CRAMER'S METHOD

Example 1. Solve this system using Cramer's method: $\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases}$.

Solution. Let's find the determinant of the given system:

$$\Delta = \begin{vmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 5\cdot 2\cdot 2 + 4\cdot 3\cdot(-1) + 1\cdot 3\cdot(-1) - 4\cdot 2\cdot(-1) - 2\cdot 1\cdot(-1) -$$

$$-3\cdot 3\cdot 5 = 20 - 12 - 3 + 8 + 2 - 45 = -30 \neq 0.$$

Its determinant is non-zero. Let's apply the formulas by Cramer:

$$\Delta_1 = \begin{vmatrix} 0 & -1 & -1 \\ 14 & 2 & 3 \\ 16 & 3 & 2 \end{vmatrix} = 0 - 48 - 42 + 32 - 0 + 28 = -30, \qquad x_1 = \frac{\Delta_1}{\Delta} = \frac{-30}{-30} = 1,$$

$$\Delta_2 = \begin{vmatrix} 5 & 0 & -1 \\ 1 & 14 & 3 \\ 4 & 16 & 2 \end{vmatrix} = 140 + 0 - 16 + 56 - 0 - 240 = -60, \qquad x_2 = \frac{\Delta_2}{\Delta} = \frac{-60}{-30} = 2,$$

$$\Delta_3 = \begin{vmatrix} 5 & -1 & 0 \\ 1 & 2 & 14 \\ 4 & 3 & 16 \end{vmatrix} = 160 - 56 + 0 - 0 - 210 + 16 = -90, \qquad x_3 = \frac{\Delta_3}{\Delta} = \frac{-90}{-30} = 3.$$

Checking by substitution x_1, x_2, x_3 into the initial system:

$$\begin{cases} 5x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 14 \\ 4x_1 + 3x_2 + 2x_3 = 16 \end{cases} \Rightarrow \begin{cases} 5\cdot 1 - 2 - 3 = 0 \\ 1 + 2\cdot 2 + 3\cdot 3 = 14 \\ 4\cdot 1 + 3\cdot 2 + 2\cdot 3 = 16 \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ 14 = 14 \\ 16 = 16 \end{cases}.$$

INVERSE MATRIX METHOD

Example 2. Let's find the solution of the system from **example 1**.

Solution. Here $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 14 \\ 16 \end{pmatrix}$, $A = \begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$.

Let's calculate the determinant of the given system:

$$\Delta = \begin{vmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 5\cdot2\cdot2 + 4\cdot3\cdot(-1) + 1\cdot3\cdot(-1) - 4\cdot2\cdot(-1) - 2\cdot1\cdot(-1) -$$

$$-3\cdot3\cdot5 = 20 - 12 - 3 + 8 + 2 - 45 = -30 \neq 0.$$

Its determinant is non-zero. Let's find the inverse matrix by cofactors:

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5,\ A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & -1 \\ 3 & 2 \end{vmatrix} = -1,\ A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} = -1,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 10,\ A_{22} = (-1)^{2+2}\begin{vmatrix} 5 & -1 \\ 4 & 2 \end{vmatrix} = 14,\ A_{32} = (-1)^{3+2}\begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} = -16,$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = -5,\ A_{23} = (-1)^{2+3}\begin{vmatrix} 5 & -1 \\ 4 & 3 \end{vmatrix} = -19,\ A_{33} = (-1)^{3+3}\begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 11.$$

$$A^{-1} = \frac{1}{-30}\begin{pmatrix} -5 & 10 & -5 \\ -1 & 14 & -19 \\ -1 & -16 & 11 \end{pmatrix}^T = \frac{1}{-30}\begin{pmatrix} -5 & -1 & -1 \\ 10 & 14 & -16 \\ -5 & -19 & 11 \end{pmatrix} = \begin{pmatrix} \frac{5}{30} & \frac{1}{30} & \frac{1}{30} \\ -\frac{10}{30} & -\frac{14}{30} & \frac{16}{30} \\ \frac{5}{30} & \frac{19}{30} & -\frac{11}{30} \end{pmatrix}.$$

The solution of given system is $X = A^{-1}\cdot B$. Then

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\cdot B = \begin{pmatrix} \frac{5}{30} & \frac{1}{30} & \frac{1}{30} \\ -\frac{10}{30} & -\frac{14}{30} & \frac{16}{30} \\ \frac{5}{30} & \frac{19}{30} & -\frac{11}{30} \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 14 \\ 16 \end{pmatrix} = \begin{pmatrix} \frac{1}{6}0 + \frac{14}{30} + \frac{16}{30} \\ -\frac{1}{3}0 - \frac{98}{15} + \frac{128}{15} \\ \frac{1}{6}0 + \frac{266}{30} - \frac{176}{30} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Thus, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

ELEMENTARY ROW OPERATIONS (OR ELEMENTARY TRANSFORMATIONS) ARE:

1) interchanging (exchanging) two different rows;
2) adding a multiple of one row to another row;
3) multiplying one row by a non-zero constant;
4) crossing out one of the same row;
5) crossing out zero row.

JORDAN–GAUSS METHOD

Example 3. Let's find the solution of the system from example 1.

