\frac{1}{sin\,x}\cdot cos\,x}{-\frac{1}{x^2}}=$$

$$=-\lim_{x\to 0}\frac{x^2\,cos\,x}{sin\,x}=-\lim_{x\to 0}\frac{2x\,cos\,x-x^2\,sin\,x}{cos\,x}=0.$$

Thus,

$$\lim_{x\to 0} ln\,y=0,$$

then

$$ln\lim_{x\to 0} y=0,\qquad \text{or}\qquad \lim_{x\to 0} y=e^0=1.$$

Example 23. Calculate the limit: $\lim\limits_{x\to 0}\left(1+x\right)^{ln\,x}$.

Solution. If $x\to 0$ then we obtain the indeterminate form 1^∞. Let's designate this function as y and take the natural logarithm:

$$y=\left(1+x\right)^{ln\,x},\ ln\,y=ln\,x\cdot ln\left(1+x\right).$$

Let's find the limit using L'Hospital's rule:

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \ln x \cdot \ln(1+x) = \lim_{x\to 0} \frac{\ln(1+x)}{\frac{1}{\ln x}} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{-\frac{1}{x\ln^2 x}} =$$

$$= -\lim_{x\to 0} \frac{x\ln^2 x}{1+x} = -\lim_{x\to 0} \frac{\ln^2 x}{\frac{1}{x}+1} = -\lim_{x\to 0} \frac{2\ln x \cdot \frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} \frac{2\ln x}{\frac{1}{x}} =$$

$$= 2\lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -2\lim_{x\to 0} x = 0.$$

Thus, $\lim_{x\to 0} \ln y = 0$. Then $\ln \lim_{x\to 0} y = 0$ or $\lim_{x\to 0} y = e^0 = 1$.

L'Hospital's rule of can be successively applied several times if the ratio of the derivatives leads to indeterminate again and derivatives satisfy the conditions of L'Hospital's rule.

Example 24. Calculate the limit: $\lim_{x\to\frac{\pi}{2}} \frac{tgx}{tg3x}$.

Solution. If $x \to \frac{\pi}{2}$ we obtain the indeterminate form $\frac{\infty}{\infty}$. Accorging to L'Hospital's rule we find: $\lim_{x\to\frac{\pi}{2}} \frac{tgx}{tg3x} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 3x}\cdot 3} = \lim_{x\to\frac{\pi}{2}} \frac{\cos^2 3x}{3\cos^2 x}$.

The obtained indeterminate form is $\frac{0}{0}$. Let’s use L'Hospital's rule again and find:

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos^2 3x}{3\cos^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{2\cos 3x(-\sin 3x)\cdot 3}{3\cdot 2\cos x(-\sin x)} = \lim_{x\to\frac{\pi}{2}} \frac{\sin 6x}{\sin 2x}.$$

We obtain the indeterminate form $\frac{0}{0}$ again. We use L'Hospital's rule:

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin 6x}{\sin 2x} = \lim_{x\to\frac{\pi}{2}} \frac{6\cdot\cos 6x}{2\cdot\cos 2x} = 3.$$

1 Conditions of the Function Monotonicity. Extremums

Theorem 1. Let the function $y = f(x)$ be continuous and differentiable on the interval $[a,b]$. In order to the function be constant on $[a,b]$ it is necessary and sufficiently that $f'(x) = 0 \ \forall x \in (a,b)$.

Theorem 2. Let the function $y = f(x)$ be continuous and differentiable on the interval $[a,b]$, then:

a) if $f'(x) > 0 \ \forall x \in (a,b)$, then $f(x)$ increases;

b) if $f'(x) < 0 \ \forall x \in (a,b)$, then $f(x)$ decreases.

