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INDEFINITE INTEGRAL. INTEGRATION BY PARTS (the 3-rd method)

$$\int u dv = uv - \int v du$$

	Kind of integral	Factor u	Factor dv
1.	$\int P_n(x) a^x dx$, where $P_n(x)$ is a polynomial	$P_n(x)$	$a^x dx$
2.	$\int P_n(x) e^{ax} dx$	$P_n(x)$	$e^{ax} dx$
3.	$\int P_n(x) \sin axdx$	$P_n(x)$	$\sin axdx$
4.	$\int P_n(x) \cos axdx$	$P_n(x)$	$\cos axdx$
5.	$\int P_n(x) \arccos axdx$	$\arccos ax$	$P_n(x)dx$
6.	$\int P_n(x) \arcsin axdx$	$\arcsin ax$	$P_n(x)dx$
7.	$\int P_n(x) arc\,tgaxdx$	$arc\,tgax$	$P_n(x)dx$
8.	$\int P_n(x) arc\,ctgaxdx$	$arc\,ctgax$	$P_n(x)dx$
9.	$\int P_n(x) \ln xdx$	$\ln x$	$P_n(x)dx$
10.	$\int e^{ax} \sin bxdx$	either e^{ax} or $\sin bx$	either $\sin bxdx$ or $e^{ax}dx$
11.	$\int e^{ax} \cos bxdx$	either e^{ax} or $\cos bx$	either $\cos bxdx$ or $e^{ax}dx$

Example 1. Find $\int x \cos 3xdx$.

We have $u = x$, $dv = \cos 3xdx$, using this table.

We get $du = dx$, $v = \int \cos 3xdx = \frac{1}{3}\sin 3x$ (we take $C = 0$).

Using the formula of integration by parts we obtain

$\int x\cos 3xdx = x\cdot\frac{1}{3}\sin 3x - \int \frac{1}{3}\sin 3xdx$ or

$$\int x\cos 3xdx = \frac{x}{3}\sin 3x - \frac{1}{3}\left(-\frac{1}{3}\cos 3x\right) + C =$$

$$= \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + C.$$

Tasks

Find integrals:

61 $\int x\sin 2xdx$. *Answer*: $-\frac{x}{2}\cos 2x + \frac{\sin 2x}{4} + C$.

62 $\int x\cos xdx$. *Answer*: $x\sin x + \cos x + C$.

63 $\int x\, arctgx\, dx$. *Answer*: $\frac{x^2+1}{2} arctg\, x - \frac{x}{2} + C$.

64 $\int xe^{-x}dx$. *Answer*: $C-e^{-x}\left(x+1\right)$.

65 $\int x^2e^{-x}dx$. *Answer*: $C-e^{-x}\left(2+2x+x^2\right)$.

66 $\int x2^x dx$. *Answer*: $\dfrac{2^x}{ln^2 2}\left(x\,ln\,2-1\right)+C$.

67 $\int\left(x^2-3x\right)ln\,xdx$. *Answer*: $\left(\dfrac{x^3}{3}-\dfrac{3}{2}x^2\right)ln\,x-\dfrac{x^3}{9}+\dfrac{3}{4}x^2+C$.

68 $\int x^2 \cos xdx$. *Answer*: $\left(x^2-2\right)\sin x+2x\cos x+C$.

69. $\int e^x \sin xdx$. *Answer*: $\dfrac{e^x\left(\sin x-\cos x\right)}{2}+C$.

70. $\int e^{ax}\cos bxdx$. *Answer*: $\dfrac{e^{ax}\left(b\sin bx+a\cos bx\right)}{a^2+b^2}+C$.

71. $\int\dfrac{arcsin\,x}{\sqrt{1+x}}dx$. *Answer*: $2\sqrt{1+x}\,arcsin\,x+4\sqrt{1-x}+C$.

72. $\int arctgxdx$

73. $\int \arcsin xdx$

74. $\int xe^{2x}dx$

75. $\int x^2\ln xdx$

INTEGRATION OF RATIONAL FUNCTIONS

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}$$

Example 2. Find $\int\frac{(3x+5)dx}{x^2+4x+5}$.

We allocate the perfect square in the denominator:

$x^2+4x+5=x^2+2\cdot 2\cdot x+4+1=(x+2)^2+1$

We make the substitution $t=x+2$, then $dt=dx$, $x=t-2$.

$$\int\frac{(3x+5)dx}{x^2+4x+5}=\int\frac{(3x+5)dx}{(x+2)^2+1}=\int\frac{3(t-2)+5}{t^2+1}dt=$$

$$=\int\frac{3t-1}{t^2+1}dt=\int\frac{3tdt}{t^2+1}-\int\frac{dt}{t^2+1}=3\cdot\frac{1}{2}\int\frac{2tdt}{t^2+1}-arctgt=$$

$$=\frac{3}{2}ln\left(t^2+1\right)-arctgt+C.$$

Let's get back to the previous variable:

$$\int\frac{(3x+5)dx}{x^2+4x+5}=\frac{3}{2}ln\left((x+2)^2+1\right)-arctg\,(x+2)+C=$$

$$=\frac{3}{2}ln\left(x^2+4x+5\right)-arctg\,(x+2)+C.$$

Example 4. Find $\int\frac{\left(x^2+2\right)dx}{x^2\left(x^4-1\right)}$.

The integrand is a proper fraction. Let's decompose the denominator by factors:

$x^2\left(x^4-1\right)=x^2\left(x^2-1\right)\left(x^2+1\right)=x^2(x-1)(x+1)\left(x^2+1\right)$.