Solution. By elementary row operations of the augmented matrix, we obtain

$$A|B=\left(\begin{array}{ccc|c}5 & -1 & -1 & 0\\ 1 & 2 & 3 & 14\\ 4 & 3 & 2 & 16\end{array}\right)\sim\left(\begin{array}{ccc|c}1 & 2 & 3 & 14\\ 5 & -1 & -1 & 0\\ 4 & 3 & 2 & 16\end{array}\right)\sim\begin{bmatrix} \\ [2]+[1]\cdot(-5)\\ [3]+[1]\cdot(-4)\end{bmatrix}\sim$$

$$\sim\left(\begin{array}{ccc|c}1 & 2 & 3 & 14\\ 0 & -5 & -10 & -40\\ 0 & -11 & -16 & -70\end{array}\right)\sim\begin{bmatrix} \\ [2]:(-5)\\ \\ \end{bmatrix}\sim\left(\begin{array}{ccc|c}1 & 2 & 3 & 14\\ 0 & 1 & 2 & 8\\ 0 & -11 & -16 & -70\end{array}\right)\sim\begin{bmatrix}[1]+[2]\cdot(-2)\\ \\ [3]+[2]\cdot 11\end{bmatrix}\sim$$

$$\sim\left(\begin{array}{ccc|c}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 8\\ 0 & 0 & 6 & 18\end{array}\right)\sim\begin{bmatrix} \\ \\ [3]:6\end{bmatrix}\sim\left(\begin{array}{ccc|c}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 8\\ 0 & 0 & 1 & 3\end{array}\right)\sim\sim\begin{bmatrix}[1]+[3]\\ [2]+[3]\cdot(-2)\\ \\ \end{bmatrix}\sim\left(\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 3\end{array}\right).$$

Then write down the received augmented matrix as this system: $\begin{cases}x_1=1\\ x_2=2\\ x_3=3\end{cases}.$

KRONECKER–CAPELLI THEOREM

1. A linear system is consistent if its basic matrix and its augmented matrix have the same rank, i. e. $rang\ A=rang\ A|B$.

A consistent system is determined if the ranks are equal to the unknowns number, i. e. $rang\ A=rang\ A|B=n$.

2. A consistent system is undetermined if the ranks are less than the unknowns number, i. e. $rang\ A=rang\ A|B<n$.

3. A linear system is inconsistent if its basic matrix and its augmented matrix have the different rank, i. e. $rang\ A\neq rang\ A|B$.

If $rang\ A=rang\ A|B=n$, then carrying out the backward way we obtain the corresponding values of unknowns.

If $rang\ A=rang\ A|B=r<n$, then we should choose *the main (basic)* unknowns, i. e. those ones which coefficients generate the unit matrix. The basic variables are remained on the left, and other $n-r$ variables are transposed to the right parts of equations. The variables placed on the right part of the system are called *free variables.* The basic variables are expressed through free ones using the backward way. The obtained equalities are the *general solution of the system.*

Assigning to free variables any numeric values, we can find corresponding values of the basic variables. Thus we can find the *particular solutions* of the initial system of equations. If free variables are assigned zero value, then the obtained particular solution is called *basic*. If the values of the basic variables are not negative, then the solution is called *supporting*.

Example 4. Investigate the compatibility of this system: $\begin{cases} x_1+2x_2-3x_3+4x_4=7 \\ 2x_1+4x_2+5x_3-x_4=2 \\ 5x_1+10x_2+7x_3+2x_4=11 \end{cases}$.

Solution. By elementary row operations of the augmented matrix, we obtain:

$$A|B=\left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 2 & 4 & 5 & -1 & 2 \\ 5 & 10 & 7 & 2 & 11 \end{array}\right) \sim \begin{bmatrix} \\ [2]+[1]\cdot(-2) \\ [3]+[1]\cdot(-5) \end{bmatrix} \sim \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 11 & -9 & -12 \\ 0 & 0 & 22 & -18 & -24 \end{array}\right) \sim$$

$$\sim \begin{bmatrix} \\ [2]:11 \\ \\ \end{bmatrix} \sim \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \\ 0 & 0 & 22 & -18 & -24 \end{array}\right) \sim \begin{bmatrix} \\ \\ [3]+[2]\cdot(-22) \end{bmatrix} \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \end{array}\right) \sim \begin{bmatrix} [1]+[2]\cdot 3 \\ \\ \end{bmatrix} \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 7 \\ 0 & 0 & 1 & -9/11 & -12/11 \end{array}\right) \sim \begin{bmatrix} [1]+[2]\cdot 3 \\ \\ \end{bmatrix} \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 17/11 & 41/11 \\ 0 & 0 & 1 & -9/11 & -12/11 \end{array}\right).$$

The initial system is equivalent to the following system: $\begin{cases} x_1+2x_2+\dfrac{17}{11}x_4=\dfrac{41}{11} \\ x_3-\dfrac{9}{11}x_4=-\dfrac{12}{11} \end{cases}$.

Let's obtain the general solution: $\begin{cases} x_1=\dfrac{41}{11}-2x_2-\dfrac{17}{11}x_4 \\ x_3=-\dfrac{12}{11}+\dfrac{9}{11}x_4 \end{cases}$,

where x_1, x_3 are basic unknowns, x_2, x_4 are free ones.

For example, obtain the particular solution, if $x_2=1, x_4=-1$:

$x_1=\dfrac{41}{11}-2+\dfrac{17}{11}$ or $x_1=\dfrac{36}{11}$, $\qquad x_3=-\dfrac{12}{11}-\dfrac{9}{11}$ or $x_3=-\dfrac{21}{11}$.

Thus $x_1 = \frac{36}{11}$, $x_2 = 1$, $x_3 = -\frac{21}{11}$, $x_4 = -1$ are the particular solution.

For example, obtain the basic solution, if $x_2 = 0, x_4 = 0$: $x_1 = \frac{41}{11}$, $x_3 = -\frac{12}{11}$.

Thus $x_1 = \frac{41}{11}$, $x_2 = 0$, $x_3 = -\frac{12}{11}$, $x_4 = 0$ are the basic solution.

In this example the basic solution is not the supporting one, because $x_3 = -\frac{12}{11} < 0$.