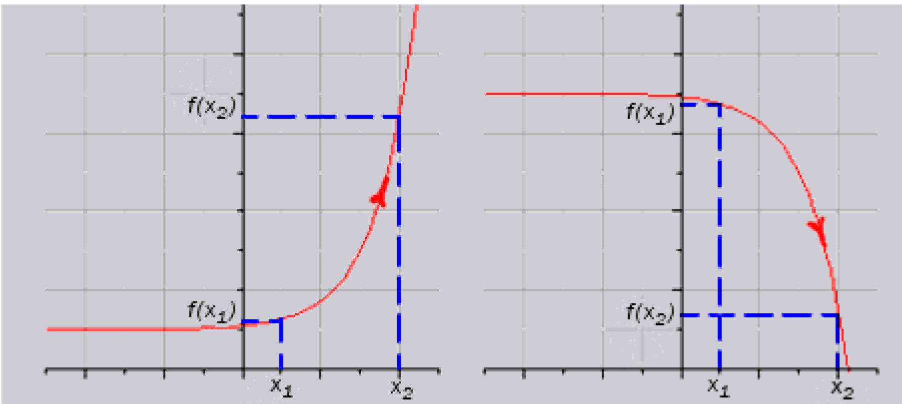
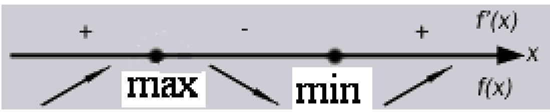


Fig. 1. Increasing function and decreasing function

Theorem 3. If the function $f(x)$ is differentiable and is increasing on the interval (a,b), then $f'(x) > 0 \ \forall x \in (a,b)$. If the function $f(x)$ decreases then $f'(x) < 0 \ \forall x \in (a,b)$.



The point x_0 is called *the point of the maximum (minimum)* of the function $y = f(x)$, if there is such a neighborhood $(x_0 - \delta, x_0 + \delta)$, in which $f(x_0)$ is the greatest (smallest) value among the values of all the points of this interval, i.e. $f(x_0) \geq f(x)$ (or $f(x_0) \leq f(x)$). The points of the maximum and minimum of a function are called the points of the *extremum* of this function.

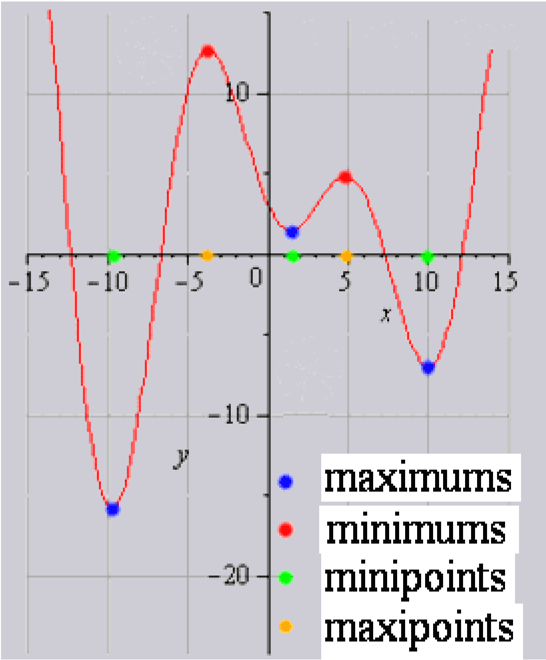
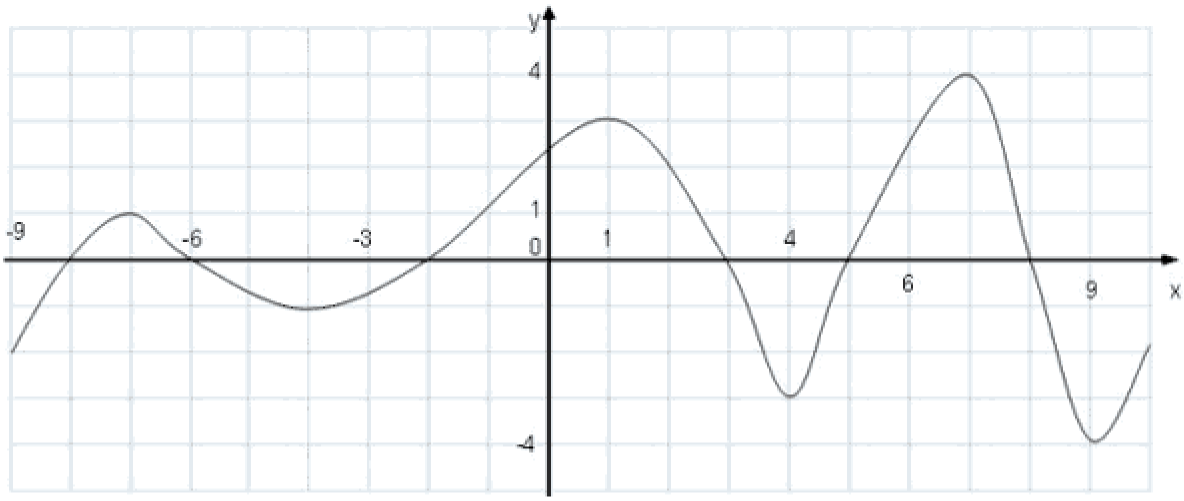