Roots $x=1$, $x=-1$ are simple roots; $x=0$ is the multiple roots (its multiplicity equals 2); x^2+1 doesn't have real roots.

The integrand is written as the sum of partial fractions

$$\frac{x^2+2}{x^2\left(x^4-1\right)}=\frac{x^2+2}{x^2(x-1)(x+1)\left(x^2+1\right)}=$$

$$=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{x+1}+\frac{Mx+N}{x^2+1}.$$

We multiply this equality by the denominator $x^2\left(x^4-1\right)$. Then

$$x^2+2=Ax(x-1)(x+1)(x^2+1)+B(x-1)(x+1)(x^2+1)+$$
$$+Cx^2(x+1)(x^2+1)+Dx^2(x-1)(x^2+1)+$$
$$+(Mx+N)x^2(x-1)(x+1).$$

For finding undetermined coefficients we use the combined method (method 1 and method 2).

We take $x=0$, then $2=B\cdot(-1)$, i.e. $B=-2$.

We take $x=1$, then $1+2=C\cdot 2\cdot 2$, i.e. $C=\frac{3}{4}$.

We take $x=-1$, then $1+2=D\cdot(-2)\cdot 2$, i.e. $D=-\frac{3}{4}$.

We equate coefficients at equal degrees of x on the left and on the right. You need to write three equations in order to find A, M, N.

$$\begin{array}{c|l} x^5 & A+C+D+M=0, \\ x^4 & B+C-D+N=0, \\ x & -A=0. \end{array}$$

From the last equation we have $A=0$. Let's substitute A, B, C, D into the first and the second equations and find $M=0$, $N=\frac{1}{2}$.

Thus, $\frac{x^2+2}{x^2(x^4-1)}=\frac{-2}{x^2}+\frac{\frac{3}{4}}{x-1}+\frac{-\frac{3}{4}}{x+1}+\frac{\frac{1}{2}}{x^2+1}$.

We get

$$\int\frac{(x^2+2)dx}{x^2(x^4-1)}=-2\int\frac{dx}{x^2}+\frac{3}{4}\int\frac{dx}{x-1}-\frac{3}{4}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{x^2+1}=$$
$$=\frac{2}{x}+\frac{3}{4}\ln|x-1|-\frac{3}{4}\ln|x+1|+\frac{1}{2}arctgx+C=$$
$$=\frac{2}{x}+\frac{3}{4}\ln\left|\frac{x-1}{x+1}\right|+\frac{1}{2}arctgx+C.$$

Tasks

Find integrals:

76. $\int\frac{5dx}{x-3}$

77. $\int\frac{(3x-1)dx}{x^2-4x+8}$

78. $\int\frac{dx}{x^2-3x+2}$

79. $\int\frac{dx}{x^2+4x-5}$

80. $\int \frac{dx}{x^2-6x+5}$

81. $\int \frac{dx}{(x^2-1)(x-2)}$

81 $\int \frac{xdx}{x^2+3x-4}$

82. $\int \frac{(x^2-x)dx}{x^2-6x+10}$

83. $\int \frac{x^4}{x^2+4}dx$.

Answer: $\frac{x^3}{3}-4x+8arctg\frac{x}{2}+C$.

84. $\int \frac{2x+7}{x^2+x-2}dx$.

Answer: $ln\left|\frac{(x-1)^3}{x+2}\right|+C$.

85. $\int \frac{3x^2+2x-3}{x^3-x}dx$.

Answer: $ln\left|\frac{x^3(x-1)}{x+1}\right|+C$.

86. $\int \frac{x+2}{x^3-2x^2}dx$.

Answer: $\frac{1}{x}+ln\left|\frac{x-2}{x}\right|+C$.

87. $\int \frac{5x^3+2}{x^3-5x^2+4x}dx$.

Answer: $5x+ln\left|\frac{x^{\frac{1}{2}}(x-4)^{\frac{161}{6}}}{(x-1)^{\frac{7}{3}}}\right|+C$.

88. .$\int \frac{3x^2+2x+1}{(x+1)^2(x^2+1)}dx$.

Answer: $ln\frac{\sqrt{x^2+1}}{|x+1|}+arctgx-\frac{1}{x+1}+C$.

89. $\int \frac{2x^2+x+4}{x^3+x^2+4x+4}dx$.

Answer: $ln\left(|x+1|\sqrt{x^2+4}\right)+C$.

90. $\int \frac{x^5+2x^3+4x+4}{x^4+2x^3+2x^2}dx$.

Answer:

$$\frac{x^2}{2}-2x-\frac{2}{x}+2\,ln\left(x^2+2x+2\right)-$$
$$-2arctg\,(x+1)+C.$$

91. $\int \frac{x^2dx}{x^4-1}$.

Answer: $\frac{1}{4}ln\left|\frac{1-x}{1+x}\right|+\frac{1}{2}arctgx+C$.

92. $\int \frac{(x^2-19x+6)dx}{(x-1)(x^2+5x+6)}$

93. $\int \frac{(x^2-x+1)dx}{x^3-3x+2}$

94. $\int \frac{(x^3-3x+4)dx}{x-2}$

95. $\int \frac{(x^3+x+1)dx}{x^2+1}$

96. $\int \frac{dx}{x^2+6x-7}$

97. $\int \frac{(6x+1)dx}{x^2-6x+8}$

98. $\int \frac{(x^2+x+2)dx}{x(x^2-1)}$

99. $\int \frac{(20-3x)dx}{x^3+3x^2-10x}$

100. $\int \frac{(7x+12)dx}{3x^2-5x-2}$

101. $\int \frac{(x^2-38x+157)dx}{(x-1)(x+4)(x-9)}$