Fig. 2. Extremums

Theorem 4. *(The necessary condition for existence of an extremum).* If the continuous function $y = f(x)$ has an extremum at the point $x = x_0$, then the derivative of the function at this point is either equal to zero or does not exist. *Points,* in which the derivative is equal to zero or does not exist, are called *critical.*



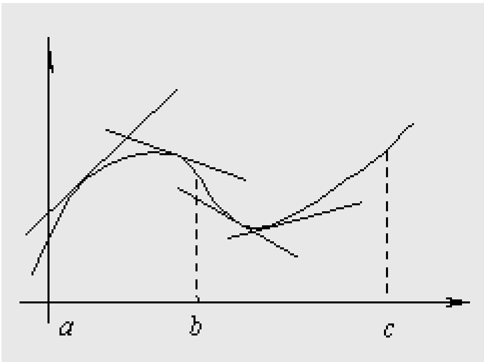
Theorem 5. *(The sufficient condition for existence of an extremum of the function by the first derivative).* Let x_0 be a critical point. Then, if the function $y = f(x)$ has its derivative $f'(x)$ in some neighborhood of the point x_0 and if the derivative $f'(x)$ changes its sign from plus to minus at

passing through the point $x = x_0$, then the function has a maximum at this point, and at changing the sign from minus to plus it has a minimum.

Theorem (*The sufficient condition for existence of an extremum of the function by the second derivative).* If the function $y = f(x)$ in some neighborhood of the point x_0 is continuous, has the second derivative and $f'(x_0) = 0$, $f''(x_0) \neq 0$, then, if $f''(x_0) > 0$, the function has the minimum at the point x_0, if $f''(x_0) < 0$, the function has the maximum at the point x_0.

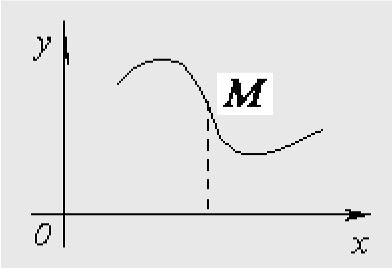
17. Convexity and Concavity of a Curve. Points of Inflection

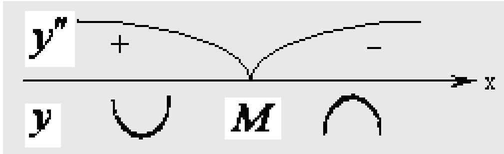
The curve $y = f(x)$ is called convex at the point x_0 if in some neighborhood of this point $(x_0 - \delta, x_0 + \delta)$ it is located below the tangent, drawn at the point x_0. If the curve is located above the tangent, it is called concave.



Theorem 1. If the function $y = f(x)$ in some neighborhood of the point x_0 is doubly continuously differentiable and $f''(x_0) \neq 0$, then the necessary and sufficient condition of the curve convexity at the point x_0 is the requirement $f''(x_0) < 0$, concavity - $f''(x_0) > 0$.

The point $M(x_1, f(x_1))$ is called *the inflection point* of the given curve $y = f(x)$, if there is such neighborhood of the point x_1, that while $x > x_1$, in this neighborhood the convexity of the curve is directed to one direction, and while $x < x_1$ in this neighborhood the concavity to the other direction.





In order to let the point $x = x_0$ be the inflection point of the given curve it is necessary that the second derivative at this point will be either equal to zero $f''(x_0) = 0$ or will not exist.

Theorem 2. (*The sufficient condition for existence of an inflection point)* Let the curve be defined by the equation $y = f(x)$. If $f''(x_0) = 0$ or $f''(x_0)$ does not exist and during passing through $x = x_0$ the derivative $f''(x_0)$ changes its sign, the point of the curve with the abscissa x_0 is a point of inflection.

18. Asymptotes of curves

The straight line $x = x_0$ is called a *vertical asymptote* if $\lim\limits_{x \to x_0 \pm 0} f(x) = \pm\infty$. At a finding vertical asymptotes the break points of the function are investigated. In these points the unilateral limits are calculated.

The asymptote of the plot of the function $y = f(x)$ is a straight line, possessing such a property that the distance between the line and the point on the curve tends to zero while the point on the curve tends to infinity $(x \to \pm\infty)$.

The equation of the *inclined asymptote* looks like $y = kx + b$. In particular, if $k = 0$, the asymptote is horizontal. If the inclined asymptote exists, k and b have to be calculated according to the following formulas:

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x}, \qquad b = \lim_{x \to \pm\infty} \left(f(x) - k \cdot x \right)$$

If, at least, one limit does not exist, the curve has no inclined asymptotes. The asymptotes can vary while $x \to +\infty$ and $x \to -\infty$.

19. The General Plan for Investigating a Function and Constructing Its Plot

1. Definition of existence of the domain of the function.

2. Investigation of the function on continuity. Finding break points of the function and defining their character. Determining vertical asymptotes.

3. Investigation of the function on parity and oddness.

4. Investigation of the function on periodicity.

5. Definition of inclined and horizontal asymptotes.

6. Investigation of the function for extremums. Finding intervals of monotonicity of the function.

7. Finding inflection points of the function, intervals of convexity and concavity.

8. Definition of intersection points with the coordinate axes.

9. Investigation of the function behaviour at infinity.

10. Plot a graph of the function.

Example 1. Investigate and plot the graph of the function: $y = \frac{x^3}{x^2 - 1}$.

1. Let's determine the domain of the function existence: $(-\infty,-1)\cup(-1,1)\cup(1,+\infty)$.

2. Let's research the continuity of the function: $x = 1$ and $x = -1$ are break points of the function, since

$$\lim_{x\to 1\pm 0} \frac{x^3}{x^2-1} = \frac{(1\pm 0)^3}{(1\pm 0)^2 - 1} = \frac{1}{1-1} = \frac{1}{0} = +\infty$$

and

$$\lim_{x\to -1\pm 0} \frac{x^3}{x^2-1} = \frac{(-1\pm 0)^3}{(-1\pm 0)^2 - 1} = \frac{1}{1-1} = \frac{1}{0} = +\infty,$$

and therefore $x = 1$ and $x = -1$ are vertical asymptotes.

3. Investigate the function on parity:

If $f(-x) = f(x)$ then a function is parity.

If $f(-x) = -f(x)$ then a function is oddness.

If $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$ then it's a function of a general view.

Let's check it.

$$f(-x) = \frac{(-x)^3}{(-x)^2 - 1} = \frac{-x^3}{x^2-1} = -\frac{x^3}{x^2-1} = -f(x)$$

4. The function is nonperiodic, i.e. there is no such value T that the equality $f(x+T)=f(x), \quad \forall x \in D(f)$.

5. The inclined asymptote has the equation

$$y = kx + b,$$

where $k = \lim\limits_{x\to\pm\infty} \frac{f(x)}{x}$ and $b = \lim\limits_{x\to\pm\infty} \left(f(x) - k\cdot x\right)$.

If $k = 0$ then the asymptote $y = b$ is called horizontal.

Let's find the inclined asymptotes:

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^3}{x\left(x^2-1\right)} = \lim_{x\to\pm\infty} \frac{x^2}{x^2-1} = \left|\frac{\infty}{\infty}\right| =$$

$$= \lim_{x\to\pm\infty} \frac{\left(x^2\right)'}{\left(x^2-1\right)'} = \lim_{x\to\pm\infty} \frac{2x}{2x-0} = \lim_{x\to\pm\infty} \frac{2x}{2x} = \lim_{x\to\pm\infty} 1 = 1$$

$$b = \lim_{x\to\pm\infty} \left(f(x) - k\cdot x\right) = \lim_{x\to\pm\infty} \left(\frac{x^3}{x^2-1} - x\right) = \lim_{x\to\pm\infty} \left(\frac{x^3 - x^3 + x}{x^2-1}\right) =$$

$$= \lim_{x\to\infty\pm} \frac{x}{x^2-1} = \left|\frac{\infty}{\infty}\right| = \lim_{x\to\infty\pm} \frac{(x)'}{\left(x^2-1\right)'} = \lim_{x\to\infty\pm} \frac{1}{2x} = 0.$$

Hence, this function has an inclined asymptote $y = 1\cdot x + 0$ or $y = x$.

Determine the intervals of monotonicity and the extremums of the function. For this purpose it is necessary to find the first derivative of the function and to determine points, in which it is to zero or does not exist.

Let's find the first derivative of the function:

$$y' = \left(\frac{x^3}{x^2-1}\right)' = \frac{3x^2\cdot(x^2-1) - 2x\cdot x^3}{(x^2-1)^2} = \frac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2} =$$

$$= \frac{x^4 - 3x^2}{(x^2-1)^2} = \frac{x^2\cdot\left(x^2-3\right)}{(x^2-1)^2} = \frac{x^2\cdot\left(x-\sqrt{3}\right)\cdot\left(x+\sqrt{3}\right)}{(x^2-1)^2}.$$

Let's find the critical points:

$$y' = 0 \qquad \text{or}$$

$$y' = \frac{x^2 \cdot \left(x - \sqrt{3}\right) \cdot \left(x + \sqrt{3}\right)}{(x^2 - 1)^2}$$

or $x^2\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)=0$ and $x^2-1\neq 0$ or $x=0$, $x=-\sqrt{3}$, $x=\sqrt{3}$, $x\neq -1,\ x\neq 1$.

Let's determine the intervals of monotonicity.

These points divide a range into 6 intervals, on each of which the first derivative keeps its sign.

If $x\in\left(-\infty;-\sqrt{3}\right)\cup\left(\sqrt{3};\infty\right)$ then $y'>0$, i.e. the function increases.

If $x\in\left(-\sqrt{3};-1\right)\cup\left(-1;0\right)\cup\left(0;1\right)\cup\left(1;\sqrt{3}\right)$ then $y'<0$, i.e. the function decreases.

Hence, the point $x=-\sqrt{3}$ is maximum and the point $x=\sqrt{3}$ is minimum. The values of the functions at these points are equal to

$$y_{\max}\left(-\sqrt{3}\right)=\frac{\left(-\sqrt{3}\right)^3}{\left(-\sqrt{3}\right)^2-1}=-\frac{3\sqrt{3}}{3-1}=-\frac{3\sqrt{3}}{2} \text{ and}$$

$$y_{\min}\left(\sqrt{3}\right)=\frac{\left(\sqrt{3}\right)^3}{\left(\sqrt{3}\right)^2-1}=\frac{3\sqrt{3}}{3-1}=\frac{3\sqrt{3}}{2}.$$

7. To define the intervals of convexity (or concavity) and the inflection points we find the second derivative:

$$y''=\left(y'\right)'=\left(\frac{x^4-3x^2}{(x^2-1)^2}\right)'=$$

$$=\frac{(4x^3-6x)(x^2-1)^2-(x^4-3x^2)4x(x^2-1)}{(x^2-1)^4}=$$

$$=\frac{\left(x^2-1\right)\left((4x^3-6x)(x^2-1)-(x^4-3x^2)4x\right)}{(x^2-1)^4}=$$

$$=\frac{4x^5-4x^3-6x^3+6x-4x^5+12x^3}{(x^2-1)^3}=$$

$$= \frac{2x^3 + 6x}{\left(x^2 - 1\right)^3} = \frac{2x\left(x^2 + 3\right)}{\left(x^2 - 1\right)^3}.$$

If $y'' = 0$ then $y'' = \frac{2x(x^2 + 3)}{(x^2 - 1)^3} = 0$ or $x = 0$, $x^2 + 3 \neq 0$ $x \neq -1$, $x \neq 1$.

We obtain that at $x \in \left(-1; 0\right) \cup \left(1; \sqrt{3}\right) \cup \left(\sqrt{3}; +\infty\right)$ the graph is convex, at $x \in \left(-\infty; -\sqrt{3}\right) \cup \left(-\sqrt{3}; -1\right) \cup \left(0; 1\right)$ the graph is concave.

The point with coordinates $x = 0$ and $y = \frac{0^3}{0^2 - 1} = 0$, i.e. the origin, is a point of the inflection.

8. Determine the points of intersection of the graph with the coordinate axes:

with the *x*-axis: $y = 0$ and $\frac{x^3}{x^2 - 1} = 0$ or $x^3 = 0$ or $x = 0$;

with the *y*-axis: $x = 0$ and $y = \frac{0^3}{0^2 - 1} = 0$.

This point is the origin $O(0; 0)$.

9. Test the function on infinity: $\lim\limits_{x \to \pm\infty} \frac{x^3}{x^2 - 1} = \pm\infty$.

10. Let’s plot a graph of the function (fig. 1).

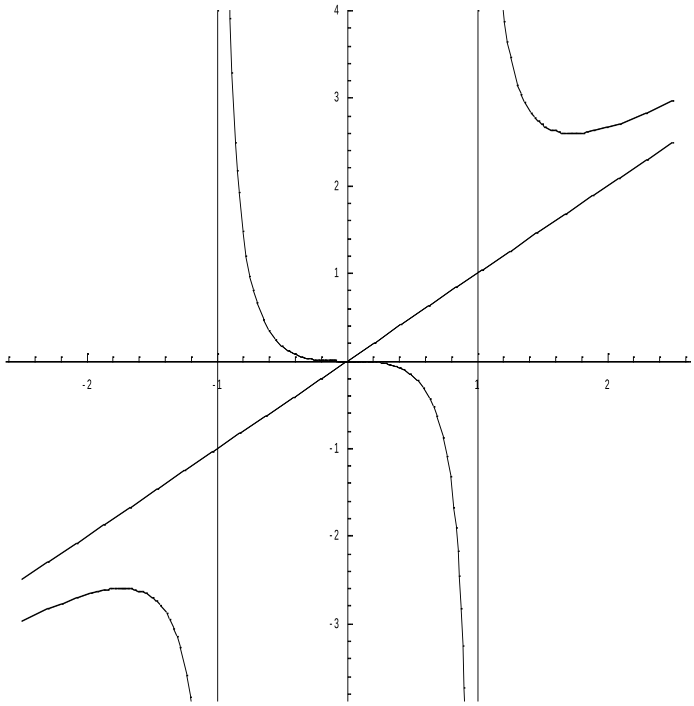


Fig. 1

20. The greatest and the least values of a function on the interval

Let the function $y = f(x)$ be continuous on the interval $[a,b]$. Then, according to Weierstrass' theorem, it achieves the greatest and the least values in this interval. These values can be achieved either on the borders of the interval or at internal points, being the extremums of the function. From this fact we conclude the following plan for defining the greatest and the least values of the function:

1. Define all the critical points, belonging to the given interval $[a,b]$.
2. Calculate the values of the function in the found critical points and on the borders of the interval.
3. Choose the greatest and the least values from the obtained values. The chosen values are required ones.

Example 2. Find the greatest and the least values of the function $f(x) = x^3 - 3x^2 + 1$ on the interval $[-1,4]$.

Solution. Let's find all the critical points using the condition

$$f'(x) = 0.$$

We have the derivative:

$$f'(x) = \left(x^3 - 3x^2 + 1\right)' = 3x^2 - 6x.$$

Let's equate it to 0 and find the critical points:

$$f'(x)=0 \text{ or } 3x^2-6x=0 \text{ or } 3x(x-2)=0.$$

We obtain

$$3x(x-2)=0 \text{ at } x=0 \text{ and } x=2.$$

Thus, the given function has two stationary points $x_1=0$ and $x_2=2$ inside the interval $[-1,4]$.

Let's calculate the function values at these points and on the borders of the interval:

$f(0)=0^3-3\cdot 0^2+1=1$,

$f(2)=2^3-3\cdot 2^2+1=8-12+1=-3$,

$f(-1)=(-1)^3-3\cdot(-1)^2+1=-1-3+1=-3$,

$f(4)=4^3-3\cdot 4^2+1=64-48+1=17$.

As we see, the function takes the greatest value on the right border of the interval $[-1,4]$ and the least value is taken at the internal point $x=2$ and on the left border of the interval, i.e.

$f_{least}=f(2)=f(-1)=-3$,

$f_{greatest}=f(4)=17$.

Theoretical questions

1. What do you call a derivative of a function?
2. Call the elementary rules of derivatives calculation.
3. Call the table of basic derivatives.
4. Describe the calculation of the derivative of the composite function.
5. What do you call the implicit function?
6. Describe the calculation of the derivative of the implicit function.
7. Describe the calculation of the derivative of the power exponential function.
8. Describe the calculation of the derivative of the function given in parametric form.
9. Describe the calculation of logarithmic differentiation.
10. Give the geometric meaning of a derivative.
11. Write down the equation of the tangent.
12. Write down the equation of the normal line.
13. Give the physical meaning of a derivative.

14. What do you call an elasticity of a function?
15. Give the economic meaning of a derivative.
16. Formulate Rolle's theorem.
17. Formulate Lagrange's theorem.
18. Formulate Cauchy's Theorem.
19. What do you call a differential of a function?
20. Describe the application of the differential to approximate calculations.
21. What do you call the second order derivative?
22. What do you call the second order differential?
23. What do you call the n^{th} order derivative?
24. What do you call the n^{th} order differential?
25. Call the evaluation of indeterminate forms by L'Hospital's rule.
26. Describe L'Hospital's rule.
27. Formulate Rolle’s theorem.
28. Formulate Lagrange’s theorem.
29. Formulate Cauchy’s theorem.
30. What do you call the point of the local maximum?
31. What point is called local minimum?
32. What do you call the extremum of the function?
33. Call the necessary condition for existence of an extremum.
34. What points are called critical?
35. Call the sufficient condition for existence of an extremum of the function by the first derivative.
36. Call the sufficient condition for existence of an extremum of the function by the second derivative.
37. What curve is called convex at the point x_0?
38. What curve is called concave at the point x_0?
39. What point is called the inflection point of the curve?
40. Call the sufficient condition for existence of an inflection point.
41. What straight line $x = x_0$ is called a vertical asymptote?
42. What straight line is called the inclined asymptote?
43. What straight line is called the horizontal asymptote?
44. Write down formula for calculation of the coefficients k and b of the inclined asymptote.

45. Call the general plan for Investigating a function and constructing its plot.

46. Call the plan for defining the greatest and the least values of the function.

47.